4. **Fourier expansion.** (p.12) Consider the function:

$$f(x) = \begin{cases} 1, & 0 < x \leq 1 \\ 0, & x = 0 \\ -1, & -1 \leq x < 0 \end{cases} \tag{6}$$

(a) Express $f(x)$ as a Fourier sine series:

$$f(x) = \sum_{n=1}^{\infty} c_n \sin n\pi x \tag{7}$$

That is, find the Fourier coefficients c_n. (Hint: exploit the orthogonality of the sine functions.)

(b) Draw the approximation to $f(x)$:

$$f_K(x) = \sum_{n=1}^{K} c_n \sin n\pi x \tag{8}$$

for $K = 5$, 10, 20, 100.

(c) Can $f(x)$ be represented as a Fourier cosine series? Explain.

Solution for problem 4. (p.3)

(a) The basic relation is the orthogonality of the sine functions:

$$\int_0^\pi \sin mx \sin nx \, \mathrm{d}x = \begin{cases} 0 & , \quad m \neq n \\ \frac{\pi}{2} & , \quad m = n \end{cases} \tag{42}$$

To use this orthogonality relation to find the kth Fourier coefficient, c_k, multiply the expansion in eq.(7) by $\sin k\pi x$ on both sides and integrate from -1 to 1:

$$\int_{-1}^1 f(x) \sin k\pi x \, \mathrm{d}x = \sum_{n=1}^\infty c_n \int_{-1}^1 \sin k\pi x \sin n\pi x \, \mathrm{d}x \tag{43}$$

which becomes:

$$2\int_0^1 f(x) \sin k\pi x \, \mathrm{d}x = 2c_k \int_0^1 \sin^2 k\pi x \, \mathrm{d}x \tag{44}$$

The integral on the left is found to be:

$$2\int_0^1 f(x) \sin k\pi x \, \mathrm{d}x = \begin{cases} 0 & , \quad k \text{ even} \\ \frac{4}{k\pi} & , \quad k \text{ odd} \end{cases} \tag{45}$$

The integral on the right of eq.(44) is:

$$2c_k \int_0^1 \sin^2 k\pi x \, \mathrm{d}x = c_k \tag{46}$$

Thus we find the kth Fourier coefficient is:

$$c_k = \begin{cases} 0 & , \quad k \text{ even} \\ \frac{4}{k\pi} & , \quad k \text{ odd} \end{cases} \tag{47}$$

(b) We now are in a position to calculate the truncated approximations to $f(x)$ for $K = 5$, 10, 20 and 100, as shown in fig. 1.

(c) The square-wave function $f(x)$ in eq.(6) is anti-symmetric with respect to the origin. The cosine functions are all symmetric with respect the origin. Hence $f(x)$ cannot be expanded in cosine functions.