Lecture Notes on Probabilistic Failure Models

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Primary source material: A. Høyland, and M. Rausand, 1994, *System Reliability Theory: Models and Statistical Methods,* Wiley, New York. Chapter 2.

Notes to the Student:

• These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

• Section 15 contains review exercises that will assist the student to master the material in the lecture and are highly recommended for review and self-study. The student is directed to the review exercises at selected places in the notes. They are not homework problems, and they do not entitle the student to extra credit.

Contents

1 Overview

¶ This lecture develops 3 main ideas:

- The probabilistic reliability function, $R(t)$.
- The probabilistic failure rate function $z(t)$.
- The mean time to failure $MTTF$.

¶ These ideas are based on more fundamental ideas:

- The time to failure is a random variable, t.
- The probability density function (pdf) of the time to failure, $f(t)$.
- The cumulative distribution function (cdf) of the time to failure, $F(t)$.
- Conditional probability.

2 Time to Failure

¶ The time to failure (TTF):
time from start of operation to 1st failure.

¶ Examples:
- Hours of operation.
- # of times a switch is operated.
- Kilometers traveled.
- # rotations of an axis.

The TTF may be:
- Continuous or discrete.
- Have units other than "clock time".

¶ We will usually consider **continuous** TTF.
t = TTF, a random variable.

$f(t)$ = pdf of t, the TTF.

$f(t)\mathrm{d}t$ = probability of failure in $[t,\, t+\mathrm{d}t]$.
$F(t)$ = cdf for t = probability of failure in $[0,\, t]$.

$$\begin{aligned} F(t) &= \int_0^t f(\tau)\,\mathrm{d}\tau && (1) \\ &= P(T \le t) && (2) \end{aligned}$$

Notation for random variables:

T = name of the random variable.
t = value of the random variable.

Thus we read '$T \le t$' as:
'The random variable for TTF takes a value no greater than t'.
¶ $F(t)$ is monotonically increasing in t.

¶ **Review exercise 1, p.38.**

$R(t)$ = reliability function = probability of **no** failure in $[0,\, t]$.

$$\begin{aligned} R(t) &= P(T > t) && (3) \\ &= 1 - F(t) && (4) \\ &= \int_t^\infty f(\tau)\,\mathrm{d}\tau && (5) \end{aligned}$$

3 Reliability and the Quantile

¶ In this section we explain the relation between the probabilistic reliability function, $R(t)$, and the concept of a statistical quantile.

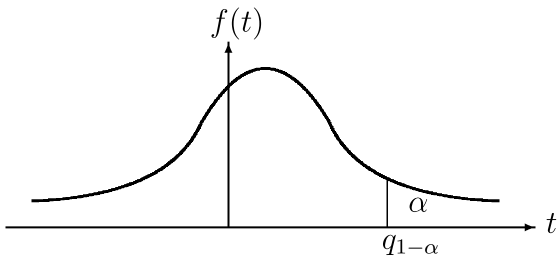


Figure 1: $1 - \alpha$ quantile of a distribution.

Fig. 1 illustrates the $1 - \alpha$ quantile of a pdf $f(t)$. Formally, the $1 - \alpha$ quantile is defined as the value of the random variable for which the probability of non-exceedence is $1 - \alpha$:

$$\text{Prob}(T \leq q_{1-\alpha}) = 1 - \alpha \tag{6}$$

Equivalently, this quantile is the value of the random variable for which the probability of exceedence is α:

$$\text{Prob}(T > q_{1-\alpha}) = \alpha \tag{7}$$

Recall the reliability function in eq.(3):

$$P(T > t) = R(t) \tag{8}$$

Comparing eqs.(7) and (8) we see that t is the $1 - R(t)$ quantile of the time-to-failure distribution.
This is important and useful since much statistical theory relates to the quantile function.

¶ Review exercise 2, p.38.

4 Failure Rate Function and 'Bath-tub Curve'

4.1 The Hazard Function

Based in part on J. Davidson, ed., 1988, *The Reliability of Mechanical Systems,* Mechanical Engineering Publications, London, sections 2.2, 6.2

• Let's start with a joke. George Burns is reported to have said:

"If you live to be one hundred, you've got it made. Very few people die past that age."

What's wrong with this statement (and why is it funny)?

• The **failure rate function** $z(t)$ is the conditional probability density of failure during $(t, t+dt)$, given no-failure up to time t:

$$z(t)\,dt = \frac{\text{prob of failure in } (t, t+dt)}{\text{prob of no failure up to } t} = \frac{f(t)\,dt}{1-F(t)} \tag{9}$$

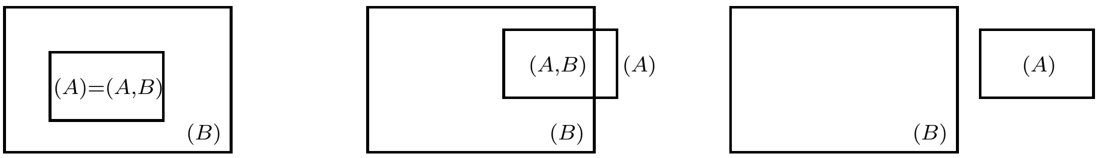


Figure 2: Venn diagrams illustrating conditional probability in eq.(10).

• We can understand this expression in terms of the basic definition of conditional probability, as illustrated in fig. 2. Let A and B be two events with joint probability function $p(A, B)$. The marginal probability of B is $p(B)$. The conditional probability of A given B is:

$$p(A|B) = \frac{p(A,B)}{p(B)} \tag{10}$$

¶ Review exercise 3, p.38.

• Now, applying this to the failure rate function we identify:

$$z(t)\,\mathrm{d}t = \text{Prob}[\text{failure in } (t, t+dt)\,|\,\text{no failure up to } t] \tag{11}$$

In other words, we identify A and B as:

$$A = \text{failure in } (t, t+dt) \tag{12}$$
$$B = \text{no failure up to } t \tag{13}$$

In this case A entails B, so $p(A, B) = p(A)$, and we obtain the final equation in (9).

• As an example, consider the exponential density:

$$f(t) = \lambda e^{-\lambda t}, \quad t \geq 0 \tag{14}$$

for which the cdf is:

$$F(t) = 1 - e^{-\lambda t} \tag{15}$$

and the reliability function is:

$$R(t) = e^{-\lambda t} \tag{16}$$

Thus the failure rate function is:

$$z(t) = \frac{f(t)}{R(t)} = \frac{\lambda e^{-\lambda t}}{e^{-\lambda t}} = \lambda \tag{17}$$

Thus the failure rate function for the exponential density is constant. This means that the probability of failure in the infinitesimal interval $(t, t + dt)$ does not depend on how long the component has survived. In other words, the item has neither increasing nor decreasing risk of failure with age.
Note also that λ has units "per time" and thus represents a "rate of failure".
¶ Review exercise 4, p.38.

• Related to this, consider the probability that the item, (still with an exponential density), will survive an additional duration τ, given that it has survived a duration t. This is the conditional reliability function:

$$\begin{aligned} \text{Prob}(\text{no failure in } \tau + t \,|\, \text{no failure in } t) &= \frac{1 - F(\tau + t)}{1 - F(t)} && (18) \\ &= \frac{e^{-\lambda(\tau+t)}}{e^{-\lambda t}} && (19) \\ &= e^{-\lambda \tau} = R(\tau) && (20) \end{aligned}$$

In other words, the probability of survival for an additional duration τ, given survival for a time t, precisely equals the survival probability for time τ. Since the failure rate function is constant, the risk rate neither increases nor decreases with time.
¶ Review exercise 5, p.38.

4.2 The Bathtub Curve

The failure rate function $z(t)\,dt$ is the conditional probability of failure in the infinitesimal interval $(t, t+dt)$ given no-failure up to time t. Typically, though of course not universally, $z(t)$ is a upward-concave function. That is, it often has an overall U-shape. In other words, the failure rate decreases early in life, is constant during the middle period of time, and increases towards the end of the useful life. These three phases can be understood as follows, illustrated in fig. 3.

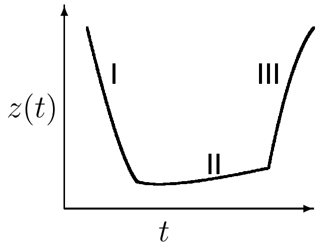


Figure 3: Schematic illustration of the bath tub curve.

1. Phase 1: high but decreasing failure rate. Early in the life of a population of components or systems, the failure rate will tend to be high due to early failure of defective items.
2. Phase 2: constant failure rate. Once the defective units have been selected out by premature failure, the population dwindles due to ‘random failures’.
3. Phase 3: increasing failure rate. Late in the life of the population the units begin to wear out, and the failure rate increases.

4.3 Relation between $f(t)$, $F(t)$, $R(t)$ and $z(t)$

• The four basic functions:

$$f(t), \quad F(t), \quad R(t), \quad z(t) \tag{21}$$

are intimately related.
Any one can be used to derive the others.
We will show that:

- $z(t)$ can be derived from any of the other functions.
- Each of the other functions can be derived from $z(t)$.

• Derive $z(t)$:

$$z(t) = \frac{f(t)}{1 - F(t)} = \frac{f(t)}{R(t)} \tag{22}$$

Also:

$$f(t) = \frac{\mathrm{d}F(t)}{\mathrm{d}t} = \frac{\mathrm{d}(1 - R(t))}{\mathrm{d}t} = -R'(t) \tag{23}$$

So:

$$z(t) = -\frac{R'(t)}{R(t)} = -\frac{\mathrm{d}\ln R(t)}{\mathrm{d}t} \tag{24}$$

• Derive $R(t)$:
Thus, because $R(0) = 1$ (Explanation: $R(t) =$ Prob$(T \geq t)$):

$$\int_0^t z(s)\,\mathrm{d}s = -\ln R(t) \tag{25}$$

or:

$$R(t) = \exp\left[-\int_0^t z(s)\,\mathrm{d}s\right] \tag{26}$$

¶ Review exercise 6, p.38.

• Derive $f(t)$:
Since:

$$z(t) = \frac{f(t)}{R(t)} \tag{27}$$

we see that:

$$f(t) = z(t)R(t) = z(t)\exp\left[-\int_0^t z(s)\,\mathrm{d}s\right] \tag{28}$$

• Derive $F(t)$:

$$F(t) = 1 - R(t) = 1 - \exp\left[-\int_0^t z(s)\,\mathrm{d}s\right] \tag{29}$$

5 Mean Time to Failure

• The MTTF is the expectation of T:

$$\mathrm{E}(T) = \int_0^\infty t f(t)\, \mathrm{d}t \tag{30}$$

t = value of the random variable.
$f(t)\,\mathrm{d}t$ = frequency of recurrence of the value t.
• Statistical expectation, $\mathrm{E}(T)$ is closely related to sample mean, $\bar{t}$:
Given a sample: $t_1, \ldots, t_N$.
The sample mean is:

$$\bar{t} = \frac{1}{N} \sum_{n=1}^{N} t_n \tag{31}$$

Group these N observations into J bins:

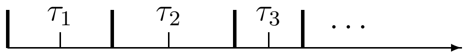


Figure 4: Bins for grouping sampled data.

τ_j = midpoint of jth bin.
n_j = number of observations in jth bin.
¶ Review exercise 7, p.38.

The population mean, eq.(31), can be approximated as:

$$\bar{t} \approx \frac{1}{N} \sum_{j=1}^{J} n_j \tau_j = \sum_{j=1}^{J} \frac{n_j}{N} \tau_j \tag{32}$$

τ_j = value of random variable.
$\lim_{N\to\infty} \frac{n_j}{N}$ = frequency of recurrence of τ_j.

As $N \to \infty$ we find:

$$\tau_j \to t \tag{33}$$
$$\frac{n_j}{N} \to f(t)\,\mathrm{d}t \tag{34}$$
$$\bar{t} \to \int_0^\infty t f(t)\,\mathrm{d}t = \mathrm{E}(t) \tag{35}$$

¶ Review exercise 8, p.38.

• The MTTF is also related to the probabilistic reliability.
Since $f(t) = -R'(t)$, eq.(30) is:

$$\begin{aligned} MTTF &= -\int_0^\infty tR'(t)\,\mathrm{d}t && (36) \\ &= -tR(t)|_0^\infty - -\int_0^\infty R(t)\,\mathrm{d}t && (37) \end{aligned}$$

This employs partial integration:

$$\int_a^b g\,\mathrm{d}f = gf|_a^b - \int_a^b f\,\mathrm{d}g \tag{38}$$

So, if $tR(t)|_0^\infty = 0$, then

$$MTTF = \int_0^\infty R(t)\,\mathrm{d}t \tag{39}$$

¶ Review exercise 9, p.38.

• Another way to evaluate the MTTF employs the Laplace transform:

$$\mathcal{L}(R) = \int_0^\infty R(t)\mathrm{e}^{-st}\,\mathrm{d}t, \quad (s = a + jb) \tag{40}$$

Thus:

$$\mathcal{L}(R)_{s=0} = \int_0^\infty R(t)\,\mathrm{d}t = MTTF \tag{41}$$

6 Poisson Process

• In this section we will derive the Poisson process for describing the probability distribution of independent failures.

• Assumptions:
  ◦ λ = average number failures/sec = constant.
  ◦ Failures occur independently of one another.

• $P_n(t)$ = probability of exactly n failures in duration t.
  We will derive the form of $P_n(t)$.
  Basic concept: **probability balance.**
  What is the random variable for which $P_n(t)$ is the probability? (Ans: n)

• What is probability of failure in $\mathrm{d}t$?
  ◦ λ = average number failures/sec = constant.
  ◦ Suppose $\lambda = 1$, $\mathrm{d}t$ = 0.001.
    Then probability of failure in $\mathrm{d}t$ = 0.001.
  ◦ Suppose $\lambda = 2$, $\mathrm{d}t$ = 0.001.
    Then probability of failure in $\mathrm{d}t = 2 \times 0.001$.
  ◦ Generalization:
    Probability of failure in $\mathrm{d}t = \lambda \mathrm{d}t$.

• Likewise, probability of no failure in $\mathrm{d}t = 1 - \lambda \mathrm{d}t$.

• Probability balance:

$$P_n(t + \mathrm{d}t) = P_n(t)(1 - \lambda \mathrm{d}t) + P_{n-1}(t)\lambda \mathrm{d}t + P_{n-2}(t)\mathcal{O}(\lambda \mathrm{d}t)^2 + \cdots \tag{42}$$

¶ Review exercise 10, p.38.

Divide by $\mathrm{d}t$ and re-arrange:

$$\frac{P_n(t + \mathrm{d}t) - P_n(t)}{\mathrm{d}t} = \lambda P_{n-1}(t) - \lambda P_n(t) + \mathcal{O}(\lambda \mathrm{d}t) + \cdots \tag{43}$$

$\mathrm{d}t \to 0$ leads to:

$$\frac{P_n(t)}{\mathrm{d}t} = \lambda \left[P_{n-1}(t) - P_n(t) \right], \quad n = 0,\, 1,\, 2,\, \ldots \tag{44}$$

We have defined:

$$P_{-1} = 0 \tag{45}$$

The initial condition is:

$$P_n(0) = \delta_{0n} \tag{46}$$

• Consider $n = 0$:

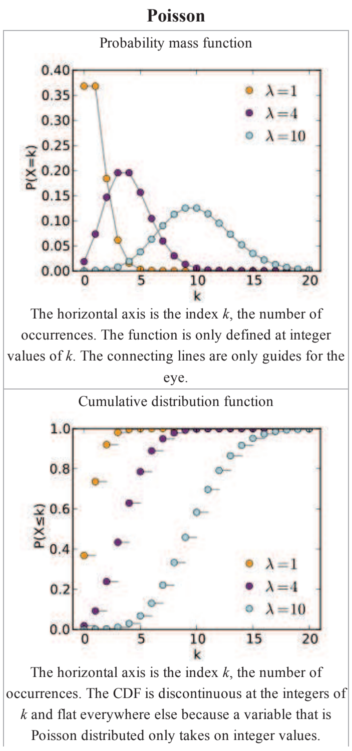
$$\dot{P}_0 = -\lambda P_0, \quad P_0(0) = 1, \quad \Longrightarrow \quad P_0(t) = \mathrm{e}^{-\lambda t} \tag{47}$$

- Consider $n = 1$:

$$\dot{P}_1 = -\lambda P_1 + \lambda P_0, \quad P_1(0) = 0, \quad \Longrightarrow \quad P_1(t) = \lambda t \mathrm{e}^{-\lambda t} \tag{48}$$

- By induction one can show:

$$P_n(t) = \frac{\mathrm{e}^{-\lambda t}(\lambda t)^n}{n!}, \quad n = 0,\ 1,\ 2,\ \ldots \tag{49}$$



The horizontal axis is the index k, the number of occurrences. The function is only defined at integer values of k. The connecting lines are only guides for the eye.

The horizontal axis is the index k, the number of occurrences. The CDF is discontinuous at the integers of k and flat everywhere else because a variable that is Poisson distributed only takes on integer values.

Figure 5: Poisson pdf and cdf. (From Wikipedia)

- What does "by induction" mean?
  - ∘ We know that eq.(49) is true for $n = 0$ and $n = 1$.
  - ∘ Suppose that eq.(49) is true for $n = 0,\ 1,\ \ldots, N$.
  - ∘ Now prove that eq.(49) is true $n = N + 1$.
  - ∘ Why does this prove eq.(49) is true for all n?
  - ∘ How to prove eq.(49) is true $n = N + 1$? Use eq.(44). Try it.

- Expectation of n:

$$\mathrm{E}(n) = \sum_{n=0}^{\infty} n P_n(t) \tag{50}$$

$$= \mathrm{e}^{-\lambda t} \sum_{n=0}^{\infty} n \frac{(\lambda t)^n}{n!} \tag{51}$$

$$= \mathrm{e}^{-\lambda t} \sum_{n=1}^{\infty} n \frac{(\lambda t)^n}{n!} \tag{52}$$

$$= \lambda t e^{-\lambda t} \sum_{n=1}^{\infty} \frac{(\lambda t)^{n-1}}{(n-1)!} \tag{53}$$

$$= \lambda t e^{-\lambda t} \underbrace{\sum_{n=0}^{\infty} \frac{(\lambda t)^{n}}{n!}}_{e^{\lambda t}} \tag{54}$$

$$= \lambda t e^{-\lambda t} e^{\lambda t} \tag{55}$$

$$= \lambda t \tag{56}$$

- Similarly, one can show that the variance of n is:

$$\mathrm{var}(n) = \mathrm{E}\left([n - \mathrm{E}(n)]^2\right) = \lambda t \tag{57}$$

¶ Review exercise 11, p.39.

7 Exponential Distribution Derived from the Poisson Process

- T_1 = random variable representing the **first occurrence** of failure in a Poisson Process. What is the distribution of T_1?

$F_{T_1}(t)$ = cumulative distribution function of T_1.

$$F_{T_1}(t) = \text{Prob}(T_1 \leq t) \tag{58}$$

Clearly:

$$F_{T_1}(t) = 0 \quad \text{for} \quad t < 0 \tag{59}$$

Also, for $t \geq 0$:

$$\begin{aligned} F_{T_1}(t) &= 1 - \text{Prob}(T_1 > t) && (60) \\ &= 1 - \underbrace{P_0(t)}_{\text{Poisson}} && (61) \\ &= 1 - \mathrm{e}^{-\lambda t} && (62) \end{aligned}$$

¶ **Review exercise 12, p.39.**

Hence the pdf of $F_{t_1}(t)$ is:

$$\begin{aligned} f_{T_1}(t) &= \frac{\mathrm{d}F_{T_1}(t)}{\mathrm{d}t} && (63) \\ &= \begin{cases} \lambda \mathrm{e}^{-\lambda t} & t \geq 0 \\ 0 & t < 0 \end{cases} && (64) \end{aligned}$$

This is the exponential distribution.

- Note: we have derived
  a **continuous** distribution on t (exponential)
  from a **discrete** distribution on n (Poisson).

- Expectation of T_1:

$$\mathrm{E}(T_1) = \int_0^\infty t f_{T_1}(t)\, \mathrm{d}t = \frac{1}{\lambda} \tag{65}$$

¶ **Review exercise 13, p.39.**

¶ The exponential distribution is very common in probabilistic reliability analysis. Why?
- Recall the two assumptions (start of section 6).
- Is this prevalence justified? When is it NOT justified? How can you know?

8 Gamma Distribution

• Now consider a generalization of $F_{T_1}(t)$.

• T_k = random variable representing the k**th occurrence** of failure in a Poisson Process. What is the distribution of T_k?

$F_{T_k}(t)$ = cumulative distribution function of T_k.

$$F_{T_k}(t) = \text{Prob}(T_k \leq t) \tag{66}$$

Clearly:

$$F_{T_k}(t) = 0 \quad \text{for} \quad t < 0 \tag{67}$$

Also, for $t \geq 0$:

$$\begin{aligned}
F_{T_k}(t) &= 1 - \text{Prob}(T_k > t) && (68)\\
&= 1 - [\text{Prob of up to } k-1 \text{ events up to } t] && (69)\\
&= 1 - [\text{Prob of} < k \text{ events up to } t] && (70)\\
&= 1 - \sum_{n=0}^{k-1} P_n(t) && (71)
\end{aligned}$$

where $P_n(t)$ is the Poisson distribution.
• The density of T_k is:

$$\begin{aligned}
f_{T_k}(t) &= \frac{\mathrm{d}}{\mathrm{d}t} F_{T_k}(t) && (72)\\
&= -\frac{\mathrm{d}}{\mathrm{d}t} \sum_{n=0}^{k-1} P_n(t) && (73)\\
&= -\frac{\mathrm{d}}{\mathrm{d}t} \sum_{n=0}^{k-1} \frac{\mathrm{e}^{-\lambda t}(\lambda t)^n}{n!} && (74)\\
&= \sum_{n=0}^{k-1} \lambda \frac{(\lambda t)^n}{n!} \mathrm{e}^{-\lambda t} - \sum_{n=0}^{k-1} n\lambda \frac{(\lambda t)^{n-1}}{n!} \mathrm{e}^{-\lambda t} && (75)\\
&= \lambda \mathrm{e}^{-\lambda t} \left[\sum_{n=0}^{k-1} \frac{(\lambda t)^n}{n!} - \sum_{n=1}^{k-1} \frac{(\lambda t)^{n-1}}{(n-1)!} \right] && (76)\\
&= \lambda \mathrm{e}^{-\lambda t} \left[\sum_{n=0}^{k-1} \frac{(\lambda t)^n}{n!} - \sum_{n=0}^{k-2} \frac{(\lambda t)^n}{n!} \right] && (77)\\
&= \lambda \mathrm{e}^{-\lambda t} \left[\frac{(\lambda t)^{k-1}}{(k-1)!} \right] && (78)
\end{aligned}$$

So:

$$f_{T_k}(t) = \frac{\lambda(\lambda t)^{k-1}\mathrm{e}^{-\lambda t}}{(k-1)!} \tag{79}$$

which is known as the **gamma distribution.**
This name refers to the **gamma function,** $\Gamma(x)$.
For integer x: $\Gamma(x) = (x-1)!$ (transparency, “SFM p.12.2”)

Mean and variance of T_k:

$$\mathrm{E}(T_k) = MTTF = \int_0^\infty t f_{T_k}(t)\,\mathrm{d}t = \frac{k}{\lambda} \tag{80}$$

$$\text{var}(T_k) = \mathrm{E}[(t-\mathrm{E}(T_k))^2] = \frac{k}{\lambda^2} \tag{81}$$

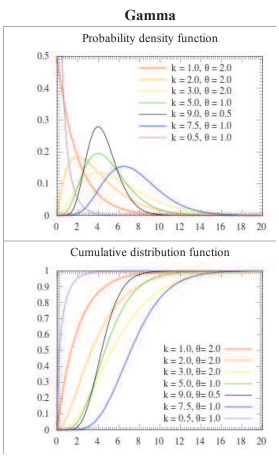


Figure 6: Gamma pdf and cdf. $\lambda = 1/\theta$. (From Wikipedia)

¶ Review exercise 14, p.39.

• **Example.** Suppose a system is subjected to a sequence of shocks which are distributed in time according to a Poisson process.

Suppose also that failure occurs on the kth shock but not before.

What is the probability distribution of the time to failure of the system?

Mean and variance of T_k are given by eqs.(80) and (81).
The pdf of the time to failure is $f_{T_k}(t)$, the gamma distribution:

$$f_{T_k}(t) = \frac{\lambda(\lambda t)^{k-1}\mathrm{e}^{-\lambda t}}{(k-1)!}, \quad t \geq 0 \tag{82}$$

¶ Review exercise 15, p.39.

The cumulative distribution function for T_k is:

$$F_{T_k}(t) = 1 - \sum_{j=0}^{k-1} \frac{(\lambda t)^j \mathrm{e}^{-\lambda t}}{j!} \tag{83}$$

Hence the probabilistic reliability function is:

$$R(t) = 1 - F_{T_k}(t) = \sum_{j=0}^{k-1} \frac{(\lambda t)^j \mathrm{e}^{-\lambda t}}{j!} \tag{84}$$

9 Reliability Analysis of Simple Serial and Parallel Systems

Based on J. Davidson, ed., 1988, *The Reliability of Mechanical Systems,* Mechanical Engineering Publications, London., chap. 9.

9.1 Serial Systems

9.1.1 Basic Theory

§ Consider a system containing two subunits, both of which must operate for the system to be functional. We can think of this as a system with two serial subunits, as in fig. 7.

$$\Longrightarrow \boxed{S_1} \Longrightarrow \boxed{S_2} \Longrightarrow$$

Figure 7: Serial network with two subunits.

§ Let P_i be the probability that the ith subunit is operational, and assume that the subunits fail independently. The probability that the system as a whole is functional is precisely the reliability: the probability of no-failure:

$$R = P_1 \times P_2 \tag{85}$$

¶ Review exercise 16, p.39.

Thus, the probability of failure of the system is:

$$F_S = 1 - R = 1 - P_1 \times P_2 \tag{86}$$

$$\Longrightarrow \boxed{S_1} \Longrightarrow \quad \cdots \quad \Longrightarrow \boxed{S_N} \Longrightarrow$$

Figure 8: Serial network with N subunits.

§ The immediate generalization to a system with N independent subunits, all of which are essential for system operation, is:

$$R = \prod_{i=1}^{N} P_i \tag{87}$$

Thus, the probability of failure of the system is:

$$F_S = 1 - R = 1 - \prod_{i=1}^{N} P_i \tag{88}$$

Note that:

$$\lim_{N\to\infty} F_S = 1 \quad \text{so} \quad \lim_{N\to\infty} R = 0 \tag{89}$$

§ What is the implication of this for complex systems?

9.1.2 Poisson Theory

Consider now a system with N essential subunits. That is, failure of any one subunit causes failure of the entire system. We can model this system as N serial subunits as in fig. 8. Also, suppose that failures in the individual subunits are distributed in time as Poisson random variables. Let the rate constant for failure in the ith subunit be λ_i, $i = 1, \ldots, N$. Then the probability of no-failure in duration t in the ith subunit is:

$$P_i = P_0(t; \lambda_i) = \mathrm{e}^{-\lambda_i t} \tag{90}$$

¶ Review exercise 17, p.39.

Thus, using eq.(87), the probability that the system is operational for a duration t is its reliability:

$$R(t) = \prod_{i=1}^{N} \mathrm{e}^{-\lambda_i t} = \exp\left[-t \sum_{i=1}^{N} \lambda_i\right] \tag{91}$$

Similarly, the probability of failure during time t is:

$$F_S(t) = 1 - R(t) = 1 - \prod_{i=1}^{N} \mathrm{e}^{-\lambda_i t} = 1 - \exp\left[-t \sum_{i=1}^{N} \lambda_i\right] \tag{92}$$

For notational convenience let us define:

$$\Lambda = \sum_{i=1}^{N} \lambda_i \tag{93}$$

The probability that one or another of the subunits will fail during an infinitesimal interval $\mathrm{d}t$ is $\Lambda \, \mathrm{d}t$. The probability that no failure occurred during the interval $(0, t)$ is, as we saw in eq.(91), $\mathrm{e}^{-\Lambda t}$. Thus the probability of survival up to time t and then failure of the system during $(t, \ t + \mathrm{d}t)$ is:

$$f(t) \, \mathrm{d}t = \mathrm{e}^{-\Lambda t} \Lambda \, \mathrm{d}t \tag{94}$$

This is simply the pdf of $F_S(t)$ in eq.(92).
We thus find the mean time to failure is:

$$\mathrm{E}(t) = \int_0^\infty t f(t) \, \mathrm{d}t = \int_0^\infty \Lambda t \mathrm{e}^{-\Lambda t} \, \mathrm{d}t \tag{95}$$

$$= \frac{1}{\Lambda} = \frac{1}{\sum_{i=1}^{N} \lambda_i} \tag{96}$$

Note that we have shown that serial Poisson systems have exponential distribution of the time to failure.
¶ Review exercise 18, p.39.

9.2 Parallel Systems

9.2.1 Basic Theory

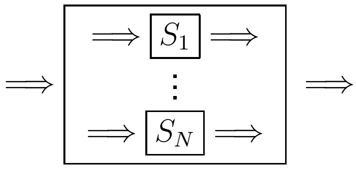


Figure 9: Parallel network with N subunits.

Consider a system which is operational if one or more of its N subunits is functional. We can think of such a system as though it were made up of N parallel subunits, as in fig. 9. As before, let P_i be the probability that the ith subunit is functional. Let $F_i = 1 - P_i$ be the probability that the ith subunit is not functional. Then the probability that the system as a whole is not functional is:

$$F_S = \prod_{i=1}^{N} F_i \tag{97}$$

That is, the situation is just the reverse of the case with serial subunits, and eq.(97) is the reverse of eq.(87).

The probability that the system is operational is its reliability:

$$R = 1 - F_S = 1 - \prod_{i=1}^{N} F_i = 1 - \prod_{i=1}^{N}(1 - P_i) \tag{98}$$

Note that:

$$\lim_{N \to \infty} R = 1 \tag{99}$$

What does this imply about the reliability of complex systems? Compare with eq.(89).

¶ Review exercise 19, p.39.

Item	Mean failure rate (FPMH)	Number of units	Total mean failure rate (FPMH)
Pressure sensor	42	2	84
Temperature sensor	12	2	24
Safety relief value	130	1	130
Shut off value	95	2	190
Heat exchanger	99	1	99
Check value	1.8	1	1.8
Pump	260	1	260
Control unit	25	1	25
Total			**814**

Table 1: Mean failure rates in Failures Per Million Hours (FPMH).

9.3 Generic-Parts-Count Reliability Analysis

§ Based on J. Davidson, ed., 1988, *The Reliability of Mechanical Systems,* Mechanical Engineering Publications, London., chap. 12.

§ The "generic parts count" method for reliability analysis of complex systems is a widely used method, and is based on the following assumptions:

1. The system is made up of distinct subunits whose failure probabilities are statistically independent.
2. The operation of the entire system is dependent on all subunits. Thus, the system is modelled as though it were serial.
3. Each subunit has a constant failure rate. Thus, the distribution of failures in each subunit is modelled as a Poisson process.

Example 1 (From Davidson 1988, chap. 12). The mean failure rates in units of failures per million hours of operation per single device are shown in table 1. The total failure rate is $\Lambda = 814 \times 10^{-6}$ failures/hour. Thus the probability that the system will survive for one month of continuous operation is:

$$R = \mathrm{e}^{-\Lambda t} = \mathrm{e}^{-814\times 10^{-6}\times 24\times 30} = 0.577 \tag{100}$$

Thus the system has a 57.7% chance of operating without failure for one 30-day month. ■

10 Pareto Distribution

• Definition: a random variable X has a Pareto distribution if its cdf is:

$$F_X(x) = 1 - \left(\frac{d}{x}\right)^c, \quad x > d, \quad c > 0, \quad d > 0 \tag{101}$$

We will explain one context in which the Pareto distribution arises.
• Let a population have an exponential TTF distribution with parameter λ:

$$f(t|\lambda) = \lambda \mathrm{e}^{-\lambda t}, \quad t \geq 0 \tag{102}$$

The parameter λ is sometimes **unknown** or **variable.**

Suppose that λ is a random variable with a gamma distribution:

$$\pi(\lambda) = \frac{\alpha}{\Gamma(k)}(\alpha\lambda)^{k-1}\mathrm{e}^{-\alpha\lambda}, \quad \lambda \geq 0, \quad \underbrace{\alpha > 0, \quad k \geq 1}_{\text{parameters}} \tag{103}$$

Recall: $\Gamma(k) = (k-1)!$ if k is an integer.

The marginal pdf of the TTF is:

$$f(t) = \int_0^\infty f(t|\lambda)\pi(\lambda)\,\mathrm{d}\lambda = \frac{k\alpha^k}{(\alpha+t)^{k+1}} \tag{104}$$

and the cdf of the TTF is:

$$F(t) = \int_0^t f(\tau)\,\mathrm{d}\tau = 1 - \left(1 + \frac{t}{\alpha}\right)^{-k} \tag{105}$$

¶ Review exercise 20, p.39.

• Now define a new random variable:

$$Y = 1 + \frac{T}{\alpha} \tag{106}$$

Y is a scaled and translated version of the TTF T.
We will show that Y has a Pareto distribution:

$$\begin{aligned} F_Y(y) &= \text{Prob}(Y \leq y) && (107)\\ &= \text{Prob}\left(\frac{T}{\alpha} + 1 \leq y\right) && (108)\\ &= \text{Prob}[T \leq \alpha(y-1)] && (109)\\ &= 1 - \left(1 + \frac{\alpha(y-1)}{\alpha}\right)^{-k} && (110)\\ &= 1 - \left(\frac{1}{y}\right)^k && (111) \end{aligned}$$

So Y has a Pareto distribution with:

$$d = 1, \quad c = k \tag{112}$$

• The reliability function, from eq.(105), is:

$$R(t) = \text{Prob}(T > t) = 1 - F(t) = \left(1 + \frac{t}{\alpha}\right)^{-k} \tag{113}$$

The MTTF is:

$$MTTF = \int_0^\infty R(t)\,\mathrm{d}t = \frac{\alpha}{k-1} \tag{114}$$

Note that:

$$MTTF = \infty \quad \text{if} \quad k = 1 \tag{115}$$

(Recall from eq.(103) that $k \ge 1$.)

• What does eq.(115) mean? For example, given two finite samples of lifetimes:
  ◦ Do these samples have MTTF's?
  ◦ What is the relation between these these two values of the sample MTTF?
  ◦ What is the relation between these sample MTTF's and the ensemble MTTF?

• The failure rate function is:

$$z(t) = \frac{f(t)}{R(t)} = \frac{k}{\alpha + t} \tag{116}$$

which decreases monotonically with increasing t.
  • What does this mean?
  • What stage-of-life would this probabilistic failure model represent?

¶ Review exercise 21, p.39.

11 Weibull Distribution

- Definition: a random variable T has a Weibull distribution if its cdf is:

$$F_T(t) = \text{Prob}(T \le t) = \begin{cases} 1 - \mathrm{e}^{-(\lambda t)^\alpha}, & t > 0 \\ 0 & t \le 0 \end{cases} \quad (117)$$

- The pdf of T is (fig. 10):

$$f_T(t) = \frac{\mathrm{d}F_T(t)}{\mathrm{d}t} = \alpha\lambda(\lambda t)^{\alpha-1}\mathrm{e}^{-(\lambda t)^\alpha}, \quad t > 0 \quad (118)$$

- The reliability function is:

$$R(t) = 1 - F_T(t) = \mathrm{e}^{-(\lambda t)^\alpha}, \quad t > 0 \quad (119)$$

- The failure rate function is (fig. 11):

$$z(t) = \frac{f(t)}{R(t)} = \alpha\lambda(\lambda t)^{\alpha-1}, \quad t > 0 \quad (120)$$

$z(t)$ **increases** in t for $\alpha > 1$.
$z(t)$ **decreases** in t for $\alpha < 1$.

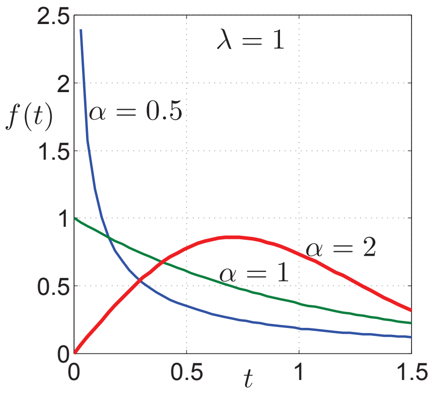


Figure 10: Weibull pdf function.

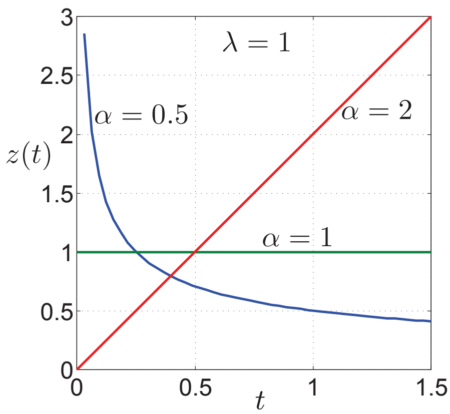


Figure 11: Weibull failure rate function.

¶ **Review exercise 22, p.39.**

- The MTTF is:

$$MTTF = \int_0^\infty R(t)\,\mathrm{d}t = \frac{1}{\lambda}\Gamma\left(1 + \frac{1}{\alpha}\right) \quad (121)$$

- The Weibull distribution arises as a **limit distribution.**

Weibull describes the distribution of the **least** or **greatest** of a large number of identically distributed non-negative random variables. Suppose:

$$x_1,\, x_2,\, \ldots, x_N \quad \text{are i.i.d.} \quad \text{and} \quad N \text{ is large} \tag{122}$$

Then:

$$\min_n x_n \sim \text{ Weibull} \tag{123}$$

And:

$$\max_n x_n \sim \text{ Weibull} \tag{124}$$

The Weibull distribution is thus sometimes called the **weakest link** distribution.

¶ Review exercise 23, p.39.

• **Example.** Consider a steel pipe with wall thickness θ which is exposed to corrosion.

The surface has N pits, where N is large.

D_i = depth of ith pit.

Failure occurs when the any pit penetrates the surface:

$$\text{Failure:} \quad \max_{1 \le i \le N} D_i = \theta \tag{125}$$

T_i = time to penetration of the ith pit, which is proportional to D_i.

The TTF is:

$$T = \min_{1 \le i \le N} T_i \tag{126}$$

If the number of pits, N, is large, then T will tend to a Weibull distribution. ∎

• **Example.** Consider a system with N components, whose lifetimes are identically distributed.

The system fails as soon as one component fails.

Let $T_1,\, \ldots, T_N$ be the TTFs of the components.

The TTF of the system is:

$$T = \min_{1 \le n \le N} T_n \tag{127}$$

The distribution of T will tend to a Weibull distribution for large N. ∎

12 Normal Distribution

• Definition: t has a **normal distribution** if its pdf is:

$$f(t) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(t-\mu)^2}{2\sigma^2}\right), \quad -\infty < t < \infty \tag{128}$$

We write:

$$t \sim \mathcal{N}(\mu, \sigma^2) \tag{129}$$

where μ and σ^2 are the mean and variance of t.
• The **standard normal distribution** is $t \sim \mathcal{N}(0,1)$. Its pdf is:

$$\phi(t) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{t^2}{2}\right), \quad -\infty < t < \infty \tag{130}$$

The cdf is denoted $\Phi(t)$.

• **Central limit theorem.** If $x_1,\ x_2,\ \ldots$ are independent, identically distributed random variables, then:

$$x = \frac{1}{\sqrt{N}} \sum_{i=1}^{N} x_i \tag{131}$$

tends to a normal distribution as $N \to \infty$.
Note: this does not depend on how the x_i are distributed, or even if they are discrete or continuous random variables.

¶ **Review exercise 24, p.39.**

• **Standardization.** If $T \sim \mathcal{N}(\mu, \sigma^2)$, then:

$$F_T(t) = \text{Prob}(T \le t) = \text{Prob}\left(\frac{T-\mu}{\sigma} \le \frac{t-\mu}{\sigma}\right) \tag{132}$$

But:

$$\frac{T-\mu}{\sigma} \sim \mathcal{N}(0,1) \tag{133}$$

Thus:

$$F_T(t) = \Phi\left(\frac{t-\mu}{\sigma}\right) \tag{134}$$

• If $T \sim \mathcal{N}(\mu, \sigma^2)$, then the **reliability function** is:

$$R(t) = 1 - F_T(t) = 1 - \Phi\left(\frac{t-\mu}{\sigma}\right) \tag{135}$$

• The **failure rate function** is:

$$z(t) = \frac{f(t)}{R(t)} = \frac{\phi\left(\frac{t-\mu}{\sigma}\right)}{\sigma\left[1 - \Phi\left(\frac{t-\mu}{\sigma}\right)\right]} \tag{136}$$

which is always increasing with t.

13 Lognormal Distribution

• Definition. A random variable T is **lognormally distributed** if:

$$Y = \ln T \quad \text{and} \quad Y \sim \mathcal{N}(\mu, \sigma^2) \tag{137}$$

• The pdf of T is:

$$f(t) = \begin{cases} \frac{1}{\sqrt{2\pi}\sigma t} \exp\left[-\frac{(\ln t - \mu)^2}{2\sigma^2}\right], & t > 0 \\ 0 & t \le 0 \end{cases} \tag{138}$$

where the parameters are $\sigma^2 > 0$ and $-\infty < \mu < \infty$.

• The MTTF and variance of T are:

$$MTTF = \exp\left(\mu + \frac{\sigma^2}{2}\right) \tag{139}$$

$$\text{var}(T) = \mathrm{e}^{2\mu}\left(\mathrm{e}^{2\sigma^2} - \mathrm{e}^{\sigma^2}\right) \tag{140}$$

• The reliability function is:

$$R(t) = \text{Prob}(T > t) = \text{Prob}(\ln T > \ln t) \tag{141}$$

$$= \text{Prob}\left(\frac{\ln T - \mu}{\sigma} > \frac{\ln t - \mu}{\sigma}\right) \tag{142}$$

$$= 1 - \Phi\left(\frac{\ln t - \mu}{\sigma}\right) \tag{143}$$

because:

$$Y = \ln T \sim \mathcal{N}(\mu, \sigma^2) \tag{144}$$

so:

$$\frac{\ln T - \mu}{\sigma} \sim \mathcal{N}(0, 1) \tag{145}$$

14 Probabilistic Reliability with Info-Gap-Uncertain PDF

14.1 The Problem

¶ **The problem:**
- Fat tails: extreme adverse outcomes too frequent.
- Conditional volatility:
  Size of fluctuations varies in time.
- Predicting percentiles:
  - ○ 95th: is sometimes okay.
  - ○ 99th: is sometimes under- or over-estimated.

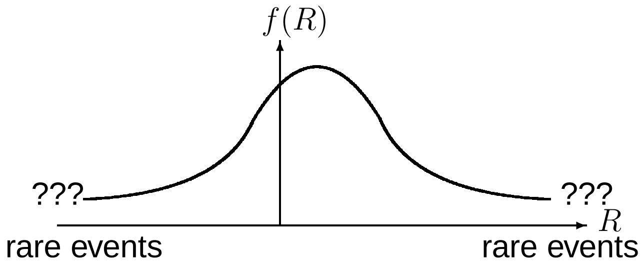


Figure 12: Uncertain tails of a distribution.

¶ **The opportunity:**
- Fat tails: extreme favorable outcomes occur too.
- Two sides to uncertainty:
  - ○ Design **robustly** against failure: robust-satisficing.
  - ○ Design **opportunely** for windfall: opportune-windfalling.

¶ **Two foci of uncertainty:**
- Statistical fluctuations:
  - ○ Randomness, “noise”.
  - ○ Estimation uncertainty.
- Info-gap uncertainty:
  - ○ Surprises.
  - ○ Structural changes.
  - ○ Historical data used to predict future.

¶ **Review exercise 25, p.39.**

¶ Related concepts:
- Statistical quantiles.
- Value at Risk in financial economics.[1]

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[1]○ Yakov Ben-Haim, 2005, Value at risk with info-gap uncertainty, *Journal of Risk Finance,* vol. 6, #5, pp.388–403.
○ Yakov Ben-Haim, 2010, *Info-Gap Economics: An Operational Introduction,* Palgrave-Macmillan.

14.2 Uncertainty and Robustness

¶ **Notation:**

$R =$ random variable, e.g. system response.
$f(R) =$ pdf for R.
- ∘ Highly uncertain.
- ∘ Large info-gaps.

$\widetilde{f}(R) =$ best estimate of $f(R)$.
- ∘ Statistical estimation.
- ∘ Historical data.

$R_\star =$ least acceptable value of R.
- ∘ Large R is desired.
- ∘ Design requirement.

¶ **Info-gap uncertainty:**

$f(R) =$ unknown true pdf for R.
$\widetilde{f}(R) =$ best estimate of $f(R)$.
$\mathcal{F}(h, \widetilde{f}) =$ info-gap model for uncertainty in $f(R)$, fig. 13.
- Unbounded family of nested sets of pdfs.
- E.g. Unknown fractional error in $f(R)$:

$$\mathcal{F}(h, \widetilde{f}) = \left\{ f(R) : \ f(R) \ge 0, \ \int_{-\infty}^{\infty} f(R)\, \mathrm{d}R = 1, \ \frac{\left| f(R) - \widetilde{f}(R) \right|}{\widetilde{f}(R)} \le h \right\}, \ h \ge 0 \qquad (146)$$

- No known worst case.
- Unbounded horizon of uncertainty, h.
- Info-gap uncertainty.

¶ **Review exercise 26, p.39.**

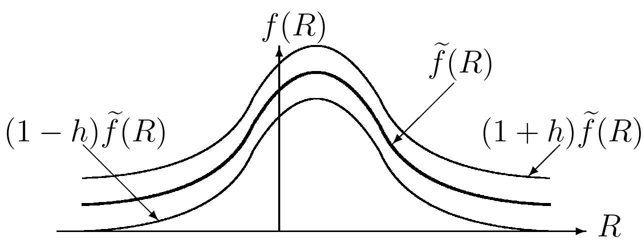


Figure 13: Uncertainty envelope at horizon of uncertainty h.

¶ **Failure probability:**

- Failure: response less than $R_\star$: $R < R_\star$.
- Given $R_\star$, the failure probability is (fig. 14):

$$P_\mathrm{f}(f) = \int_{-\infty}^{R_\star} f(R)\,\mathrm{d}R \tag{147}$$

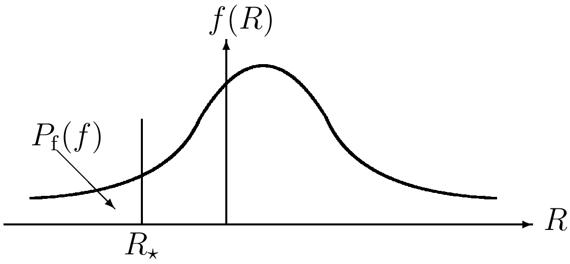


Figure 14: $q(c, f)$ is the cth quantile of $f(R)$.

- Probabilistic design requirement:

$$P_\mathrm{f}(f) \le c \tag{148}$$

  E.g., $c = 0.01$.
- **Problem:** $f(R)$ highly uncertain, especially on its tails.
- Estimated failure probability:

$$P_\mathrm{f}(\widetilde{f}) = \int_{-\infty}^{R_\star} \widetilde{f}(R)\,\mathrm{d}R \tag{149}$$

¶ **Robustness question:**
How wrong can $\widetilde{f}(R)$ be, without failure probability exceeding c?

¶ **Review exercise 27, p.40.**

¶ **Design implication:** large robustness implies desirable design.

¶ **Info-gap robustness: Formulation.**

- $q(c,f) = c$th **quantile** of $f(R)$:

$$\int_{-\infty}^{q(c,f)} f(R)\,\mathrm{d}R = c \tag{150}$$

¶ **Review exercise 28, p.40.**

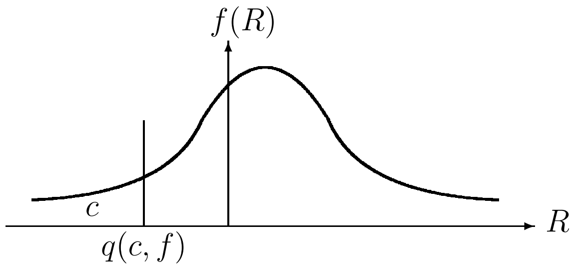


Figure 15: $q(c,f)$ is the cth quantile of $f(R)$.

- $R_\star =$ least acceptable response.
- Estimated response: $q(c,\widetilde{f})$
  If $\widetilde{f}$ is correct, then the prob of response less than $q(c,\widetilde{f})$ is less than c.
- True response: $q(c,f)$
- Requirement: R exceeds $R_\star$ with probability no less than $1-c$:

$$q(c,f) \geq R_\star \tag{151}$$

This is because:

$$q(c,f) \geq R_\star \quad \Longleftrightarrow \quad c \geq \int_{-\infty}^{R_\star} f(R)\,\mathrm{d}R \tag{152}$$

- Robustness of $R_\star$ with confidence c is:
  - Max tolerable info-gap in f.
  - Max h at which true response at confidence $1-c$, is no less than $R_\star$:

$$\widehat{h}(R_\star, c) = \max\left\{h:\ \left(\min_{f\in\mathcal{F}(h,\widetilde{f})} q(c,f)\right) \geq R_\star\right\} \tag{153}$$

¶ **Review exercise 29, p.40.**

14.3 Deriving the Robustness Function

¶ We now derive the robustness function, defined in eq.(153) on p.32 as:

$$\widehat{h}(R_\star, c) = \max\left\{h : \left(\min_{f\in\mathcal{F}(h,\widetilde{f})} q(c,f)\right) \geq R_\star\right\} \tag{154}$$

where $q(c, f)$ is the cth quantile of $f(R)$.
The robustness is the max horizon of uncertainty so that the cth quantile of f exceeds $R_\star$.

¶ Recall the definition of the quantile:

$$\int_{-\infty}^{q(c,f)} f(R)\,\mathrm{d}R = c \tag{155}$$

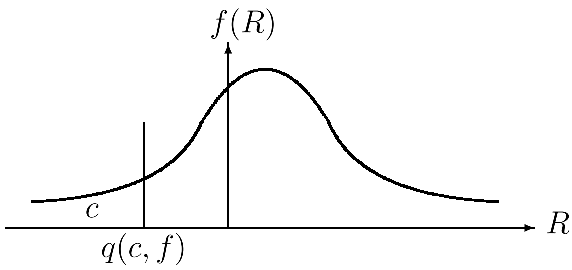


Figure 16: $q(c, f)$ is the cth quantile of $f(R)$.

¶ So, the min in eq.(154) occurs when the lower tail of $f(R)$ is as fat as possible:[2]

$$f(R) = (1 + h)\widetilde{f}(R) \tag{156}$$

¶ Review exercise 30, p.40.

[2]This is true for very small c.

¶ The following two inequalities are equivalent, as illustrated in fig. 16:

$$q(c, f) \geq R_\star \iff c \geq \int_{-\infty}^{R_\star} f(R)\,\mathrm{d}R \tag{157}$$

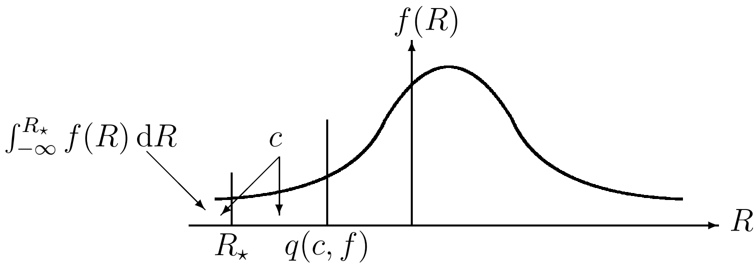


Figure 17: $q(c, f)$ is the cth quantile of $f(R)$.

¶ Using eqs.(156) and (157) the robustness is the max h at which:

$$c \geq \int_{-\infty}^{R_\star} (1 + h)\tilde{f}(R)\,\mathrm{d}R \tag{158}$$

Solving this for h yields the robustness:

$$\widehat{h}(R_\star, c) = \frac{c}{\displaystyle\int_{-\infty}^{R_\star} f(R)\,\mathrm{d}R} - 1 \tag{159}$$

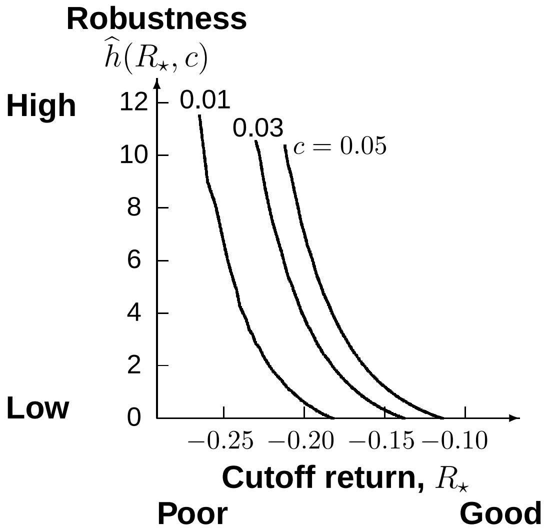
or zero if this is negative.

14.4 Numerical Example

¶ **Estimated pdf,** $\widetilde{f}(R)$, is normal: $\mu = 0.05$, $\sigma^2 = 0.01$.

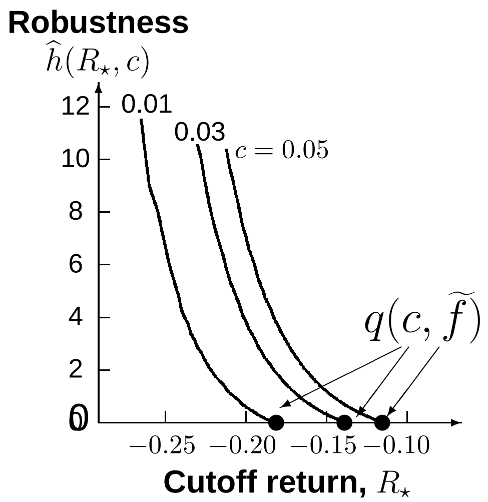
¶ **Trade-off:**
Robustness decreases as aspiration increases (transparency):

$$R_{\star 1} < R_{\star 2} < 0 \quad \text{implies} \quad \widehat{h}(R_{\star 2}, c) \le \widehat{h}(R_{\star 1}, c) \tag{160}$$



¶ **Best-model response** is unreliable:

$$\widehat{h}(R_\star^o, c) = 0 \quad \text{if} \quad R_\star^o = q(c, \widetilde{f}) \tag{161}$$



14.5 Opportuneness

¶ **Two faces of uncertainty:**
- Pernicious: threatening failure.
- Propitious: enabling windfall.

¶ We will now briefly study the propitious side of uncertainty.

¶ Recall: $q(c, f)$ is cth quantile of $f(R)$. Units: response, R.

¶ **Robustness,** re-stated (eq.(153), p.32):

$$\widehat{h}(R_\star, c) = \max\left\{h : \left(\min_{f\in\mathcal{F}(h,\widetilde{f})} q(c, f)\right) \geq R_\star\right\} \tag{162}$$

- $R_\star$ is the least acceptable response.
- $\widehat{h}(R_\star, c)$ is the max horizon of uncertainty in the pdf, $f(R)$, up to which response at least as large as $R_\star$ is **guaranteed** with probability no less than $1 - c$.
- **Greatest horizon of uncertainty at which tolerable response is guaranteed.**

¶ **Opportuneness:**
- **Lowest horizon of uncertainty at which wonderful response is possible.**
- Windfall response: $R^\star$, greater than $R_\star$.

$$\widehat{\beta}(R^\star, c) = \min\left\{h : \left(\max_{f\in\mathcal{F}(h,\widetilde{f})} q(c, f)\right) \geq R^\star\right\} \tag{163}$$

- $\widehat{\beta}(R^\star, c)$ is the min horizon of uncertainty in the pdf, $f(R)$, up to which response at least as large as $R_\star$ is **possible** with probability no less than $1 - c$.

¶ **Immunity functions:**
- Robustness: immunity against failure. Bigger $\widehat{h}$ is better.
- Opportuneness: immunity against windfall. Big $\widehat{\beta}$ is bad.

¶ **Summary**
- Quantile: statisical assessment of risk.
- Foci of uncertainty:
  - ◦ Estimation error: randomness.
  - ◦ Info-gap uncertainty, surprise, past/future.
- Info-gap robustness:
  - ◦ Managing info-gaps.
  - ◦ Supplementing statistical quantile.
- Info-gap opportuneness:

○ Exploiting info-gaps.

15 Review Exercises

§ The exercises in this section are not homework problems, and they do not entitle the student to credit. They will assist the student to master the material in the lecture and are highly recommended for review and self-study.

1. (a) In analogy to eq.(1), p.4, write an integral expression for the probability of failure in the time interval from t_1 to t_2.
   (b) Write an integral expression for the probability of failure in the interval $[t_1, t_2]$ or $[t_3, t_4]$ where $t_2 < t_3$.
   (c) Write an integral expression for the probability of failure in the interval $[t_1, t_2]$ or $[t_3, t_4]$ where $t_4 > t_2 > t_3 > t_1$.

2. Section 3, p.5:
   (a) For any random variable x, a median of the distribution of x is a value, q, such that the following two conditions hold:[3]

$$\text{Prob}(x \le q) \ge \frac{1}{2} \quad \text{and} \quad \text{Prob}(x \ge q) \ge \frac{1}{2} \tag{164}$$

   Is a median a quantile? If so, what is its "$1 - \alpha$" value?
   (b) If $f(t)$ is a symmetric probability density function on the interval $[-a, a]$, what is the difference between its mean and its median?
   (c) Compare the mean and the median of the exponential distribution, $f(t) = \lambda \mathrm{e}^{-\lambda t}$, $t \ge 0$. Which is larger? Why?

3. Explain the denominator of eq.(10), p.6. Also consider the special case that B always occurs.

4. Derive eq.(15), p.7, from eq.(14).

5. Explain eq.(18), p.7, in terms of eq.(10), p.6.

6. Why does $R(0) = 1$ make sense on p.9?

7. Given a sample, $t_1, \ldots, t_N$, what would be a good choice of bins—size, location and number of bins—in fig. 4, p.10?

8. Why is eq.(32), p.10, an approximation? Under what conditions is it a good approximation?

9. Use eq.(39), p.11, to give a "reliability" explanation of why a large MTTF is desirable. Alternatively, if the MTTF is large, use eq.(39) to explain the sense in which the system is reliable, and the limits of this interpretation.

10. (a) Explain why the probability on the righthand side of eq.(42), p.12, "balances" the probability on the left hand side.

[3] DeGroot, Morris H., *Probability and Statistics,* 2nd ed., Addison-Wesley, Reading, MA, 1986, p.207.

(b) In eq.(42) the value of n can either stay the same or increase; it cannot decrease. Now suppose that it can also decrease due to "repair" as opposed to failure. In analogy to λ, Let μ be the average number of repairs/sec. Thus the probability of repair in duration $\mathrm{d}t$ is $\mu \mathrm{d}t$. Suppose that repair and failure are independent. Derive a probability balance equation in analogy to eq.(42).

11. Reasoning as in eqs.(50)–(56), p.13, derive an expression for $\mathrm{E}(n^2)$.
12. Explain eq.(60), p.15.
13. Derive eq.(65), p.15.
14. For what value of k does f_{T_k} in eq.(79), p.17, become the exponential distribution?
15. (a) The mean and variance in eqs.(80) and (81), p.17, increase as k increases. Does this make sense? Why?
    (b) How does the relative error, $\sigma/$MTTF, vary with k? Does this make sense? Why?
16. Explain eq.(85), p.19.
17. Explain eq.(90), p.20.
18. Following eq.(93), p.20, we said that the probability that one or another of the subunits will fail during an infinitesimal interval $\mathrm{d}t$ is $\Lambda\,\mathrm{d}t$. Explain. What assumptions underlie this assertion?
19. (a) Compare eqs.(97), p.21, and (88), p.19. Explain the difference.
    (b) Compare eqs.(98), p.21, and (87), p.19. Explain the difference.
20. (a) Explain the integral in eq.(104), p.23.
    (b) There are two random variables in eqs.(102) and (103), p.23, t and λ. Explain how this can arise as a mixture of a continuum of different exponential distributions, each with its own exponent.
21. What is the meaning of the decreasing failure rate function in eq.(116), p.24?
22. What is the meaning of the decreasing or increasing failure rate function in eq.(120), p.25? Why does this make the Weibull distribution useful for explaining a wide range of phenomena?
23. Explain the "weakest link" metaphor as it is applied to the Weibull distribution on p.26.
24. Why does the central limit theorem, p.27, motivate calling the normal distribution "normal"?
25. Explain the similarity and difference between the two foci of uncertainty on p.29.
26. (a) Why is there no known worst case in the info-gap model of eq.(146), p.30? Could there be a worst case which simply is not known?
    (b) Explain that the sets of the info-gap model of eq.(146) become more inclusive as h increases. Explain why this gives h its meaning as an "horizon of uncertainty".

27. Can we answer the robustness question on p.31 even if we don't know how wrong $\widetilde{f}(R)$ actually is?

28. Explain why eq.(150), p.32, is a definition of the quantile function.

29. Explain eq.(153), p.32, in your own words.

30. Why does the solution in eq.(156), p.33, depend on c being very small?