# Lecture 2

# Info-Gap Robustness of a Beam

## with

# Uncertain Load

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# Contents

**1 Info-Gap Robustness of a Beam With an Uncertain Load (besancon2016lec02-002.tex) 3**

**2 Statically Loaded Beam: Continued (besancon2016lec02-002.tex) 7**
2.1 Load-Uncertainty Envelope . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 7
2.2 Fourier Representation of a Function . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 10
2.3 Geometry of Ellipsoids . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . . 12
2.4 Fourier Ellipsoid Bounded Uncertain Load . . . . . . . . . . . . . . . . . . . . . . . . . . . . 15

**3 Conclusion (besancon2016lec02-002.tex) 18**

# 1 Info-Gap Robustness of a Beam With an Uncertain Load

(Source: Yakov Ben-Haim, 1996, *Robust Reliability in the Mechanical Sciences,* Springer, sections 3.1, 3.2.)

¶ 3 components of reliability analysis:
1. A system model.
2. A failure criterion.
3. An uncertainty model.

¶ We will consider info-gap models of uncertainty and develop, in a preliminary example, the idea of **info-gap robustness.**

¶ Consider a:
- Uniform simply-supported beam.
- Uncertain distributed load density function, $\phi(x)$ [N/m].

¶ We wish to
- Analyze the reliability of the beam given very fragmentary information.
- Optimize the design of the beam by enhancing the reliability.
- Evaluate the impact of different levels and types of information.

¶ What we **do know** about the load:
- $\widetilde{\phi}(x) =$ nominal load density function, [N/m].
- Substantial deviation from the nominal load is bounded along the beam.

¶ What we **do not know** about the load:
- The precise realization of the load density, $\phi(x)$.
- The bound on the deviation of the true from the nominal load.

¶ The disparity between what we
**do know** and what we **need to know**
for a fully competent design or analysis
is an **information gap.**

¶ We represent the load uncertainty with an info-gap model:

$$\mathcal{U}(h,\widetilde{\phi}) = \left\{ \phi(x) : \ \left| \phi(x) - \widetilde{\phi}(x) \right| \le h \right\}, \quad h \ge 0 \tag{1}$$

This is an info-gap **uncertainty model.**

¶ Note the two levels of uncertainty in an info-gap model:
- At fixed $h$: true load profile $\phi(x)$ is unknown.
- Horizon of uncertainty — $h$ — is unknown.

¶ **2 properties of all info-gap models:**
- *Contraction:*

$$\mathcal{U}(0) = \left\{ \widetilde{\phi}(x) \right\} \tag{2}$$

- *Nesting:*

$$h < h' \quad \Longrightarrow \quad \mathcal{U}(h) \subseteq \mathcal{U}(h') \tag{3}$$

¶ **System model:**
- Static bending moment as a function of load profile: $M(x)$.
- For simple-simple beam one finds:

$$M(x) = -\frac{L-x}{L} \int_0^x \phi(u) u \, \mathrm{d}u - \frac{x}{L} \int_x^L \phi(u)(L-u) \, \mathrm{d}u \tag{4}$$

where $L$ is the length of the beam.

¶ The **failure criterion:**
The beam fails if the bending moment $M(x)$ exceeds the critical value $M_\mathrm{c}$:

$$\max_{0 \le x \le L} |M(x)| > M_\mathrm{c} \tag{5}$$

¶ We evaluate the **robustness,** $\widehat{h}$, by combining
System model, uncertainty model, and failure criterion:
The **robustness** is:
The greatest info-gap, $h$,
such that the **system model**
does not violate the **failure criterion**
for any load profile up to **uncertainty** $h$.
We can express $\widehat{h}$ as:

$$\begin{aligned} \widehat{h} &= \text{maximum tolerable uncertainty} && (6) \\ &= \max \{ h : \text{ failure cannot occur} \} && (7) \\ &= \max \left\{ h : \left( \max_{0 \le x \le L} |M(x)| \right) \le M_\mathrm{c} \text{ for all } \phi(x) \text{ in } \mathcal{U}(h, \widetilde{\phi}) \right\} && (8) \\ &= \max \left\{ h : \left( \max_{\phi \in \mathcal{U}(h, \widetilde{\phi})} \max_{0 \le x \le L} |M(x)| \right) \le M_\mathrm{c} \right\} && (9) \end{aligned}$$

We can invert the order of the maxima inside the set.

¶ We begin by evaluating:

$$\max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} |M(x)| = \max\left(\max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} M(x),\ \left|\min_{\phi\in\mathcal{U}(h,\widetilde{\phi})} M(x)\right|\right) \tag{10}$$

¶ To find these extrema note that:

- Other than $\phi(u)$, the integrands of both integrals in eq.(4) on p.4 have the same sign everywhere.
- Thus, extremal $M(x)$ is obtained by choosing
  $\phi(x) = \widetilde{\phi}(x) + h$ or $\phi(x) = \widetilde{\phi}(x) - h$.
- **We consider a special case:** $\widetilde{\phi}(x) =$ positive constant.
- The results:

$$\max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} M(x) = -\frac{(h-\widetilde{\phi})x(L-x)}{2} \tag{11}$$

$$\min_{\phi\in\mathcal{U}(h,\widetilde{\phi})} M(x) = -\frac{(h+\widetilde{\phi})x(L-x)}{2} \tag{12}$$

Hence:

$$\max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} |M(x)| = \frac{(h+\widetilde{\phi})x(L-x)}{2} \tag{13}$$

¶ We are now ready to evaluate the second optimization, on $x$,
in the expression for the robustness, eq.(9) on p.4.
We find the maximum at $x = L/2$, resulting in:

$$\max_{0\le x\le L}\ \max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} |M(x)| = \frac{(h+\widetilde{\phi})L^2}{8} \tag{14}$$

¶ The robustness is the greatest $h$
at which the maximum bending moment $M(x)$
does not exceed the critical value $M_{\rm c}$.
We find:

$$\underbrace{\frac{(h+\widetilde{\phi})L^2}{8}}_{\text{max bending moment}} = \underbrace{M_{\rm c}}_{\text{critical moment}} \implies \widehat{h} = \frac{8M_{\rm c}}{L^2} - \widetilde{\phi} \tag{15}$$

**Design implications:** the robustness, $\widehat{h}$, increases as:

- The beam length $L$ decreases.
- The nominal load $\widetilde{\phi}$ decreases.
- The critical bending moment $M_{\rm c}$ increases.

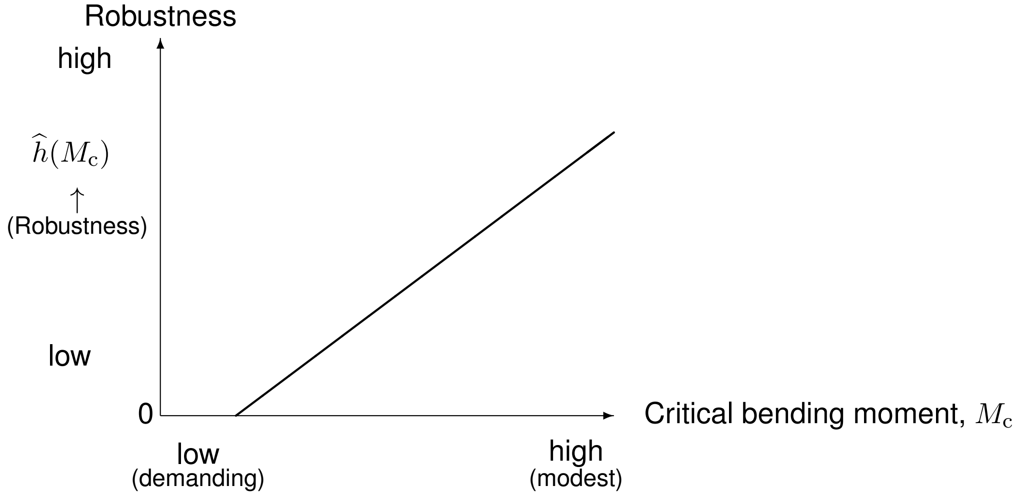


Figure 1: Robustness curve.

¶ **Two Properties:** Trade-off and zeroing (see fig. 1).

¶ **Trade off:** robustness vs performance.

- $\widehat{h}(M_c)$ *gets worse* (decreases) as $M_c$ *gets better* (decreases).
- This is sometimes called the pessimist's theorem. Why?
- The slope of the robustness curve expresses the cost of robustness. Why?

¶ **Zeroing:** Estimated performance has zero robustness:

$$\widehat{h}(M_c) = 0 \quad \text{if} \quad M_c = \frac{\widetilde{\phi} L^2}{8} = \text{estimated bending moment} \tag{16}$$

# 2 Statically Loaded Beam: Continued

¶ Knowledge is:
- Power.
- Robustness against surprise and uncertainty.

## 2.1 Load-Uncertainty Envelope

¶ **Different prior information; different uncertainty.** Examples:
- Hidden load on left half of beam.
- Flow perpendicular to beam; increasing turbulence in middle region.

¶ Let us now consider different prior information.
Rather than the uniform-bound info-gap model of eq.(1) on p.3,
suppose we have information which indicates that
the uncertain deviation $\phi(x) - \widetilde{\phi}(x)$ varies within an envelope:

$$\mathcal{U}(h, \widetilde{\phi}) = \left\{ \phi(x) : \ \left| \phi(x) - \widetilde{\phi}(x) \right| \leq h\psi(x) \right\}, \quad h \geq 0 \tag{17}$$

where we **know:**
$\widetilde{\phi}(x) =$ nominal load profile.
$\psi(x) =$ load-uncertainty envelope.
and we **do not know:**
$\phi(x) =$ actual load profile.
$h =$ uncertainty parameter, horizon of uncertainty.

¶ **Examples of envelope function,** $\psi(x)$:
- Hidden load on left half of beam.

$$\psi(x) = \begin{cases} 1, & 0 \leq x \leq L/2 \\ 0, & L/2 < x \leq L \end{cases} \tag{18}$$

- Flow perpendicular to beam; increasing turbulence in middle region.

$$\psi(x) = \sin \frac{\pi x}{L} \tag{19}$$

¶ As usual with an info-gap model, there are two levels of uncertainty:
- Unknown realization $\phi(x)$ at info-gap $h$.
- Unknown horizon of uncertainty, $h$.

¶ As before:
- The system model is eq.(4) on p.4.
- The failure criterion is eq.(5) on p.4.

¶ To find the maximum absolute bending moment
we evaluate the max and the min of $M_\phi(x)$.
The max (least negative) is obtained with the lowest possible load profile,
while
The min (most negative) is obtained with the greatest possible load profile.
We find:

$$
\begin{aligned}
M_1(x) &= \min_{\phi \in \mathcal{U}(h,\widetilde{\phi})} M(x) && (20)\\
&= -\frac{L-x}{L}\int_0^x \left[\widetilde{\phi}(u) + h\psi(u)\right] u \, \mathrm{d}u \\
&\quad -\frac{x}{L}\int_x^L \left[\widetilde{\phi}(u) + h\psi(u)\right] (L-u) \, \mathrm{d}u && (21)\\
M_2(x) &= \max_{\phi \in \mathcal{U}(h,\widetilde{\phi})} M(x) && (22)\\
&= -\frac{L-x}{L}\int_0^x \left[\widetilde{\phi}(u) - h\psi(u)\right] u \, \mathrm{d}u \\
&\quad -\frac{x}{L}\int_x^L \left[\widetilde{\phi}(u) - h\psi(u)\right] (L-u) \, \mathrm{d}u && (23)
\end{aligned}
$$

We can express these succintly as:

$$
\begin{aligned}
M_1(x) &= M_{\widetilde{\phi}}(x) + hM_\psi(x) && (24)\\
M_2(x) &= M_{\widetilde{\phi}}(x) - hM_\psi(x) && (25)
\end{aligned}
$$

where $M_{\widetilde{\phi}}(x)$ and $M_\psi(x)$ are defined implicitly in eqs.(21) and (23).

¶ Let us consider a **special case:**

The nominal load increases towards the center of the beam:

$$\widetilde{\phi}(x) = \widetilde{\phi} \sin \frac{\pi x}{L} \qquad (26)$$

where $\widetilde{\phi}$ is a known positive constant.

The uncertainty in the load increases towards the center of the beam:

$$\psi(x) = \sin \frac{\pi x}{L} \qquad (27)$$

¶ Note that $\phi(x)$, $\widetilde{\phi}(x)$ and $h$ all have the same units.

The functions in eqs.(24) and (25) become:

$$M_{\widetilde{\phi}}(x) = -\frac{L^2\widetilde{\phi}}{\pi^2}\sin\frac{\pi x}{L} \tag{28}$$

$$M_{\psi}(x) = \frac{M_{\widetilde{\phi}}(x)}{\widetilde{\phi}} \tag{29}$$

¶ The least and greatest bending moments at point $x$ are:

$$M_1(x) = -(\widetilde{\phi}+h)\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} \tag{30}$$

$$M_2(x) = -(\widetilde{\phi}-h)\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} \tag{31}$$

¶ From this we find that the greatest absolute bending moment occurs at the midpoint of the beam:

$$\max_{0\le x\le L}\ \max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} |M(x)| = \frac{(\widetilde{\phi}+h)L^2}{\pi^2} \tag{32}$$

¶ To find the robustness, we equate the maximum bending moment to the critical moment and solve for $h$:

$$\frac{(\widetilde{\phi}+h)L^2}{\pi^2} = M_{\rm c} \quad\Longrightarrow\quad \widehat{h} = \frac{\pi^2 M_{\rm c}}{L^2} - \widetilde{\phi} \tag{33}$$

This is quite similar to the uniform-bound case, eq.(15) on p.5.

¶ **The two info-gap models we have studied are:**

$$\mathcal{U}(h,\widetilde{\phi}) = \left\{\phi(x):\ \left|\phi(x)-\widetilde{\phi}(x)\right| \le h\right\},\quad h\ge 0 \tag{34}$$

(Eq.(1) on p. 3.) with robustness (eq.15), p.5:

$$\widehat{h} = \frac{8M_{\rm c}}{L^2} - \widetilde{\phi} \tag{35}$$

$$\mathcal{U}(h,\widetilde{\phi}) = \left\{\phi(x):\ \left|\phi(x)-\widetilde{\phi}(x)\right| \le h\psi(x)\right\},\quad h\ge 0 \tag{36}$$

(Eq.(17) on p. 7) with robustness in eq.(33):

$$\widehat{h} = \frac{\pi^2 M_{\rm c}}{L^2} - \widetilde{\phi} \tag{37}$$

• Both of these uncertainty models entail **unbounded rate of variation.**

• We sometimes have information which constrains the rate of variation of the uncertain function. We will now develop the tools needed to exploit this information.

## 2.2 Fourier Representation of a Function

¶ We interrupt our study of this example to briefly introduce the Fourier representation of a function. We will use Fourier representations in a new type of info-gap model.

**¶ Motivation:**

- The info-gap models of eqs.(1), p.3, and (17), p.7, allow unbounded rate of variation.
- We now have new information that constrains the rate of variation.

¶ Let $\phi(x)$ be an arbitrary but piece-wise continuous function defined on the interval $-L \leq x \leq L$. Then $\phi(x)$ can be represented as:

$$\phi(x) = \sum_{n=0}^{\infty} \left[ b_n \sin \frac{n\pi x}{L} + c_n \cos \frac{n\pi x}{L} \right] \tag{38}$$

¶ Let $\phi(x)$ be an arbitrary but piece-wise continuous function defined on the interval $0 \leq x \leq L$. Then $\phi(x)$ can be represented as:

$$\phi(x) = \sum_{n=0}^{\infty} c_n \cos \frac{n\pi x}{L} \tag{39}$$

¶ How to choose the Fourier coefficients $c_0$, $c_1$, ... in eq.(39)?
Exploit orthogonality:

$$\int_0^{\pi} \cos mx \cos nx \, \mathrm{d}x = \left\{ \begin{array}{ll} \frac{\pi}{2} & m = n \\ 0 & m \neq n \end{array} \right. \tag{40}$$

To do this, multiply both sides of eq.(39) by $\cos \frac{k\pi x}{L}$ and integrate from 0 to $L$:

$$\int_0^L \phi(x) \cos \frac{k\pi x}{L} \, \mathrm{d}x = \sum_{n=0}^{\infty} c_n \int_0^L \cos \frac{k\pi x}{L} \cos \frac{n\pi x}{L} \, \mathrm{d}x \tag{41}$$

$$= \frac{c_k L}{2} \tag{42}$$

So, if we know the function $\phi(x)$ we can calculate the Fourier coefficients of its expansion:

$$c_k = \frac{2}{L} \int_0^L \phi(x) \cos \frac{k\pi x}{L} \, \mathrm{d}x \tag{43}$$

¶ These Fourier coefficients have many interesting and important properties. First of all, they minimize the mean squared error between $\phi(x)$ and its expansion. That is, the $c_n$ minimize:

$$S^2 = \int_0^L \left( \phi(x) - \sum_{n=0}^{\infty} c_n \cos \frac{n\pi x}{L} \right)^2 \mathrm{d}x \tag{44}$$

In fact,

$$\lim_{N \to \infty} S^2 = 0 \tag{45}$$

Another important property relates to truncated expansions:

$$\phi(x) = \sum_{n=0}^{N} c_n \cos \frac{n\pi x}{L} \, \mathrm{d}x \tag{46}$$

Regardless of the order of the expansion, $N$:
- Orthogonality yields the same Fourier coefficients, $c_k$.
- These coefficients minimize the mean squared error of the truncated expansion.

¶ Band-limited function:

$$\phi(x) = \sum_{n=n_1}^{n_2} c_n \cos \frac{n\pi x}{L} \tag{47}$$

$$= c^T \gamma(x) \tag{48}$$

¶ Uncertainty in $\phi(x)$ is represented as uncertainty in Fourier coefficients $c$.
- For instance: $c$ in ellipsoid of known shape and unknown size:

$$\mathcal{U}(h, \widetilde{c}) = \left\{ \phi(x) = c^T \gamma(x) : \ (c - \widetilde{c})^T W (c - \widetilde{c}) \le h^2 \right\}, \quad h \ge 0 \tag{49}$$

## 2.3 Geometry of Ellipsoids

¶ **Motivation:**

- Suppose we have limited 2-dimensional data about an uncertain phenomenon:

$$(c_1, c_2)_i, \ i = 1, \ldots, n \tag{50}$$

- These data, when plotted, spread over an ellipse-like cluster around (0,0).
- Future data might extend beyond this cluster.
- How to represent our uncertainty?

¶ **Preliminary question:**

- Consider the $c_1 \times c_2$ plane.
- What shape is described by: $c_1^2 + c_2^2 = h^2$? Circle.
- What shape is described by: $ac_1^2 + bc_2^2 = h^2$, where $a, b > 0$? Ellipse.
- What shape is described by: $ac_1^2 + gc_1c_2 + bc_2^2 = h^2$, where $a, b > 0$?
  Ellipse if the coefficients define a positive definite matrix.

¶ We need one more digression before we proceed with our example: Geometry of ellipsoids.
The question we study in this subsection is:
What are the **directions and lengths** of the principal axes of an ellipsoid?

¶ If: $c$ is an $N$-vector and $W$ is a real, symmetric, positive definite matrix,
then an ellipsoid of $c$-vectors of dimension $N$ is defined by:

$$c^T W c = h^2 \tag{51}$$

where $h$ is any positive real number.

¶ Simple examples:

$$h^2 = c_1^2 w_1 + c_2^2 w_2, \quad W = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}, \quad w_i > 0 \tag{52}$$

$$h^2 = 2c_1^2 + c_1 c_2 + 2c_2^2, \quad W = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{53}$$

¶ To answer our question, we must solve an optimization problem.
We must find vectors $c$ which have two properties:

- Length is extremal.
- Lie on the boundary of the ellipsoid.

¶ To optimize the length of $c$, it is sufficient to optimize the square of the length of $c$.
So we must optimize:

$$c^T c \tag{54}$$

Let's try differential calculus:

$$0 = \frac{\mathrm{d}c^T c}{\mathrm{d}c} = 2c \quad \Longrightarrow \quad c = 0 \tag{55}$$

That's the minimum. What's the maximum? $c^T c$ is unbounded. We need the constraint.

¶ To solve this problem we will use the method of **Lagrange multipliers.**

¶ A $c$-vector lies on the ellipsoid if eq.(51) is satisfied.
Expressing this slightly differently, the constraint on $c$ is:

$$h^2 - c^T W c = 0 \tag{56}$$

¶ Define the objective function:

$$H = c^T c - \lambda(h^2 - c^T W c) \tag{57}$$

If we find all $c$-vectors which
optimize $H$ subject to the constraint,
we will have solved the problem.

¶ Condition for extremum of $H$:

$$0 = \frac{\partial H}{\partial c} \quad = \quad 2c - 2\lambda W c \tag{58}$$

$$\Longrightarrow \quad (I - \lambda W)c = 0 \tag{59}$$

which means that:
$c =$ is an eigenvector of $W$.
$\frac{1}{\lambda} =$ the corresponding eigenvalue.

¶ Define the eigenvalues and orthonormal eigenvectors of $W$:

$$W v_i = \mu_i v_i, \quad i = 1, \ldots, N \tag{60}$$

where:

$$0 < \mu_1 \leq \cdots \leq \mu_N \quad \text{and} \quad v_m^T v_n = \delta_{mn} \tag{61}$$

where $\delta_{mn}$ is the Kronecker delta function:

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \neq n \end{cases} \tag{62}$$

¶ Now, since $c$ must be an eigenvector of $W$, we know that:

$$c = r v_i \tag{63}$$

for some non-zero $r$ and for any $i = 1, \ldots, N$.

Hence the constraint on $c$ is:

$$h^2 = c^T W c = r^2 v_i^T W v_i = r^2 \mu_i \quad \Longrightarrow \quad r = \pm \frac{h}{\sqrt{\mu_i}} \tag{64}$$

¶ Thus the optimizing $c$-vectors are:

$$c = \pm \frac{h}{\sqrt{\mu_i}} v_i, \quad i = 1, \ldots, N \tag{65}$$

From this we see that:
The **directions** of the principal semi-axes are:

$$\pm v_1, \ldots, \pm v_N \tag{66}$$

The **lengths** of the principal semi-axes are:

$$\frac{h}{\sqrt{\mu_1}}, \ldots, \frac{h}{\sqrt{\mu_N}} \tag{67}$$

## 2.4 Fourier Ellipsoid Bounded Uncertain Load

Based on *Robust Reliability in the Mechanical Sciences,* section 3.2.4.

¶ We now consider a different type of prior information about the uncertain load profile $\phi(x)$.

¶ About $\phi(x)$ we **know:**
• Load vanishes at ends: $\phi(0) = \phi(L) = 0$.
• $\phi(x)$ is constrained to specific known spatial frequencies.
• The amplitudes of these frequencies are bounded by an ellipsoid of known shape.

¶ About $\phi(x)$ we **do not know:**
• The precise amplitudes of the Fourier coefficients.
• The size of the ellipsoid.

¶ In other words, a load profile is represented by:

$$\phi(x) = \sum_{n=n_1}^{n_2} c_n \sin\frac{n\pi x}{L} \tag{68}$$

$$= c^T\sigma(x) \tag{69}$$

where:
$c =$ vector of unknown Fourier coefficients.
$\sigma(x) =$ vector of known corresponding sine functions.

¶ The uncertainty in $\phi(x)$ is represented by the following Fourier ellipsoid bound info-gap model:

$$\mathcal{U}(h, 0) = \left\{\phi(x) = c^T\sigma : \ c^T W c \le h^2\right\}, \quad h \ge 0 \tag{70}$$

where $W$ is a known, real, symmetric, positive definite matrix.

¶ The system model is obtained by combining eq.(4) on p.4 for the bending moment with eq.(69):

$$M(x) = c^T \underbrace{\left[-\frac{L-x}{L}\int_0^x u\sigma(u)\,\mathrm{d}u - \frac{x}{L}\int_x^L (L-u)\sigma(u)\,\mathrm{d}u\right]}_{\zeta(x)} \tag{71}$$

$$= c^T\zeta(x) \tag{72}$$

¶ As before, failure occurs if the bending moment exceeds a critical value, as expressed in eq.(5) on p.4.

For an example of a Fourier ellipsoid model see: Yakov Ben-Haim and Isaac Elishakoff, Non-Probabilistic models of uncertainty in the non-linear buckling of shells with general imperfections: Theoretical estimates of the knockdown factor. *A.S.M.E. Journal of Applied Mechanics,* Vol. 56, pp 403–410, 1989.

¶ In order to find the robustness, eq.(9), p.4, we must solve the following optimization:

$$\max M(x) \quad \text{for} \quad c^T W c \le h^2 \tag{73}$$

which is equivalent to:

$$\max c^T \zeta \quad \text{for} \quad c^T W c \le h^2 \tag{74}$$

To do this we employ the Cauchy inequality:

$$\left(x^T y\right)^2 \le \left(x^T x\right)\left(y^T y\right) \tag{75}$$

with equality iff:

$$x \propto y \tag{76}$$

Let us write:

$$c^T \zeta = \left(W^{1/2} c\right)^T \left(W^{-1/2} \zeta\right) \tag{77}$$

Applying Cauchy's inequality to the expression on the right:

$$\left(c^T \zeta\right)^2 \le \left[\left(W^{1/2} c\right)^T \left(W^{1/2} c\right)\right]\left[\left(W^{-1/2} \zeta\right)^T \left(W^{-1/2} \zeta\right)\right] \tag{78}$$

$$= \underbrace{\left[c^T W c\right]}_{\le h^2} \left[\zeta^T W^{-1} \zeta\right] \tag{79}$$

From this we conclude that:

$$\max_{c \in \mathcal{U}(h,0)} M(x) = h \sqrt{\zeta(x)^T W^{-1} \zeta(x)} \tag{80}$$

¶ We can now express the robustness as the greatest value of the uncertainty parameter $h$ at which the bending moment does not exceed the critical value. We find:

$$\widehat{h} = \frac{M_{\rm c}}{\max_{0 \le x \le L} \sqrt{\zeta(x)^T W^{-1} \zeta(x)}} \tag{81}$$

¶ Let us consider a **special case:**
$W$ is the identity matrix, so the uncertainty ellipsoid is a sphere.

¶ Now $\zeta^T W \zeta$ becomes:

$$\zeta^T(x) \zeta(x) = \frac{L^4}{\pi^4} \sum_{n=n_1}^{n_2} \frac{1}{n^4} \sin^2 \frac{n \pi x}{L} \tag{82}$$

The terms in this sum decrease rapidly with $n$.
Hence the maximum is dominated by the first term:

$$\max_{0 \le x \le L} \sqrt{\zeta(x)^T \zeta(x)} \approx \max_{0 \le x \le L} \sqrt{\frac{L^4}{\pi^4} \frac{1}{n_1^4} \sin^2 \frac{n_1 \pi x}{L}} \tag{83}$$

$$= \frac{L^2}{n_1^2 \pi^2} \tag{84}$$

From eq.(81) we find the robustness to be:

$$\widehat{h} \approx \frac{n_1^2 \pi^2 M_c}{L^2} \tag{85}$$

¶ Comparing this with the robustness for the uniform-bound info-gap model, with $\widetilde{\phi} = 0$, eq.(15) on p.5:

$$\widehat{h} = \frac{8 M_c}{L^2} \tag{86}$$

we see that the reliability is substantially enhanced
by constraining the spatial modes of the load function.

# 3 Conclusion

§ 3 components of reliability analysis:

1. A system model.
2. A failure criterion.
3. An uncertainty model.

§ Robustness:

- Maximum tolerable uncertainty.
- Basis for design selection.
- Combination of the 3 components.