31. **Non-linear force-deflection relation** (p.172). Equilibrium of a 1-dimensional system is specified by:

$$xf = k + kf^2 \tag{103}$$

where x is the deflection, f is the force, which is known to be non-negative, and k is a known positive constant. The system is safe if $x \leq x_{\rm c}$. The nominal force is $\widetilde{f}$, for which the system is safe. The uncertainty of the actual force is described by an info-gap model:

$$\mathcal{U}(h, \widetilde{f}) = \left\{ f : \ f \geq 0, \ \left| \frac{f - \widetilde{f}}{\widetilde{f}} \right| \leq h \right\}, \quad h \geq 0 \tag{104}$$

(a) Derive the robustness function if $\widetilde{f} = 1$.

(b) Derive the robustness function if $\widetilde{f} > 1$. Discuss the qualitative difference between the two solutions.

Solution for problem 31 (p.29) **Non-linear force-deflection relation**. The deflection is:

$$x(f) = \frac{k}{f} + kf \tag{997}$$

which is asymmetric as seen in fig. 36.

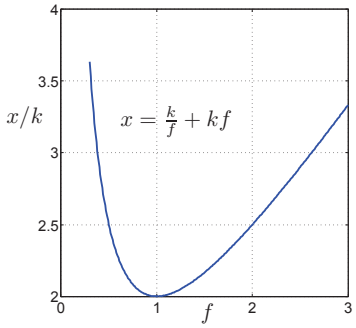

Figure 36: Deflection-force relation for problem 31.

The robustness is defined as:

$$\widehat{h}(x_c) = \max\left\{h : \left(\max_{f\in\mathcal{U}(h,\widetilde{f})} x(f)\right) \le x_c\right\} \tag{998}$$

Differentiating x wrt f in eq.(997):

$$\frac{\mathrm{d}x}{\mathrm{d}f} = k\left(1 - \frac{1}{f^2}\right) \tag{999}$$

The inner maximum occurs either at $(1-h)\widetilde{f}$ or $(1+h)\widetilde{f}$. Compare the corresponding values of x:

$$\frac{k}{(1-h)\widetilde{f}} + k(1-h)\widetilde{f} \quad ? \quad \frac{k}{(1+h)\widetilde{f}} + k(1+h)\widetilde{f} \tag{1000}$$

$$\frac{1}{(1-h)\widetilde{f}} - \frac{1}{(1+h)\widetilde{f}} \quad ? \quad 2h\widetilde{f} \tag{1001}$$

$$\frac{1}{(1-h)(1+h)} \quad ? \quad \widetilde{f}^2 \tag{1002}$$

$$\frac{1}{1-h^2} \quad ? \quad \widetilde{f}^2 \tag{1003}$$

(a) The lefthand side of eq.(1003) exceeds 1, and $\widetilde{f} = 1$, so we conclude that '?' is in fact '$\ge$'. Thus the global maximum of $x(f)$ occurs for $f = (1-h)\widetilde{f}$. This is what we expect from the steeper slope of the left hand branch of the curve in fig. 36. Substituting $(1-h)\widetilde{f}$ in eq.(997), equating to x_c, and solving for h yields the robustness, as we now show. Eq.(997) becomes:

$$\frac{k}{(1-h)\widetilde{f}} + k(1-h)\widetilde{f} = x_c \tag{1004}$$

$$\iff \quad (1-h)^2\widetilde{f}^2 - \frac{x_c}{k}(1-h)\widetilde{f} + 1 = 0 \tag{1005}$$

$$\iff \quad (1-h)\widetilde{f} = \frac{x_c}{2k} \pm \frac{1}{2}\sqrt{\left(\frac{x_c}{k}\right)^2 - 4} \tag{1006}$$

Define $\rho = \frac{x_c}{k}$. Recall that $\widetilde{f} = 1$ so, when $h = 0$, we see that $\rho = 2$. Choose '$-$' for the '$\pm$' in eq.(1006):

$$(1-h)\widetilde{f} = \frac{x_c}{2k} - \frac{1}{2}\sqrt{\left(\frac{x_c}{k}\right)^2 - 4} \tag{1007}$$

Differentiation of the righthand side of eq.(1007) with respect to ρ shows that the righthand side decreases as ρ increases from 2. This is what we expect from the left hand branch of the curve in fig. 36. Thus the value of h that satisfies eq.(1007) *increases* as ρ *increases,* as we expect from the definition of the robustness in eq.(998). Solving eq.(1007) yields the robustness:

$$\widehat{h}(x_c) = 1 - \frac{x_c}{2k\widetilde{f}} + \frac{1}{2\widetilde{f}}\sqrt{\frac{{x_c}^2}{k^2} - 4} \tag{1008}$$

or zero if this is negative. Recall that $\widetilde{f} = 1$. The discussion following eq.(1007) shows that this expression for $\widehat{h}(x_c)$ in fact increases as x_c increases.

(b) The lefthand side of eq.(1003) exceeds 1, and $\widetilde{f} > 1$, so the value of '?' depends on h. Let h_s denote the value of h at which equality holds in eq.(1003). Thus:

$$h_s = \sqrt{1 - \frac{1}{\widetilde{f}^{\,2}}} \tag{1009}$$

which is between 0 and 1. Thus we find that the inner maximum in the robustness, which we denote $m(h)$, is:

$$m(h) = \begin{cases} \dfrac{k}{(1+h)\widetilde{f}} + k(1+h)\widetilde{f} & \text{if } h \le h_s \\ \dfrac{k}{(1-h)\widetilde{f}} + k(1-h)\widetilde{f} & \text{else} \end{cases} \tag{1010}$$

$m(h)$ is continuous but has a kink at $h = h_s$. The continuity results from the fact that '?' is '=' in eq.(1000) when $h = h_s$, so the two lines in eq.(1010) are equal when $h = h_s$.

$m(h)$ is the inverse of the robustness function, $\widehat{h}(x_c)$. We see that $\widehat{h}(x_c)$ has a kink at $x_c = m(h_s)$, unlike in part (a). This can result in curve crossing for different choices of k and $\widetilde{f}$, as seen in fig. 37.

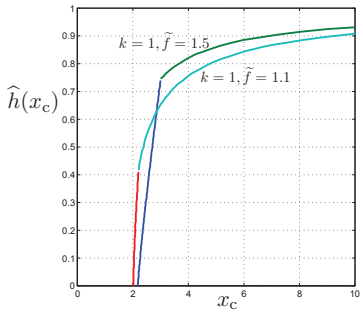

Figure 37: Robustness curve for problem 31(b).