

Lecture 4
Vibration Suppression
with
Uncertain Load
~**Info-Gap Analysis**~

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1 Design of a Vibrating Cantilever

1.1 Design Problem

¶ We now consider an example:

Vibration control in a cantilever subject to uncertain dynamic excitation.

¶ The cantilever: rigid beam which is clamped at one end.

See transparency of: • Galileo's cantilever.

- Atomic force microscope.

¶ The cantilever is the paradigm for:

- Tall building.
- Radio tower.
- Crane (agoran).
- Airplane wing.
- Turbine blade.
- Diving board.
- Canon barrel.
- Atomic force microscope.
- etc.

¶ Central goal in design of the cantilever:

Control of vibration resulting from external loads.

¶ Two basic approaches:

1. Prevent vibration by stiffening the beam.
2. Absorb vibration by dissipating energy.

¶ These design concepts are **not** mutually exclusive.

They can be implemented together.

¶ These design concepts are relevant in different circumstances as we will see.

1.2 Robustness Function

¶ We will use the **robustness function** to evaluate the design options.

¶ Later we will consider the **opportuneness function**.

¶ As usual, the three components of the analysis are:

1. System model.
2. Failure (or performance) criterion.
3. Uncertainty model.

¶ We use a simple **system model**:

Rigid vibration around the clamped base.

$\theta(t)$ = angle of deflection of beam [radian].

$u(t)$ = moment of force at base, [Nm].

Equation of motion:

$$J \frac{d^2\theta(t)}{dt^2} + c \frac{d\theta(t)}{dt} + k\theta = u(t) \quad (1)$$

J = moment of inertia of beam wrt rotation at base, $\int_0^L m(x)x^2 dx$.

c = damping coefficient.

k = rotational stiffness coefficient, [Nm/radian].

¶ Solution of eq. of motion, for:

- Zero initial conditions, $\theta(0) = \dot{\theta}(0) = 0$
- Subcritical damping, $\zeta^2 < 1$:

$$\theta_u(t) = \int_0^t u(\tau) f(t - \tau) d\tau \quad (2)$$

$f(t)$ = impulse response function:

$$f(t) = \frac{1}{J\omega_d} e^{-\zeta\omega t} \sin \omega_d t \quad (3)$$

$\omega^2 = k/J$ = squared natural frequency.

$\zeta = \frac{c}{2J\omega}$ = dimensionless damping coefficient.

$\omega_d = \omega\sqrt{1 - \zeta^2}$ = damped natural frequency.

¶ We now consider the **uncertainty model**.

What we **know** about the load is:

- The nominal load, $\tilde{u}(t)$.
- The actual loads are transient:
 - May vary rapidly,
 - May attain large deviations from the nominal load.
 - No sustained deviation from the nominal load

We will model load uncertainty with the **cumulative energy bound** info-gap model:

$$\mathcal{U}(h, \tilde{u}) = \left\{ u(t) : \int_0^\infty [u(t) - \tilde{u}(t)]^2 dt \leq h^2 \right\}, \quad h \geq 0 \quad (4)$$

¶ The **performance criterion**: Deflection must not exceed critical value:

$$|\theta(t)| \leq \theta_c \quad (5)$$

In terms of reward functions, define:

$$R(q, u) = |\theta(t)| \quad (6)$$

u = uncertain load.

q = design concept, as expressed in damping c and stiffness k .

¶ The robustness function can be defined as:

$$\hat{h}(q, \theta_c) = \max \left\{ h : \left(\max_{u \in \mathcal{U}(h, \tilde{u})} |\theta_u(t)| \right) \leq \theta_c \right\} \quad (7)$$

$\hat{h}(q, \theta_c)$ is the maximum tolerable info-gap.

¶ We now evaluate:

$$\max_{u \in \mathcal{U}(h, \tilde{u})} |\theta_u(t)| \quad (8)$$

¶ Note that $\theta_u(t)$ in eq.(2) on p.4 can be re-written:

$$\theta_u(t) = \int_0^t u(\tau) f(t - \tau) d\tau \quad (9)$$

$$= \int_0^t [u(\tau) - \tilde{u}(\tau)] f(t - \tau) d\tau + \underbrace{\int_0^t \tilde{u}(\tau) f(t - \tau) d\tau}_{\tilde{\theta}(t)} \quad (10)$$

where $\tilde{\theta}(t)$ = nominal deflection.

We need the Schwarz inequality:

$$\left(\int_a^b f(t)g(t) dt \right)^2 \leq \int_a^b f(t)^2 dt \int_a^b g(t)^2 dt \quad (11)$$

with equality iff:

$$f(t) = cg(t) \quad (12)$$

for any non-zero constant c .

Now notice that the first integral in eq.(10) on p.5 is bounded:

$$\left(\int_0^t [u(\tau) - \tilde{u}(\tau)] f(t - \tau) d\tau \right)^2 \leq \underbrace{\left(\int_0^t [u(\tau) - \tilde{u}(\tau)]^2 d\tau \right)}_I \underbrace{\left(\int_0^t f^2(t - \tau) d\tau \right)}_{II} \quad (13)$$

¶ Note:

- From the info-gap model we know that: Integral I $\leq h^2$.
- Integral II is known.
- The info-gap model allows us to choose $u(\tau)$ such that:

$$u(\tau) - \tilde{u}(\tau) \propto f(t - \tau) \quad (14)$$

- Thus, from eqs.(10) and (13):

$$\max_{u \in \mathcal{U}(h, \tilde{u})} |\theta_u(t)| = h \sqrt{\int_0^t f^2(\tau) d\tau} + |\tilde{\theta}(t)| \quad (15)$$

¶ We can now express the robustness function:

- Equate $\max |\theta_u(t)|$ to θ_c .
- Solve for h , yielding \hat{h} :

$$h \sqrt{\int_0^t f^2(\tau) d\tau} + |\tilde{\theta}(t)| = \theta_c \implies \hat{h}(q, \theta_c) = \frac{\theta_c - |\tilde{\theta}(t)|}{\sqrt{\int_0^t f^2(\tau) d\tau}} \quad (16)$$

unless this is negative, in which case $\hat{h} = 0$.

1.3 Numerical Example

¶ We will consider a specific example. Nominal input $\tilde{u}(t)$ is square:

$$\tilde{u}(t) = \begin{cases} \tilde{u}_o, & 0 \leq t \leq T \\ 0, & t > T \end{cases} \quad (17)$$

The nominal response can be calculated:

$$\tilde{\theta}(t) = \theta_{\tilde{u}}(t) = \frac{(1 - \zeta^2)\tilde{u}_o}{J\omega_d} \gamma(t) \quad (18)$$

where $\gamma(t)$ is a known function.

For notational convenience we represent integral II in eq.(13) on p.6 as:

$$\sqrt{\int_0^t f^2(t - \tau) d\tau} = \frac{1 - \zeta^2}{2J\omega_d^{3/2}} \phi(t) \quad (19)$$

where $\phi(t)$ is a known function.

Now the robustness function can be expressed:

$$\hat{h}(q, \theta_c) = \frac{2J\theta_c\omega^2\sqrt{\omega_d} - 2\sqrt{\omega_d}|\tilde{u}_o\gamma(t)|}{\omega\phi(t)} \quad (20)$$

Recall: $q =$ decision vector $= (c, k)$, which is embedded in ω and ω_d .

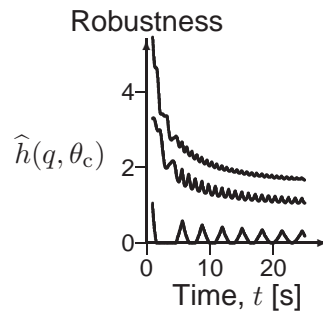


Figure 1: Robustness versus time for three values of the natural frequency $\omega = 1, 3$ and 4 (bottom to top). Negligible damping: $\zeta = 0.01$. $1 = J\theta_c = \tilde{u}_0$. $T = 5$.

¶ $\hat{h}(q, \theta_c)$ vs. t is plotted in fig. 1

For various natural frequencies: $\omega = 1, 3$ and 4 (bottom to top).

With negligible damping: $\zeta = 0.01$.

- \hat{h} oscillates but tends to decrease over time.
- At low stiffness ($\omega = 1$) the robustness periodically vanishes.
- At moderate and high stiffness ($\omega = 3, 4$)
 \hat{h} oscillates but does not reach zero for the duration shown.
- The transition from rapid to slow decrease in \hat{h}
occurs about at $t = T$ (end of nominal input).

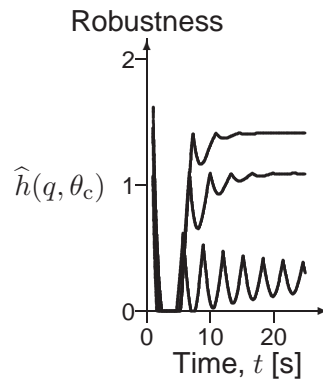


Figure 2: Robustness versus time for three values of the damping ratio $\zeta = 0.03, 0.3, 0.5$ (bottom to top). Fixed natural frequency $\omega = 1$. $1 = J\theta_c = \tilde{u}_0$. $T = 5$.

¶ Now consider fig. 2, which shows

$\hat{h}(q, \theta_c)$ vs. t for various damping ratios:

$\zeta = 0.03, 0.3$ and 0.5

at low stiffness: $\omega = 1$.

- Lowest curve is quite similar to lowest curve in fig. 1.

- With large damping ($\zeta = 0.3$ or 0.5):

\hat{h} is small for $t \leq T$

\hat{h} is large and nearly constant thereafter.

¶ Comparing figs. 1 and 2:

- Fig. 1 is based on the “stiffness” design concept, with negligible damping.
- Fig. 2 is based on the “dissipation” design concept, with negligible stiffness.
- The choice of a design concept depends on the time frame of interest:
 - $t < T$ calls for “stiffness” design.
 - $t > T$ calls for “dissipation” design.
 - $t > 0$ calls for combined “stiffness” and “dissipation” design.

1.4 Opportuneness Function

¶ We now consider the opportuneness function.

Windfall reward: angular deflection θ_w **much less** (much better) than the survival requirement, θ_c :

$$\theta_w < \tilde{\theta} < \theta_c \tag{21}$$

¶ Immunity to windfall, $\hat{\beta}(q, \theta_w)$: the **least** info-gap at which windfall is **possible**.

¶ Analogous to eq.(7) on p. 5:

$$\hat{\beta}(q, \theta_w) = \min \left\{ h : \min_{u \in \mathcal{U}(h, \tilde{u})} |\theta_u(t)| \leq \theta_w \right\} \tag{22}$$

¶ **Smaller is better** for $\hat{\beta}$. Unlike \hat{h} , for which **bigger is better**.

¶ Proceeding as in eq.(15) on p. 6 we find:

$$\min_{u \in \mathcal{U}(h, \tilde{u})} |\theta_u(t)| = -h \sqrt{\int_0^t f^2(\tau) d\tau} + |\tilde{\theta}(t)| \tag{23}$$

Equating this to θ_w and solving for h yields the opportuneness function, as in eq.(16) on p. 6:

$$-h \sqrt{\int_0^t f^2(\tau) d\tau} + |\tilde{\theta}(t)| = \theta_w \implies \hat{\beta}(q, \theta_w) = \frac{|\tilde{\theta}(t)| - \theta_w}{\sqrt{\int_0^t f^2(\tau) d\tau}} \tag{24}$$

unless this is negative, in which case $\hat{\beta} = 0$.

Why does $\hat{\beta} = 0$ in this case?

$\hat{\beta} < 0$ only if $|\tilde{\theta}(t)| < \theta_w$.

This means that the **nominal response** $|\tilde{\theta}(t)|$ is less than the **windfall response** θ_w .

Hence windfall is possible even without uncertainty: The immunity to windfall is zero.

¶ Compare $\widehat{\beta}(q, \theta_w)$ to the robustness in eq.(16) on p. 6:

$$\widehat{h}(q, \theta_c) = \frac{\theta_c - |\widetilde{\theta}(t)|}{\sqrt{\int_0^t f^2(\tau) d\tau}} \quad (25)$$

We see that the immunity functions are related as:

$$\widehat{\beta}(q, \theta_w) = -\widehat{h}(q, \theta_c) + \frac{\theta_c - \theta_w}{\sqrt{\int_0^t f^2(\tau) d\tau}} \quad (26)$$

¶ We now consider **antagonism** and **sympathy** of the immunity functions.

¶ The immunity functions $\widehat{\beta}(q, \theta_w)$ and $\widehat{h}(q, \theta_c)$ are **sympathetic** if they can be improved simultaneously.

They are **antagonistic** if either can be improved only at the expense of the other.

¶ For example, we can vary ω . The immunity functions are **antagonistic** if:

$$\underbrace{\frac{\partial \widehat{h}(q, \theta_c)}{\partial \omega} > 0}_{\text{improving with } \omega} \quad \text{and} \quad \underbrace{\frac{\partial \widehat{\beta}(q, \theta_w)}{\partial \omega} > 0}_{\text{degenerating with } \omega} \quad (27)$$

or if:

$$\underbrace{\frac{\partial \widehat{h}(q, \theta_c)}{\partial \omega} < 0}_{\text{degenerating with } \omega} \quad \text{and} \quad \underbrace{\frac{\partial \widehat{\beta}(q, \theta_w)}{\partial \omega} < 0}_{\text{improving with } \omega} \quad (28)$$

¶ On the other hand, the immunity functions are **sympathetic** if:

$$\underbrace{\frac{\partial \widehat{h}(q, \theta_c)}{\partial \omega} > 0}_{\text{improving with } \omega} \quad \text{and} \quad \underbrace{\frac{\partial \widehat{\beta}(q, \theta_w)}{\partial \omega} < 0}_{\text{improving with } \omega} \quad (29)$$

or if:

$$\underbrace{\frac{\partial \widehat{h}(q, \theta_c)}{\partial \omega} < 0}_{\text{degenerating with } \omega} \quad \text{and} \quad \underbrace{\frac{\partial \widehat{\beta}(q, \theta_w)}{\partial \omega} > 0}_{\text{degenerating with } \omega} \quad (30)$$

¶ In short, the immunity functions are **sympathetic** wrt ω if and only if:

$$\frac{\partial \widehat{h}(q, \theta_c)}{\partial \omega} \frac{\partial \widehat{\beta}(q, \theta_w)}{\partial \omega} < 0 \quad (31)$$

¶ Return to eq.(26) on p. 10.

- Question: Under what conditions will \hat{h} and $\hat{\beta}$ always be sympathetic?
- Answer: If and only if their optima coincide. See fig. 3.

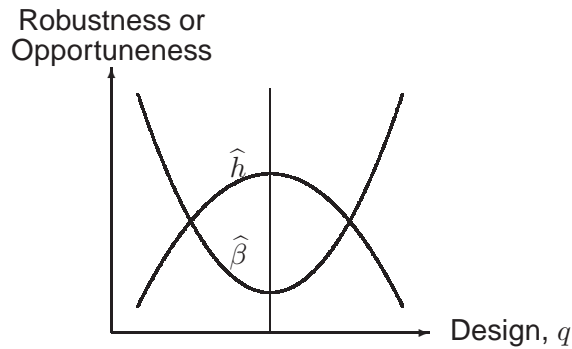


Figure 3: Sympathetic robustness and opportunity curves.

¶ When will this occur? Iff

$$\frac{\partial \hat{\beta}}{\partial q} = 0 = \frac{\partial \hat{h}}{\partial q} \tag{32}$$

From eq.(26) we see that this will happen only if, at the same q , we also have:

$$\frac{\partial D}{\partial q} = 0 \tag{33}$$

where we define:

$$D = \frac{\theta_c - \theta_w}{\sqrt{\int_0^t f^2(\tau) d\tau}} \tag{34}$$

“Usually” this will not happen, which means that, instead of fig. 3, we will have fig. 4.

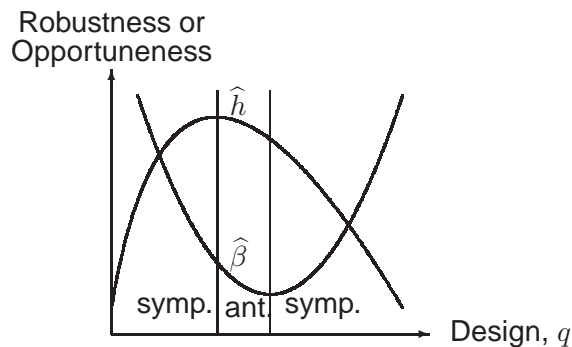


Figure 4: Robustness and opportunity curves which are both sympathetic and antagonistic.

