Lecture Notes on Robustness and Opportuneness

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Primary source material: Yakov Ben-Haim, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* 2nd edition, Academic Press. Chapter 3.

Notes to the Student:

• These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

• Section 18 contains review exercises that will assist the student to master the material in the lecture and are highly recommended for review and self-study. The student is directed to the review exercises at selected places in the notes. They are not homework problems, and they do not entitle the student to extra credit.

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1 Preliminary Example: Reliability of a Beam With an Uncertain Load

(Source: Y. Ben-Haim, *Robust Reliability in the Mechanical Sciences,* sections 3.1, 3.2.)

¶ 3 components of reliability analysis:
1. A system model.
2. A failure criterion.
3. An uncertainty model.

¶ We will consider info-gap models of uncertainty and develop, in a preliminary example, the idea of **info-gap robustness.**

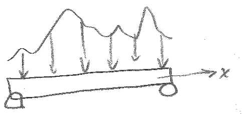

Figure 1: Simply-supported beam.

¶ Consider a:
- Uniform simply-supported beam, fig. 1.
- Uncertain distributed load density function, $\phi(x)$ [N/m].

¶ We wish to
- Analyze the reliability of the beam given very fragmentary information.
- Optimize the design of the beam by enhancing the reliability.
- Evaluate the impact of different levels and types of information.

¶ What we **do know** about the load:
- $\widetilde{\phi}(x) =$ nominal load density function, [N/m].
- Substantial deviation from the nominal load is bounded along the beam.

¶ What we **do not know** about the load:
- The precise realization of the load density, $\phi(x)$.
- The bound on the deviation of the true from the nominal load.

¶ The disparity between what we
do know and what we **need to know**
for a fully competent design or analysis
is an **information gap.**

¶ We represent the load uncertainty with an info-gap model:

$$\mathcal{U}(h,\widetilde{\phi}) = \left\{ \phi(x) : \ \left| \phi(x) - \widetilde{\phi}(x) \right| \le h \right\}, \quad h \ge 0 \tag{1}$$

This is an info-gap **uncertainty model.**

¶ Note the two levels of uncertainty in an info-gap model:
- At fixed h: true load profile $\phi(x)$ is unknown.
- Horizon of uncertainty — h — is unknown.

¶ **2 properties of all info-gap models:**
- *Contraction:*

$$\mathcal{U}(0) = \left\{ \widetilde{\phi}(x) \right\} \tag{2}$$

- *Nesting:*

$$h < h' \quad \Longrightarrow \quad \mathcal{U}(h) \subseteq \mathcal{U}(h') \tag{3}$$

¶ **System model:**
- Static bending moment as a function of load profile: $M(x)$.
- For simple-simple beam one finds:

$$M(x) = -\frac{L-x}{L} \int_0^x \phi(u) u \,\mathrm{d}u - \frac{x}{L} \int_x^L \phi(u)(L-u)\,\mathrm{d}u \tag{4}$$

where L is the length of the beam. (**Review exercise 1 on p.97.**)

¶ The **failure criterion:**
The beam fails if the absolute bending moment, $|M(x)|$, exceeds the critical value M_{c}:

$$\max_{0 \le x \le L} |M(x)| > M_{\mathrm{c}} \tag{5}$$

¶ We evaluate the **robustness,** $\widehat{h}$, by combining
System model, uncertainty model, and failure criterion:
The **robustness** is:
The greatest info-gap, h,
such that the **system model**
does not violate the **failure criterion**
for any load profile up to **uncertainty** h.
We can express $\widehat{h}$ as:

$$\begin{aligned}
\widehat{h} &= \text{maximum tolerable uncertainty} && (6)\\
&= \max \{ h : \ \text{failure cannot occur} \} && (7)\\
&= \max \left\{ h : \ \left(\max_{0 \le x \le L} |M(x)| \right) \le M_{\mathrm{c}} \text{ for all } \phi(x) \text{ in } \mathcal{U}(h,\widetilde{\phi}) \right\} && (8)\\
&= \max \left\{ h : \ \left(\max_{\phi \in \mathcal{U}(h,\widetilde{\phi})} \ \max_{0 \le x \le L} |M(x)| \right) \le M_{\mathrm{c}} \right\} && (9)
\end{aligned}$$

We can invert the order of the maxima inside the set.

¶ We begin by evaluating:

$$\max_{\phi \in \mathcal{U}(h,\widetilde{\phi})} |M(x)| = \max \left(\max_{\phi \in \mathcal{U}(h,\widetilde{\phi})} M(x), \ \left| \min_{\phi \in \mathcal{U}(h,\widetilde{\phi})} M(x) \right| \right) \tag{10}$$

¶ To find these extrema note that:

- Other than $\phi(u)$, the integrands of both integrals in eq.(4) on p.4 have the same sign everywhere.
- Thus, extremal $M(x)$ is obtained by choosing
  $\phi(x) = \widetilde{\phi}(x) + h$ or $\phi(x) = \widetilde{\phi}(x) - h$.
- **We consider a special case:** $\widetilde{\phi}(x) =$ positive constant.
- The results:

$$\max_{\phi \in \mathcal{U}(h,\widetilde{\phi})} M(x) = -\frac{(h - \widetilde{\phi})x(L-x)}{2} \tag{11}$$

$$\min_{\phi \in \mathcal{U}(h,\widetilde{\phi})} M(x) = -\frac{(h + \widetilde{\phi})x(L-x)}{2} \tag{12}$$

Hence:

$$\max_{\phi \in \mathcal{U}(h,\widetilde{\phi})} |M(x)| = \frac{(h + \widetilde{\phi})x(L-x)}{2} \tag{13}$$

- **Review exercise 2 on p.97.**

¶ We are now ready to evaluate the second optimization, on x,
in the expression for the robustness, eq.(9) on p.4.
We find the maximum at $x = L/2$, resulting in:

$$\max_{0 \le x \le L} \max_{\phi \in \mathcal{U}(h,\widetilde{\phi})} |M(x)| = \frac{(h + \widetilde{\phi})L^2}{8} \tag{14}$$

¶ The robustness is the greatest h
at which the maximum absolute bending moment $|M(x)|$
does not exceed the critical value $M_{\rm c}$.
We find:

$$\underbrace{\frac{(h + \widetilde{\phi})L^2}{8}}_{\text{max bending moment}} = \underbrace{M_{\rm c}}_{\text{critical moment}} \implies \widehat{h} = \frac{8M_{\rm c}}{L^2} - \widetilde{\phi} \tag{15}$$

Design implications: the robustness, $\widehat{h}$, increases as:

- The beam length L decreases.
- The nominal load $\widetilde{\phi}$ decreases.
- The critical bending moment $M_{\rm c}$ increases.

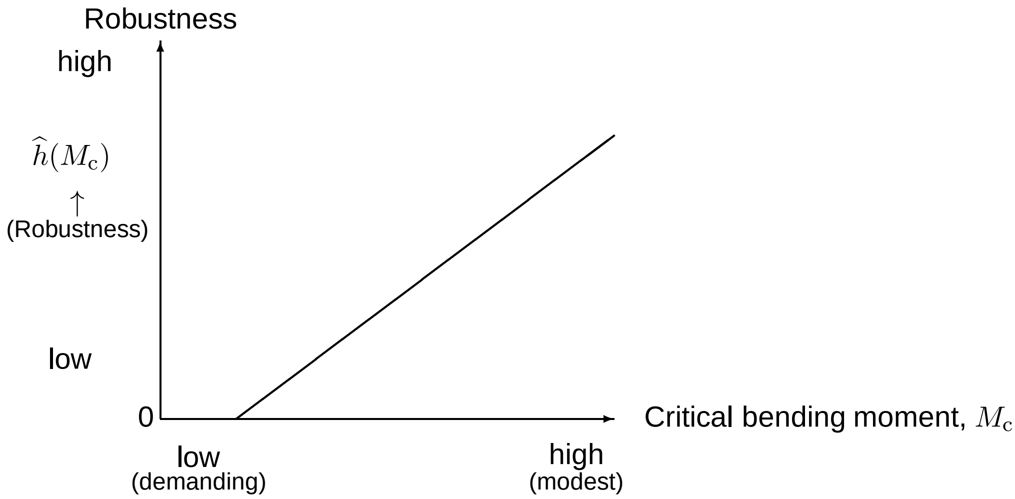

Figure 2: Robustness curve.

¶ **Two Properties:** Trade-off and zeroing (see fig. 2).

¶ **Trade off:** robustness vs performance.
- $\widehat{h}(M_c)$ *gets worse* (decreases) as M_c *gets more demanding* (decreases).
- This is sometimes called the pessimist's theorem. Why?
- The slope of the robustness curve expresses the cost of robustness. Why?

¶ **Zeroing:** Estimated performance has zero robustness:

$$\widehat{h}(M_c) = 0 \quad \text{if} \quad M_c \le \frac{\widetilde{\phi} L^2}{8} = \text{estimated bending moment} \tag{16}$$

- **Review exercise 3 on p.97.**

2 Statically Loaded Beam: Continued

¶ Knowledge is:
- Power.
- Robustness against surprise and uncertainty.

2.1 Load-Uncertainty Envelope

¶ Let us now consider **different prior information.**
Rather than the uniform-bound info-gap model of eq.(1) on p.4,
suppose we have information which indicates that
the uncertain deviation $\phi(x) - \widetilde{\phi}(x)$ varies within an envelope:

$$\mathcal{U}(h, \widetilde{\phi}) = \left\{ \phi(x) : \ \left| \phi(x) - \widetilde{\phi}(x) \right| \le h\psi(x) \right\}, \quad h \ge 0 \tag{17}$$

where we **know:**
$\widetilde{\phi}(x) =$ nominal load profile.
$\psi(x) =$ load-uncertainty envelope.
and we **do not know:**
$\phi(x) =$ actual load profile.
$h =$ uncertainty parameter, horizon of uncertainty.

¶ **Examples of envelope function,** $\psi(x)$:
- Hidden load on left half of beam.

$$\psi(x) = \begin{cases} 1, & 0 \le x \le L/2 \\ 0, & L/2 < x \le L \end{cases} \tag{18}$$

- Flow perpendicular to beam; increasing turbulence in middle region.

$$\psi(x) = \sin \frac{\pi x}{L} \tag{19}$$

¶ As usual with an info-gap model, there are two levels of uncertainty:
- Unknown realization $\phi(x)$ at info-gap h.
- Unknown horizon of uncertainty, h.

¶ As before:
- The system model is eq.(4) on p.4.
- The failure criterion is eq.(5) on p.4.

¶ To find the maximum absolute bending moment
we evaluate the max and the min of $M_\phi(x)$.
The max (least negative) is obtained with the lowest possible load profile,
while
The min (most negative) is obtained with the greatest possible load profile.
We find:

$$\begin{aligned} M_1(x) &= \min_{\phi\in\mathcal{U}(h,\widetilde{\phi})} M(x) & (20)\\ &= -\frac{L-x}{L}\int_0^x \left[\widetilde{\phi}(u)+h\psi(u)\right] u\,\mathrm{d}u \\ &\quad -\frac{x}{L}\int_x^L \left[\widetilde{\phi}(u)+h\psi(u)\right](L-u)\,\mathrm{d}u & (21)\\ M_2(x) &= \max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} M(x) & (22)\\ &= -\frac{L-x}{L}\int_0^x \left[\widetilde{\phi}(u)-h\psi(u)\right] u\,\mathrm{d}u \\ &\quad -\frac{x}{L}\int_x^L \left[\widetilde{\phi}(u)-h\psi(u)\right](L-u)\,\mathrm{d}u & (23)\end{aligned}$$

- **Review exercise 4 on p.97.**

We can express these succintly as:

$$\begin{aligned} M_1(x) &= M_{\widetilde{\phi}}(x) + hM_\psi(x) & (24)\\ M_2(x) &= M_{\widetilde{\phi}}(x) - hM_\psi(x) & (25)\end{aligned}$$

where $M_{\widetilde{\phi}}(x)$ and $M_\psi(x)$ are defined implicitly in eqs.(21) and (23).

¶ Let us consider a **special case:**

The nominal load increases towards the center of the beam:

$$\widetilde{\phi}(x) = \widetilde{\phi}\sin\frac{\pi x}{L} \quad (26)$$

where $\widetilde{\phi}$ is a known positive constant.

The uncertainty in the load increases towards the center of the beam:

$$\psi(x) = \sin\frac{\pi x}{L} \quad (27)$$

¶ Note that $\phi(x)$, $\widetilde{\phi}(x)$ and h all have the same units.

The functions in eqs.(24) and (25) become:

$$\begin{aligned} M_{\widetilde{\phi}}(x) &= -\frac{L^2\widetilde{\phi}}{\pi^2}\sin\frac{\pi x}{L} & (28)\\ M_\psi(x) &= \frac{M_{\widetilde{\phi}}(x)}{\widetilde{\phi}} & (29)\end{aligned}$$

¶ The least and greatest bending moments at point x are:

$$\begin{aligned} M_1(x) &= -(\widetilde{\phi}+h)\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} & (30)\\ M_2(x) &= -(\widetilde{\phi}-h)\frac{L^2}{\pi^2}\sin\frac{\pi x}{L} & (31)\end{aligned}$$

¶ From this we find that the greatest absolute bending moment occurs at the midpoint of the beam:

$$\max_{0\le x\le L}\ \max_{\phi\in\mathcal{U}(h,\widetilde{\phi})} |M(x)| = \frac{(\widetilde{\phi}+h)L^2}{\pi^2} \tag{32}$$

¶ To find the robustness, we equate the maximum bending moment to the critical moment and solve for h:

$$\frac{(\widetilde{\phi}+h)L^2}{\pi^2} = M_{\rm c} \quad\Longrightarrow\quad \widehat{h} = \frac{\pi^2 M_{\rm c}}{L^2} - \widetilde{\phi} \tag{33}$$

This is quite similar to the uniform-bound case, eq.(15) on p.5.

- **Review exercise 5 on p.97.**

¶ **The two info-gap models we have studied are:**

$$\mathcal{U}(h,\widetilde{\phi}) = \left\{\phi(x):\ \left|\phi(x)-\widetilde{\phi}(x)\right| \le h\right\},\quad h\ge 0 \tag{34}$$

(Eq.(1) on p. 4.) with robustness (eq.15), p.5:

$$\widehat{h} = \frac{8M_{\rm c}}{L^2} - \widetilde{\phi} \tag{35}$$

$$\mathcal{U}(h,\widetilde{\phi}) = \left\{\phi(x):\ \left|\phi(x)-\widetilde{\phi}(x)\right| \le h\psi(x)\right\},\quad h\ge 0 \tag{36}$$

(Eq.(17) on p. 7) with robustness in eq.(33):

$$\widehat{h} = \frac{\pi^2 M_{\rm c}}{L^2} - \widetilde{\phi} \tag{37}$$

• Both of these uncertainty models entail **unbounded rate of variation.**

• We sometimes have information which constrains the rate of variation of the uncertain function. We will now develop the tools needed to exploit this information.

2.2 Fourier Representation of a Function

¶ We interrupt our study of this example to briefly introduce the Fourier representation of a function. We will use Fourier representations in a new type of info-gap model.

¶ **Motivation:**

- The info-gap models of eqs.(1), p.4, and (17), p.7, allow unbounded rate of variation.
- We now have new information that constrains the rate of variation.

¶ Let $\phi(x)$ be an arbitrary but piece-wise continuous function defined on the interval $-L \le x \le L$. Then $\phi(x)$ can be represented as:

$$\phi(x) = \sum_{n=0}^{\infty} \left[b_n \sin \frac{n\pi x}{L} + c_n \cos \frac{n\pi x}{L} \right] \tag{38}$$

¶ Let $\phi(x)$ be an arbitrary but piece-wise continuous function defined on the interval $0 \le x \le L$. Then $\phi(x)$ can be represented as:

$$\phi(x) = \sum_{n=0}^{\infty} c_n \cos \frac{n\pi x}{L} \tag{39}$$

¶ How to choose the Fourier coefficients c_0, c_1, ... in eq.(39)?
Exploit orthogonality:

$$\int_0^{\pi} \cos mx \cos nx \, \mathrm{d}x = \begin{cases} \frac{\pi}{2} & m = n \\ 0 & m \ne n \end{cases} \tag{40}$$

To do this, multiply both sides of eq.(39) by $\cos \frac{k\pi x}{L}$ and integrate from 0 to L:

$$\int_0^L \phi(x) \cos \frac{k\pi x}{L} \, \mathrm{d}x = \sum_{n=0}^{\infty} c_n \int_0^L \cos \frac{k\pi x}{L} \cos \frac{n\pi x}{L} \, \mathrm{d}x \tag{41}$$

$$= \frac{c_k L}{2} \tag{42}$$

So, if we know the function $\phi(x)$ we can calculate the Fourier coefficients of its expansion:

$$c_k = \frac{2}{L} \int_0^L \phi(x) \cos \frac{k\pi x}{L} \, \mathrm{d}x \tag{43}$$

¶ **Review exercise 6 on p.97.**

¶ These Fourier coefficients have many interesting and important properties. First of all, they minimize the mean squared error between $\phi(x)$ and its expansion. That is, the c_n minimize:

$$S^2 = \int_0^L \left(\phi(x) - \sum_{n=0}^{\infty} c_n \cos \frac{n\pi x}{L} \right)^2 \mathrm{d}x \tag{44}$$

In fact,

$$\lim_{N \to \infty} S^2 = 0 \tag{45}$$

Another important property relates to truncated expansions:

$$\phi(x) = \sum_{n=0}^{N} c_n \cos \frac{n\pi x}{L} \, \mathrm{d}x \tag{46}$$

Regardless of the order of the expansion, N:
• Orthogonality yields the same Fourier coefficients, c_k.
• These coefficients minimize the mean squared error of the truncated expansion.

¶ Band-limited function:

$$\begin{aligned} \phi(x) &= \sum_{n=n_1}^{n_2} c_n \cos \frac{n\pi x}{L} && (47) \\ &= c^T \gamma(x) && (48) \end{aligned}$$

¶ Uncertainty in $\phi(x)$ is represented as uncertainty in Fourier coefficients c.
• For instance: c in ellipsoid of known shape and unknown size:

$$\mathcal{U}(h, \widetilde{c}) = \left\{ \phi(x) = c^T \gamma(x) : \ (c - \widetilde{c})^T W (c - \widetilde{c}) \leq h^2 \right\}, \quad h \geq 0 \tag{49}$$

¶ **Example:** ps1 #4. Discuss this problem in class.

2.3 Geometry of Ellipsoids

¶ **Motivation:**

- Suppose we have limited 2-dimensional data about an uncertain phenomenon:

$$(c_1, c_2)_i,\ i = 1, \ldots, n \tag{50}$$

- These data, when plotted, spread over an ellipse-like cluster around (0,0).
- Future data might extend beyond this cluster.
- How to represent our uncertainty?

¶ **Preliminary question:**

- Consider the $c_1 \times c_2$ plane.
- What shape is described by: $c_1^2 + c_2^2 = h^2$? Circle.
- What shape is described by: $ac_1^2 + bc_2^2 = h^2$, where $a, b > 0$? Ellipse.
- What shape is described by: $ac_1^2 + gc_1c_2 + bc_2^2 = h^2$, where $a, b > 0$?
  Ellipse if the coefficients define a positive definite matrix.

¶ We need one more digression before we proceed with our example: Geometry of ellipsoids.
The question we study in this subsection is:
What are the **directions and lengths** of the principal axes of an ellipsoid?

¶ If: c is an N-vector and W is a real, symmetric, positive definite matrix,
then an ellipsoid of c-vectors of dimension N is defined by:

$$c^T W c = h^2 \tag{51}$$

where h is any positive real number.

¶ Simple examples:

$$h^2 = c_1^2 w_1 + c_2^2 w_2, \quad W = \begin{pmatrix} w_1 & 0 \\ 0 & w_2 \end{pmatrix}, \quad w_i > 0 \tag{52}$$

$$h^2 = 2c_1^2 + c_1 c_2 + 2c_2^2, \quad W = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \tag{53}$$

¶ **Review exercise 7 on p.97.**

¶ To answer our question, we must solve an optimization problem.
We must find vectors c which have two properties:

- Length is extremal.
- Lie on the boundary of the ellipsoid.

¶ To optimize the length of c, it is sufficient to optimize the square of the length of c.
So we must optimize:

$$c^T c \tag{54}$$

Let's try differential calculus:

$$0 = \frac{\mathrm{d}c^T c}{\mathrm{d}c} = 2c \quad \Longrightarrow \quad c = 0 \tag{55}$$

That's the minimum. What's the maximum? $c^T c$ is unbounded. We need the constraint.

¶ To solve this problem we will use the method of **Lagrange multipliers.**

¶ A c-vector lies on the ellipsoid if eq.(51) is satisfied.
Expressing this slightly differently, the constraint on c is:

$$h^2 - c^T W c = 0 \tag{56}$$

¶ Define the objective function:

$$H = c^T c - \lambda(h^2 - c^T W c) \tag{57}$$

If we find all c-vectors which
optimize H subject to the constraint,
we will have solved the problem.

¶ Condition for extremum of H:

$$0 = \frac{\partial H}{\partial c} \quad = \quad 2c - 2\lambda W c \tag{58}$$

$$\Longrightarrow \quad (I - \lambda W)c = 0 \tag{59}$$

which means that:
$c =$ is an eigenvector of W.
$\frac{1}{\lambda} =$ the corresponding eigenvalue.

¶ Define the eigenvalues and orthonormal eigenvectors of W:

$$W v_i = \mu_i v_i, \quad i = 1, \ldots, N \tag{60}$$

where:

$$0 < \mu_1 \le \cdots \le \mu_N \quad \text{and} \quad v_m^T v_n = \delta_{mn} \tag{61}$$

where δ_{mn} is the Kronecker delta function:

$$\delta_{mn} = \begin{cases} 1 & m = n \\ 0 & m \ne n \end{cases} \tag{62}$$

¶ **Review exercise 8 on p.97.**
¶ Now, since c must be an eigenvector of W, we know that:

$$c = r v_i \tag{63}$$

for some non-zero r and for any $i = 1, \ldots, N$.

Hence the constraint on c is:

$$h^2 = c^T W c = r^2 v_i^T W v_i = r^2 \mu_i \quad \Longrightarrow \quad r = \pm \frac{h}{\sqrt{\mu_i}} \tag{64}$$

¶ Thus the optimizing c-vectors are:

$$c = \pm \frac{h}{\sqrt{\mu_i}} v_i, \quad i = 1, \ldots, N \tag{65}$$

From this we see that:
The **directions** of the principal semi-axes are:

$$\pm v_1, \ldots, \pm v_N \tag{66}$$

The **lengths** of the principal semi-axes are:

$$\frac{h}{\sqrt{\mu_1}}, \ldots, \frac{h}{\sqrt{\mu_N}} \tag{67}$$

¶ **Example:** ps1 #5 (a), (b) for review.

2.4 Fourier Ellipsoid Bounded Uncertain Load

Based on *Robust Reliability in the Mechanical Sciences,* section 3.2.4.

¶ We now consider a different type of prior information about the uncertain load profile $\phi(x)$.

¶ About $\phi(x)$ we **know:**
- Load vanishes at ends: $\phi(0) = \phi(L) = 0$.
- $\phi(x)$ is constrained to specific known spatial frequencies.
- The amplitudes of these frequencies are bounded by an ellipsoid of known shape.

¶ About $\phi(x)$ we **do not know:**
- The precise amplitudes of the Fourier coefficients.
- The size of the ellipsoid.

¶ In other words, a load profile is represented by:

$$\phi(x) = \sum_{n=n_1}^{n_2} c_n \sin\frac{n\pi x}{L} \tag{68}$$

$$= c^T\sigma(x) \tag{69}$$

where:
$c =$ vector of unknown Fourier coefficients.
$\sigma(x) =$ vector of known corresponding sine functions.

¶ The uncertainty in $\phi(x)$ is represented by the following Fourier ellipsoid bound info-gap model:

$$\mathcal{U}(h,0) = \left\{\phi(x) = c^T\sigma :\ c^TWc \le h^2\right\}, \quad h \ge 0 \tag{70}$$

where W is a known, real, symmetric, positive definite matrix.[1]

¶ The system model is obtained by combining eq.(4) on p.4 for the bending moment with eq.(69):

$$M(x) = c^T \underbrace{\left[-\frac{L-x}{L}\int_0^x u\sigma(u)\,\mathrm{d}u - \frac{x}{L}\int_x^L (L-u)\sigma(u)\,\mathrm{d}u\right]}_{\zeta(x)} \tag{71}$$

$$= c^T\zeta(x) \tag{72}$$

¶ As before, failure occurs if the bending moment exceeds a critical value, as expressed in eq.(5) on p.4.

¶ In order to find the robustness, eq.(9), p.4, we must solve the following optimization:

$$\max M(x) \quad \text{for} \quad c^TWc \le h^2 \tag{73}$$

[1] For an example of a Fourier ellipsoid model see: Yakov Ben-Haim and Isaac Elishakoff, Non-Probabilistic models of uncertainty in the non-linear buckling of shells with general imperfections: Theoretical estimates of the knockdown factor. *A.S.M.E. Journal of Applied Mechanics,* Vol. 56, pp 403–410, 1989.

which is equivalent to:

$$\max c^T \zeta \quad \text{for} \quad c^T W c \le h^2 \tag{74}$$

To do this we employ the Cauchy inequality:

$$\left(x^T y\right)^2 \le \left(x^T x\right)\left(y^T y\right) \tag{75}$$

with equality iff:

$$x \propto y \tag{76}$$

Let us write:

$$c^T \zeta = \left(W^{1/2} c\right)^T \left(W^{-1/2} \zeta\right) \tag{77}$$

Applying Cauchy's inequality to the expression on the right:

$$\left(c^T \zeta\right)^2 \le \left[\left(W^{1/2} c\right)^T \left(W^{1/2} c\right)\right]\left[\left(W^{-1/2} \zeta\right)^T \left(W^{-1/2} \zeta\right)\right] \tag{78}$$

$$= \underbrace{\left[c^T W c\right]}_{\le h^2} \left[\zeta^T W^{-1} \zeta\right] \tag{79}$$

From this we conclude that:

$$\max_{c \in \mathcal{U}(h,0)} M(x) = h\sqrt{\zeta(x)^T W^{-1} \zeta(x)} \tag{80}$$

¶ We can now express the robustness as the greatest value of the uncertainty parameter h at which the bending moment does not exceed the critical value. We find:

$$\widehat{h} = \frac{M_{\rm c}}{\max_{0 \le x \le L} \sqrt{\zeta(x)^T W^{-1} \zeta(x)}} \tag{81}$$

¶ **Review exercise 9 on p.98.**

¶ Let us consider a **special case:**
W is the identity matrix, so the uncertainty ellipsoid is a sphere.

¶ Now $\zeta^T W \zeta$ becomes:

$$\zeta^T(x)\zeta(x) = \frac{L^4}{\pi^4} \sum_{n=n_1}^{n_2} \frac{1}{n^4} \sin^2 \frac{n\pi x}{L} \tag{82}$$

The terms in this sum decrease rapidly with n.
Hence the maximum is dominated by the first term:

$$\max_{0 \le x \le L} \sqrt{\zeta(x)^T \zeta(x)} \approx \max_{0 \le x \le L} \sqrt{\frac{L^4}{\pi^4} \frac{1}{n_1^4} \sin^2 \frac{n_1 \pi x}{L}} \tag{83}$$

$$= \frac{L^2}{n_1^2 \pi^2} \tag{84}$$

From eq.(81) we find the robustness to be:

$$\widehat{h} \approx \frac{n_1^2 \pi^2 M_{\rm c}}{L^2} \tag{85}$$

¶ Comparing this with the robustness for the uniform-bound info-gap model, with $\widetilde{\phi} = 0$, eq.(15) on p.5:

$$\widehat{h} = \frac{8 M_{\rm c}}{L^2} \tag{86}$$

we see that the reliability is substantially enhanced
by constraining the spatial modes of the load function.

¶ **Review exercise 10 on p.98.**

3 Two Faces of Uncertainty

¶ Uncertainty has two faces:
- Pernicious: threatening failure, entailing risk.
- Propitious: promising windfall, sweeping reward.

¶ In making decisions we wish to:
- protect against pernicious uncertainty,

and
- facilitate propitious uncertainty.

¶ In evaluating decisions under uncertainty we wish to assess:
- risks

and
- opportunities.

¶ This we do with 2 immunity functions *(funkziot amidut)*:
- Robustness function *(funkziat hasinut)*:
  immunity against failure.
- Opportuneness function *(funkziat hizdamnut)*:
  immunity against windfall.

4 Robustness and Opportuneness: A First Look

(Y. Ben-Haim, *Info-Gap Decision Theory*, section 3.1.1)

¶ Recall that an info-gap model is a **family:**

$$\mathcal{U}(h, \widetilde{u}), \quad h \geq 0 \tag{87}$$

of **nested sets:**

$$h < h' \quad \Longrightarrow \quad \mathcal{U}(h, \widetilde{u}) \subset \mathcal{U}(h', \widetilde{u}) \tag{88}$$

Thus info-gap uncertainty increases with
increasing h.
So: h is called the **uncertainty parameter.**

¶ The **robustness function** is the
greatest level of info-gap uncertainty
at which
failure cannot occur.

The **opportuneness function** is the
least level of info-gap uncertainty
at which
sweeping success can (but does not have to) occur.

The **robustness** function addresses **pernicious** uncertainty.
The **opportuneness** function addresses **propitious** uncertainty.

¶ We can begin to quantify these
immunity functions
as follows.

¶ Let $q =$ **decision vector,** containing:
— design parameters.
— operational options.
— time of initiation.
— etc.

¶ Let u be an uncertain vector belonging to an info-gap model:

$$\mathcal{U}(h, \widetilde{u}), \quad h \geq 0 \tag{89}$$

The **robustness function** is:

$$\widehat{h}(q) = \max\{h: \text{ minimal requirements are satisfied } \textbf{for all } u \in \mathcal{U}(h, \widetilde{u})\} \tag{90}$$

The **opportuneness function** is:

$$\widehat{\beta}(q) = \min\{h: \text{ sweeping success is enabled } \textbf{for some } u \in \mathcal{U}(h, \widetilde{u})\} \tag{91}$$

¶ $\widehat{h}(q)$ and $\widehat{\beta}(q)$ are
dual functions
or
complementary functions.

For $\widehat{h}(q)$: **bigger is better.**
For $\widehat{\beta}(q)$: **big is bad.**

¶ $\widehat{h}(q)$ entails a **maximization:**
Not of performance or outcome of decision.
Rather: ○ Immunity to uncertainty is maximized.
○ Performanced is **satisficed.**

¶ To **satisfice** (OED):
"To decide on and pursue a course of action
that will satisfy the minimal requirements
needed to achieve a particular goal."
(Herb Simon, psychologist and economist.)

¶ $\widehat{\beta}(q)$ entails a **minimization:**
Not of damage resulting from unknown events.
Rather: minimize level of uncertainty needed to
enable **windfall.**

¶ We can define **windfalling** as:
To decide on and pursue a course of action that will
minimize the immunity to propitious uncertainty
in an attempt to enable highly ambitious goals.

5 Immunity Functions

(Y. Ben-Haim, *Info-Gap Decision Theory*, Section 3.1.2)

¶ Often the success of a decision is expressed by a
scalar **reward function** *(funkziat toelet)*: $R(q, u)$
which depends on:
$q =$ decision vector.
$u =$ uncertain vector in an info-gap model.
E.g. $R(q, u) =$
- Degree of stability.
- Rate of mixing.
- Duration of life.
- Profit.

For all these entities a **large value** if $R(q, u)$ is desirable.

¶ Given a reward function, $R(q, u)$, the
minimal requirement in eq.(90) on p.20 is:
$R(q, u) \geq r_{\rm c}$
where $r_{\rm c}$ = critical, survival level of reward.

Likewise, the **sweeping success** in eq.(91) on p.20 is:
$R(q, u) \geq r_{\rm w}$
where $r_{\rm w}$ = windfall reward.
and
$r_{\rm w} \gg r_{\rm c}$.

¶ We can now define $\widehat{h}$ and $\widehat{\beta}$ more precisely.

¶ The **robustness function** is:

$$\widehat{h}(q, r_{\rm c}) = \max \left\{ h : \min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \geq r_{\rm c} \right\} \tag{92}$$

We can analyze this as follows:

$$\widehat{h}(q, r_{\rm c}) = \underbrace{\max \left\{ h : \right.}_{\substack{\text{max} \\ \text{uncertainty} \\ h \text{ so that}}} \quad \underbrace{\min_{u \in \mathcal{U}(h, \widetilde{u})}}_{\substack{\text{all } u \text{ up to} \\ \text{uncertainty} \\ h \text{ guarantee}}} \quad \underbrace{R(q, u) \geq r_{\rm c}}_{\substack{\text{minimal} \\ \text{requirement} \\ \text{for} \\ \text{survival}}} \left. \right\}$$

$\widehat{h}(q, r_{\rm c})$ is the	maximum tolerable h
so that	all u up to uncertainty h
satisfy the	minimal requirement for survival.

¶ The **Opportuneness function** is:

$$\widehat{\beta}(q, r_{\rm w}) = \min \left\{ h : \max_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \geq r_{\rm w} \right\} \tag{93}$$

We can analyze this as follows:

$$\widehat{\beta}(q, r_{\rm w}) = \underbrace{\min \left\{ h : \right.}_{\substack{\text{least} \\ \text{uncertainty} \\ h \text{ so that}}} \quad \underbrace{\max_{u \in \mathcal{U}(h, \widetilde{u})}}_{\substack{\text{some } u \\ \text{up to} \\ \text{uncertainty} \\ h \text{ enables}}} \quad \underbrace{R(q, u) \geq r_{\rm w}}_{\substack{\text{sweeping} \\ \text{success} \\ \text{or windfall}}} \quad \left. \right\}$$

$\widehat{\beta}(q, r_{\rm w})$ is the	least h
so that	some u up to uncertainty h
enables the	possibility of windfall success.

¶ Note that $\widehat{h}$ and $\widehat{\beta}$ are
extrema of sets of h-values.

Define the sets:

$$\mathcal{A}(q, r_{\rm c}) = \left\{ h : \min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \geq r_{\rm c} \right\} \tag{94}$$

$$\mathcal{B}(q, r_{\rm w}) = \left\{ h : \max_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \geq r_{\rm w} \right\} \tag{95}$$

Thus:

$$\widehat{h}(q, r_{\rm c}) = \text{LUB } \mathcal{A}(q, r_{\rm c}) \tag{96}$$

$$\widehat{\beta}(q, r_{\rm w}) = \text{GLB } \mathcal{B}(q, r_{\rm w}) \tag{97}$$

Also, if:

$$\mathcal{A}(q, r_{\rm c}) = \emptyset \tag{98}$$

then define:

$$\widehat{h}(q, r_{\rm c}) = 0 \tag{99}$$

because eq.(98) implies:

- No immunity to failure.
- Infinitestimal variation entails possibility of failure.

Likewise, if:

$$\mathcal{B}(q, r_{\mathrm{w}}) = \emptyset \tag{100}$$

then define:

$$\widehat{\beta}(q, r_{\mathrm{w}}) = \infty \tag{101}$$

because eq.(100) implies:

- No value of h is large enough
  to enable windfall r_{w}.
- The immunity to windfall is unbounded.

¶ Up to now we have considered
reward functions $R(q,u)$ for which
large reward is desirable.

¶ In some situations, **small** $R(q,u)$ is preferred over
large $R(q,u)$.
E.g. $R(q,u)$ is measure of
- **in**stability of the system.
- Financial **loss.**
- **Delay** in implementation.

¶ If small $R(q,u)$ is preferred over large $R(q,u)$
then we define the immunity functions:

$$\widehat{h}(q,r_{\rm c}) = \max\left\{h : \max_{u\in\mathcal{U}(h,\widetilde{u})} R(q,u) \le r_{\rm c}\right\} \tag{102}$$

$$\widehat{\beta}(q,r_{\rm w}) = \min\left\{h : \min_{u\in\mathcal{U}(h,\widetilde{u})} R(q,u) \le r_{\rm w}\right\} \tag{103}$$

where:

$$r_{\rm w} \ll r_{\rm c} \tag{104}$$

¶ Note that in both formulations,
- eqs.(92) and (93), (pp.23, 24)
- eqs.(102) and (103), (p.27)

"Bigger is better" for $\widehat{h}(q,r_{\rm c})$
"Big is bad" for $\widehat{\beta}(q,r_{\rm w})$

6 Design of a Vibrating Cantilever

(Y. Ben-Haim, *Info-Gap Decision Theory*, sec. 3.3.1)

6.1 Design Problem

¶ We now consider an example:
Vibration control in a cantilever subject to uncertain dynamic excitation.

¶ The cantilever: rigid beam which is clamped at one end.
See transparency of: • Galileo's cantilever.
• Atomic force microscope.

¶ The cantilever is the paradigm for:
- Tall building.
- Radio tower.
- Crane (agoran).
- Airplane wing.
- Turbine blade.
- Diving board.
- Canon barrel.
- Atomic force microscope.
- etc.

¶ Central goal in design of the cantilever:
Control of vibration resulting from external loads.

¶ Two basic approaches:
1. Prevent vibration by stiffening the beam.
2. Absorb vibration by dissipating energy.

¶ These design concepts are **not** mutually exclusive.
They can be implemented together.
¶ These design concepts are relevant in different circumstances as we will see.

6.2 Robustness Function

¶ We will use the **robustness function** to evaluate the design options.

¶ Later we will consider the **opportuneness function.**

¶ As usual, the three components of the analysis are:
1. System model.
2. Failure (or performance) criterion.
3. Uncertainty model.

¶ We use a simple **system model:**
Vibration of a rigid beam around the spring-clamped base.
$\theta(t) =$ angle of deflection of beam [radian].
$u(t) =$ moment of force at base, [Nm].
Equation of motion:

$$J\frac{\mathrm{d}^2\theta(t)}{\mathrm{d}t^2} + c\frac{\mathrm{d}\theta(t)}{\mathrm{d}t} + k\theta = u(t) \tag{105}$$

$J =$ moment of inertia of beam wrt rotation at base, $\int_0^L m(x)x^2\,\mathrm{d}x$.
$c =$ damping coefficient.
$k =$ rotational stiffness coefficient, [Nm/radian].

¶ Solution of eq. of motion, for:
- Zero initial conditions, $\theta(0) = \dot{\theta}(0) = 0$
- Subcritical damping, $\zeta^2 < 1$:

$$\theta_u(t) = \int_0^t u(\tau)f(t-\tau)\,\mathrm{d}\tau \tag{106}$$

$f(t) =$ impulse response function:

$$f(t) = \frac{1}{J\omega_{\mathrm{d}}}\mathrm{e}^{-\zeta\omega t}\sin\omega_{\mathrm{d}}t \tag{107}$$

$\omega^2 = k/J =$ squared natural frequency.
$\zeta = \frac{c}{2J\omega} =$ dimensionless damping coefficient.
$\omega_{\mathrm{d}} = \omega\sqrt{1-\zeta^2} =$ damped natural frequency.

¶ We now consider the **uncertainty model.**
What we **know** about the load is:

- The nominal load, $\widetilde{u}(t)$.
- The actual loads are transient:
  - ○ May vary rapidly,
  - ○ May attain large deviations from the nominal load.
  - ○ No sustained deviation from the nominal load

We will model load uncertainty with the **cumulative energy bound** info-gap model:

$$\mathcal{U}(h,\widetilde{u}) = \left\{ u(t) : \int_0^\infty [u(t) - \widetilde{u}(t)]^2 \, \mathrm{d}t \le h^2 \right\}, \quad h \ge 0 \tag{108}$$

¶ **Review exercise 11, p.98.**

¶ The **performance criterion:** Deflection must not exceed critical value:

$$|\theta(t)| \le \theta_{\mathrm{c}} \tag{109}$$

In terms of reward functions, define:

$$R(q,u) = |\theta(t)| \tag{110}$$

$u =$ uncertain load.
$q =$ design concept, as expressed in damping c and stiffness k.

¶ The robustness function can be defined as in eq.(102) on p.27:

$$\widehat{h}(q,\theta_{\mathrm{c}}) = \max \left\{ h : \left(\max_{u \in \mathcal{U}(h,\widetilde{u})} |\theta_u(t)| \right) \le \theta_{\mathrm{c}} \right\} \tag{111}$$

$\widehat{h}(q,\theta_{\mathrm{c}})$ is the maximum tolerable info-gap.

¶ We now evaluate:

$$\max_{u \in \mathcal{U}(h,\widetilde{u})} |\theta_u(t)| \tag{112}$$

¶ Note that $\theta_u(t)$ in eq.(106) on p.29 can be re-written:

$$\theta_u(t) = \int_0^t u(\tau) f(t-\tau) \, \mathrm{d}\tau \tag{113}$$

$$= \int_0^t [u(\tau) - \widetilde{u}(\tau)] \, f(t-\tau) \, \mathrm{d}\tau + \underbrace{\int_0^t \widetilde{u}(\tau) f(t-\tau) \, \mathrm{d}\tau}_{\widetilde{\theta}(t)} \tag{114}$$

where $\widetilde{\theta}(t) =$ nominal deflection.

We need the Schwarz inequality:

$$\left(\int_a^b f(t) g(t) \, \mathrm{d}t \right)^2 \le \int_a^b f(t)^2 \, \mathrm{d}t \int_a^b g(t)^2 \, \mathrm{d}t \tag{115}$$

with equality iff:

$$f(t) = cg(t) \tag{116}$$

for any non-zero constant c.

Now notice that the first integral in eq.(114) on p.30 is bounded:

$$\left(\int_0^t [u(\tau) - \widetilde{u}(\tau)] \, f(t-\tau) \, \mathrm{d}\tau \right)^2 \leq \underbrace{\left(\int_0^t [u(\tau) - \widetilde{u}(\tau)]^2 \, \mathrm{d}\tau \right)}_{\mathrm{I}} \underbrace{\left(\int_0^t f^2(t-\tau) \, \mathrm{d}\tau \right)}_{\mathrm{II}} \tag{117}$$

¶ **Review exercise 12, p.98.**

¶ Note:

- From the info-gap model we know that: Integral I $\leq h^2$.
- Integral II is known.
- The info-gap model allows us to choose $u(\tau)$ such that:

$$u(\tau) - \widetilde{u}(\tau) \propto f(t-\tau) \tag{118}$$

Thus the Schwarz inequality implies that the righthand side of eq.(117) is a least upper bound.

- Thus, from eqs.(114) and (117):

$$\max_{u \in \mathcal{U}(h, \widetilde{u})} |\theta_u(t)| = h \sqrt{\int_0^t f^2(\tau) \, \mathrm{d}\tau} + \left| \widetilde{\theta}(t) \right| \tag{119}$$

¶ **Review exercise 13, p.98.**

¶ **Review exercise 14, p.98.**

¶ We can now express the robustness function:

- Equate $\max |\theta_u(t)|$ to θ_c.
- Solve for h, yielding $\widehat{h}$:

$$h \sqrt{\int_0^t f^2(\tau) \, \mathrm{d}\tau} + \left| \widetilde{\theta}(t) \right| = \theta_\mathrm{c} \quad \Longrightarrow \quad \boxed{\widehat{h}(q, \theta_\mathrm{c}) = \frac{\theta_\mathrm{c} - \left| \widetilde{\theta}(t) \right|}{\sqrt{\int_0^t f^2(\tau) \, \mathrm{d}\tau}}} \tag{120}$$

unless this is negative, in which case $\widehat{h} = 0$.

¶ **Review exercise 15, p.98.**

6.3 Numerical Example

¶ We will consider a specific example. Nominal input $\widetilde{u}(t)$ is square:

$$\widetilde{u}(t) = \begin{cases} \widetilde{u}_o, & 0 \leq t \leq T \\ 0, & t > T \end{cases} \tag{121}$$

The nominal response can be calculated:

$$\widetilde{\theta}(t) = \theta_{\widetilde{u}}(t) = \frac{(1-\zeta^2)\widetilde{u}_o}{J\omega_{\mathrm{d}}}\gamma(t) \tag{122}$$

where $\gamma(t)$ is a known function.
For notational convenience we represent integral II in eq.(117) on p.31 as:

$$\sqrt{\int_0^t f^2(t-\tau)\,\mathrm{d}\tau} = \frac{1-\zeta^2}{2J\omega_{\mathrm{d}}^{3/2}}\phi(t) \tag{123}$$

where $\phi(t)$ is a known function.

Now the robustness function can be expressed:

$$\widehat{h}(q, \theta_{\mathrm{c}}) = \frac{2J\theta_{\mathrm{c}}\omega^2\sqrt{\omega_{\mathrm{d}}} - 2\sqrt{\omega_{\mathrm{d}}}|\widetilde{u}_o\gamma(t)|}{\omega\phi(t)} \tag{124}$$

Recall: $q =$ decision vector = $(c,\, k)$, which is embedded in ω and ω_{d}.

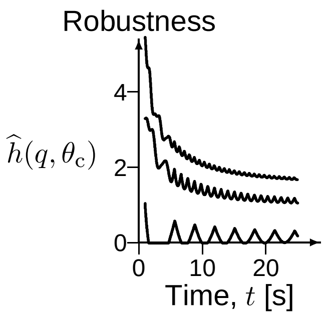

Figure 3: Robustness versus time for three values of the natural frequency $\omega =$ 1, 3 and 4 (bottom to top). Negligible damping: $\zeta = 0.01$. $1 = J\theta_{\mathrm{c}} = \widetilde{u}_0$. $T = 5$.

¶ $\widehat{h}(q, \theta_{\mathrm{c}})$ vs. t is plotted in fig. 3
For various natural frequencies: $\omega =$ 1, 3 and 4 (bottom to top).
With negligible damping: $\zeta = 0.01$.

- $\widehat{h}$ oscillates but tends to decrease over time.
- At low stiffness ($\omega = 1$) the robustness periodically vanishes.
- At moderate and high stiffness ($\omega = 3$, 4)
  $\widehat{h}$ oscillates but does not reach zero for the duration shown.
- The transition from rapid to slow decrease in $\widehat{h}$
  occurs about at $t = T$ (end of nominal input).

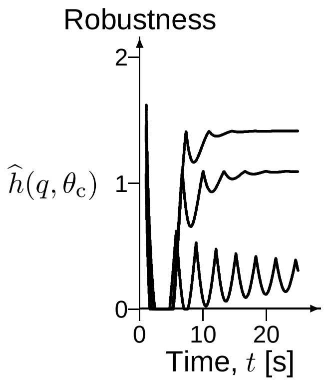

Figure 4: Robustness versus time for three values of the damping ratio $\zeta =$ 0.03, 0.3, 0.5 (bottom to top). Fixed natural frequency $\omega =$ 1. $1 = J\theta_{\rm c} = \widetilde{u}_0$. $T = 5$.

¶ Now consider fig. 4, which shows
$\widehat{h}(q,\theta_{\rm c})$ vs. t for various damping ratios:
$\zeta =$ 0.03, 0.3 and 0.5
at low stiffness: $\omega = 1$.

• Lowest curve is quite similar to lowest curve in fig. 3.
• With large damping ($\zeta =$ 0.3 or 0.5):
$\widehat{h}$ is small for $t \leq T$
$\widehat{h}$ is large and nearly constant thereafter.

¶ Comparing figs. 3 and 4:
- Fig. 3 is based on the "stiffness" design concept, with negligible damping.
- Fig. 4 is based on the "dissipation" design concept, with negligible stiffness.
- The choice of a design concept depends on the time frame of interest:
  - $t < T$ calls for "stiffness" design.
  - $t > T$ calls for "dissipation" design.
  - $t > 0$ calls for combined "stiffness" and "dissipation" design.

6.4 Opportuneness Function

¶ We now consider the opportuneness function.

Windfall reward: angular deflection $\theta_{\rm w}$ **much less** (much better) than the survival requirement, $\theta_{\rm c}$:

$$\theta_{\rm w} < \widetilde{\theta} < \theta_{\rm c} \tag{125}$$

¶ Immunity to windfall, $\widehat{\beta}(q, \theta_{\rm w})$: the **least** info-gap at which windfall is **possible.**

¶ Analogous to eq.(111) on p. 30:

$$\widehat{\beta}(q, \theta_{\rm w}) = \min \left\{ h : \min_{u \in \mathcal{U}(h, \widetilde{u})} |\theta_u(t)| \le \theta_{\rm w} \right\} \tag{126}$$

¶ **Smaller is better** for $\widehat{\beta}$. Unlike $\widehat{h}$, for which **bigger is better.**

¶ **Review exercise 16, p.98.**

¶ Proceeding as in eq.(119) on p. 31 we find:

$$\min_{u \in \mathcal{U}(h, \widetilde{u})} |\theta_u(t)| = -h \sqrt{\int_0^t f^2(\tau)\, {\rm d}\tau} + \left| \widetilde{\theta}(t) \right| \tag{127}$$

Equating this to $\theta_{\rm w}$ and solving for h yields the opportuneness function, as in eq.(120) on p. 31:

$$-h \sqrt{\int_0^t f^2(\tau)\, {\rm d}\tau} + \left| \widetilde{\theta}(t) \right| = \theta_{\rm w} \quad \Longrightarrow \quad \boxed{\widehat{\beta}(q, \theta_{\rm w}) = \frac{\left| \widetilde{\theta}(t) \right| - \theta_{\rm w}}{\sqrt{\int_0^t f^2(\tau)\, {\rm d}\tau}}} \tag{128}$$

unless this is negative, in which case $\widehat{\beta} = 0$.

Why does $\widehat{\beta} = 0$ in this case?

$\widehat{\beta} < 0$ only if $\left| \widetilde{\theta}(t) \right| < \theta_{\rm w}$.
This means that the **nominal response** $|\widetilde{\theta}(t)|$
is less than the **windfall response** $\theta_{\rm w}$.
Hence windfall is possible even without uncertainty: The immunity to windfall is zero.

¶ **Review exercise 17, p.98.**

¶ Compare $\widehat{\beta}(q, \theta_{\rm w})$ to the robustness in eq.(120) on p. 31:

$$\widehat{h}(q, \theta_{\rm c}) = \frac{\theta_{\rm c} - \left|\widetilde{\theta}(t)\right|}{\sqrt{\int_0^t f^2(\tau)\,{\rm d}\tau}} \tag{129}$$

We see that the immunity functions are related as:

$$\widehat{\beta}(q, \theta_{\rm w}) = -\widehat{h}(q, \theta_{\rm c}) + \frac{\theta_{\rm c} - \theta_{\rm w}}{\sqrt{\int_0^t f^2(\tau)\,{\rm d}\tau}} \tag{130}$$

¶ **Review exercise 18, p.98.**

¶ We now consider **antagonism** and **sympathy** of the immunity functions.

¶ The immunity functions $\widehat{\beta}(q, \theta_{\rm w})$ and $\widehat{h}(q, \theta_{\rm c})$ are
sympathetic if they can be improved simultaneously.
They are **antagonistic** if either can be improved only at the expense of the other.

¶ **Review exercise 19, p.98.**

¶ For example, we can vary ω. The immunity functions are **antagonistic** if:

$$\underbrace{\frac{\partial \widehat{h}(q, \theta_{\rm c})}{\partial \omega} > 0}_{\text{improving with } \omega} \quad \text{and} \quad \underbrace{\frac{\partial \widehat{\beta}(q, \theta_{\rm w})}{\partial \omega} > 0}_{\text{degenerating with } \omega} \tag{131}$$

or if:

$$\underbrace{\frac{\partial \widehat{h}(q, \theta_{\rm c})}{\partial \omega} < 0}_{\text{degenerating with } \omega} \quad \text{and} \quad \underbrace{\frac{\partial \widehat{\beta}(q, \theta_{\rm w})}{\partial \omega} < 0}_{\text{improving with } \omega} \tag{132}$$

¶ On the other hand, the immunity functions are **sympathetic** if:

$$\underbrace{\frac{\partial \widehat{h}(q, \theta_{\rm c})}{\partial \omega} > 0}_{\text{improving with } \omega} \quad \text{and} \quad \underbrace{\frac{\partial \widehat{\beta}(q, \theta_{\rm w})}{\partial \omega} < 0}_{\text{improving with } \omega} \tag{133}$$

or if:

$$\underbrace{\frac{\partial \widehat{h}(q, \theta_{\rm c})}{\partial \omega} < 0}_{\text{degenerating with } \omega} \quad \text{and} \quad \underbrace{\frac{\partial \widehat{\beta}(q, \theta_{\rm w})}{\partial \omega} > 0}_{\text{degenerating with } \omega} \tag{134}$$

¶ In short, the immunity functions are **sympathetic** wrt ω if and only if:

$$\frac{\partial \widehat{h}(q, \theta_{\rm c})}{\partial \omega} \frac{\partial \widehat{\beta}(q, \theta_{\rm w})}{\partial \omega} < 0 \tag{135}$$

¶ Return to eq.(130) on p. 35.

- Question: Under what conditions will $\widehat{h}$ and $\widehat{\beta}$ always be sympathetic?
- Answer: If and only if their optima coincide. See fig. 5.

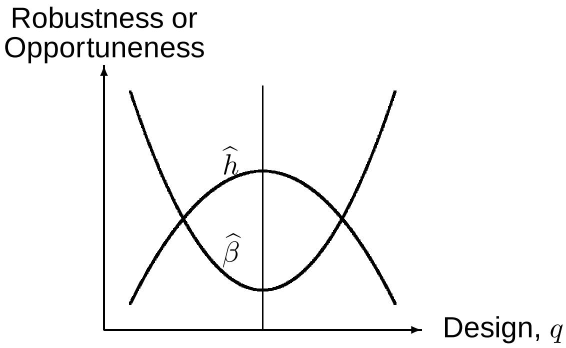

Figure 5: Sympathetic robustness and opportuneness curves.

¶ When will this occur? Iff

$$\frac{\partial\widehat{\beta}}{\partial q} = 0 = \frac{\partial\widehat{h}}{\partial q} \tag{136}$$

From eq.(130) we see that this will happen only if, at the same q, we also have:

$$\frac{\partial D}{\partial q} = 0 \tag{137}$$

where we define:

$$D = \frac{\theta_{\rm c} - \theta_{\rm w}}{\sqrt{\int_0^t f^2(\tau)\,{\rm d}\tau}} \tag{138}$$

"Usually" this will not happen, which means that, instead of fig. 5, we will have fig. 6.

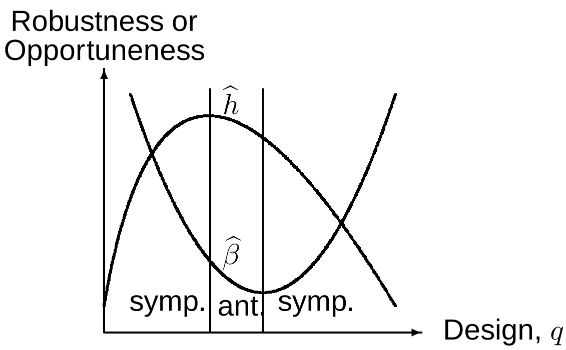

Figure 6: Robustness and opportuneness curves which are both sympathetic and antagonistic.

7 Generic Decision Algorithms

¶ We have defined the immunity functions:
$\widehat{h}(q, r_{\rm c})$ and $\widehat{\beta}(q, r_{\rm w})$
on the basis of:

- an info-gap model of uncertainty, $\mathcal{U}(h, \widetilde{u})$, $h \geq 0$.
- a scalar reward function, $R(q, u)$.

We will now show that $\widehat{h}(q, r_{\rm c})$ and $\widehat{\beta}(q, r_{\rm w})$
can be defined with a:
generic decision algorithm.

¶ $D(q, u) =$ generic decision algorithm
whose value is the “answer” or “response” to
the “input” $u \in \mathcal{U}(h, \widetilde{u})$ for some h.
$q =$ decision vector specifying the structure of D.

¶ Decisions may be an inference about a system, e.g.:

- Is it safe? Yes or no.
- Is the max response $\leq$ a critical value?

Or the decision algorithm may:

- Select one from among several hypotheses about the system or environment.
- Select one from among several design options.
- Select one from among several operational alternatives.

¶ The robustness of a decision algorithm can be formulated in several different ways.

¶ One possibility:

$$\begin{aligned}
\widehat{h}(q) &= \text{greatest info-gap uncertainty such that} \\
&\quad \text{the } \textbf{actual design} = \text{ the } \textbf{nominal design.} && (139) \\
&= \text{max info-gap at which } D(q,u) \text{ is stable.} && (140) \\
&= \max\left\{h:\ D(q,u) = D(q,\widetilde{u}) \quad \text{for all} \quad u \in \mathcal{U}(h,\widetilde{u})\right\} \\
&= \text{max info-gap at which} && (141) \\
&\quad \text{the } \textbf{best available decision } D(q,\widetilde{u}) \\
&\quad \text{is the same as} \\
&\quad \text{the } \textbf{most realistic decision } D(q,u). && (142)
\end{aligned}$$

¶ An immediate extension:

$$\begin{aligned}
\widehat{h}(q) &= \max\left\{h:\ \|D(q,u) - D(q,\widetilde{u})\| \le r_{\mathrm{c}}\ \forall\ u \in \mathcal{U}(h,\widetilde{u})\right\} && (143) \\
&= \text{max info-gap at which} \\
&\quad D(q,\widetilde{u}) \textbf{ errs no more} \text{ than } r_{\mathrm{c}}. && (144) \\
&= \text{max info-gap at which} \\
&\quad D(q,u) \textbf{ dithers no more} \text{ than } r_{\mathrm{c}}. && (145)
\end{aligned}$$

¶ Let us identify when **decision robustness** $\widehat{h}(q, r_{\mathrm{c}})$
is a relevant measure of **correctness** or **validity**
of the decision itself.
The discussion has 3 parts.

1. We assume that
$\mathcal{U}(h, \widetilde{u})$, $h \geq 0$, **accurately represents uncertain variation**
in the system or environment.
This means that $\mathcal{U}(h, \widetilde{u})$, $h \geq 0$ is rich enough to include,
at some h,
a realistic representation of the system or environment.

2. Hence, **large robustness** $\widehat{h}(q, r_{\mathrm{c}})$ means that
the **nominal decision** $D(q, \widetilde{u})$
is the same as the **true decision** $D(q, u)$
for a large range of real systems.

3. In summary:
if $\mathcal{U}(h, \widetilde{u})$ represents realistic variation
then large $\widehat{h}(q, r_{\mathrm{c}})$ warrants the decision $D(q, \widetilde{u})$.

¶ We can also define the opportuneness function as a generic decision:

$$\widehat{\beta}(q, r_{\mathrm{w}}) = \min \{h : \ \|D(q, u) - D(q, \widetilde{u})\| \leq r_{\mathrm{w}} \text{ for some } u \in \mathcal{U}(h, \widetilde{u})\} \tag{146}$$

This is the same as the $\widehat{\beta}$ defined earlier.

8 Multi-criterion Reward

¶ In some situations there may be:
multiple relevant reward criteria or functions:
$R_i(q,u), \quad i = 1,\, 2,\, \ldots$.
Each reward function may have its own
critical threshold $r_{\mathrm{c},i}, \quad i = 1,\, 2,\, \ldots$.
and
windfall threshold $r_{\mathrm{w},i}, \quad i = 1,\, 2,\, \ldots$.
Immunity functions can be defined for each criterion:
$\widehat{h}_i(q, r_{\mathrm{c},i}), \quad \widehat{\beta}_i(q, r_{\mathrm{w},i})$.

¶ There are various ways to **combine** the immunity functions.
One combination of robustness functions is to define:

$$\widehat{h}_i(q, r_{\mathrm{c}}) = \text{overall robustness.} \quad r_{\mathrm{c}} = (r_{\mathrm{c},1},\, r_{\mathrm{c},2},\, \ldots) \tag{147}$$

$$= \text{robustness of most vulnerable criterion.} \tag{148}$$

$$= \min_i \widehat{h}_i(q, r_{\mathrm{c},i}) \tag{149}$$

We have used this in project management and other examples.
¶ In a similar vein a **combined** opportuneness function is:

$$\widehat{\beta}_i(q, r_{\mathrm{w}}) = \text{overall opportuneness.} \quad r_{\mathrm{w}} = (r_{\mathrm{w},1},\, r_{\mathrm{w},2},\, \ldots) \tag{150}$$

$$= \text{opportuneness of least opportune criterion.} \tag{151}$$

$$= \max_i \widehat{\beta}_i(q, r_{\mathrm{w},i}) \tag{152}$$

¶ There are other ways of combining multiple criteria,
some of which we will encounter.

9 Three Components of Info-gap Decision Models

¶ A decision model always has three components:
- A system model.
- A performance requirement.
- An uncertainty model.

¶ A **system model** is represented by the
reward or performance function $R(q, u)$.
This function expresses the relation between
input (from the environment, etc.)
and
output (result of action, decision, etc.).
The choice of the reward function is not unique,
but depends on the issues which are relevant.

¶ The **performance requirement** is of the form:

$$R(q, u) \geq r \quad \text{or} \quad R(q, u) \leq r.$$

where:
r = critical level of reward (robust satisficing).
or
r = windfall level of reward (opportune windfalling).

¶ The **uncertainty model** is an info-gap model, $\mathcal{U}(h, \widetilde{u}),\ h \geq 0$.
There may be more than one info-gap model.

¶ It is important to stress the role of
$q =$ decision or design vector.

10 Preferences

¶ We have noted that, for the robustness function, $\widehat{h}(q, r_c)$:

bigger is better.

- This implies that, for any two choices of the decision vector, q:

  $q \succ q'$

  if $\widehat{h}(q, r_c) > \widehat{h}(q', r_c)$.
- This establish a **preference ordering** on decision options at specified demanded performance, r_c.
- The preference orderings may be different at different r_c values.

¶ We can define a **robust-optimal decision** $\widehat{q}_c(r_c)$:

$$\widehat{h}(\widehat{q}_c(r_c),\, r_c) = \max_{q \in \mathcal{Q}} \widehat{h}(q, r_c) \tag{153}$$

where $\mathcal{Q}$ = set of available options.

¶ Note: optimal action $\widehat{q}_c(r_c)$ depends on demanded performance r_c.

¶ Since both:

- the preference ordering, "$\succ$" and
- the optimal action $\widehat{q}_c(r_c)$

depend on the choice of the demanded performance r_c,
we see that

info-gap decision theory does **not determine**
a unique 'rational decision'.

Rather, $\widehat{h}(q, r_c)$ is a quantitative **decision support tool**
with which we evaluate and explore options.

¶ We have noted that, for the opportuneness function, $\widehat{\beta}(q, r_{\rm w})$:

big is bad.

- This implies that, for any two choices of the decision vector, q:
  $q \succ q'$
  if $\widehat{\beta}(q, r_{\rm w}) < \widehat{\beta}(q', r_{\rm w})$.
- This establish a **preference ordering** on decision options at specified windfall performance, $r_{\rm w}$.
- The preference orderings may be different at different $r_{\rm w}$ values.
- The opportuneness-windfall preference ordering may differ from the robust-satisficing preference ordering.

¶ We can define a **windfall-optimal decision** $\widehat{q}_{\rm w}(r_{\rm w})$:

$$\widehat{\beta}(\widehat{q}_{\rm w}(r_{\rm w}),\, r_{\rm w}) = \min_{q\in\mathcal{Q}} \widehat{\beta}(q, r_{\rm w}) \tag{154}$$

where $\mathcal{Q}$ = set of available options.

¶ Note: optimal action $\widehat{q}_{\rm c}(r_{\rm w})$ depends on windfall performance $r_{\rm w}$.

11 Trade-offs

¶ We use the immunity functions, $\widehat{h}(q, r_{\rm c})$ and $\widehat{\beta}(q, r_{\rm w})$,
to explore options and form preferences.
Several rather different trade-offs arise.

¶ One trade-off is robustness vs. reward:

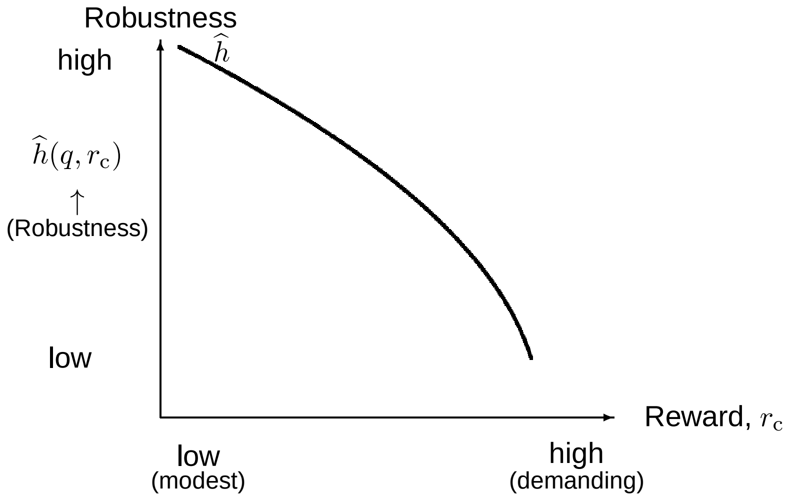

Figure 7: Robustness curve.

¶ In this figure: large r_c is better than small r_c.

- When this is true:
  The robustness vs. reward curve
  decreases monotonically with increasing critical reward.
  (As in fig. 7.)
- When small r_c is better than large r_c:
  The robustness vs. reward curve
  increases monotonically with increasing critical reward.
- The generalization:
  The robustness vs. reward curve
  decreases monotonically with increasing demanded performance.

¶ The trade-off:
High reward (great demands on performance)
is obtained in exchange for
low robustness to uncertainty.

¶ The position of the robustness curve indicates a type of **gambling.**
Consider 2 strategies whose $\widehat{h}$-functions are:

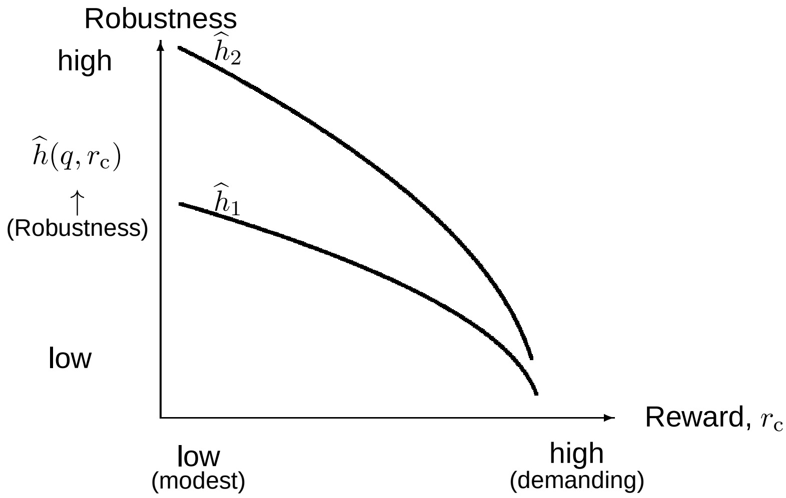

Figure 8: Robustness curve.

¶ We interpret these strategies as 'bold' and 'cautious':

- The upper strategy, $\widehat{h}_2(q, r_c)$, is **bold:**
  - At any demanded reward r_c,
    $\widehat{h}_2$ tolerates more uncertainty than $\widehat{h}_1$.
  - At any ambient uncertainty, h,
    $\widehat{h}_2$ can demand more reward than $\widehat{h}_1$.
- The upper strategy, $\widehat{h}_2(q, r_c)$, would look **risky**, **rash**,
  from the perspective of the lower strategy, $\widehat{h}_1(q, r_c)$,
  which is **cautious.**

¶ The opportuneness function also shows a trade-off:

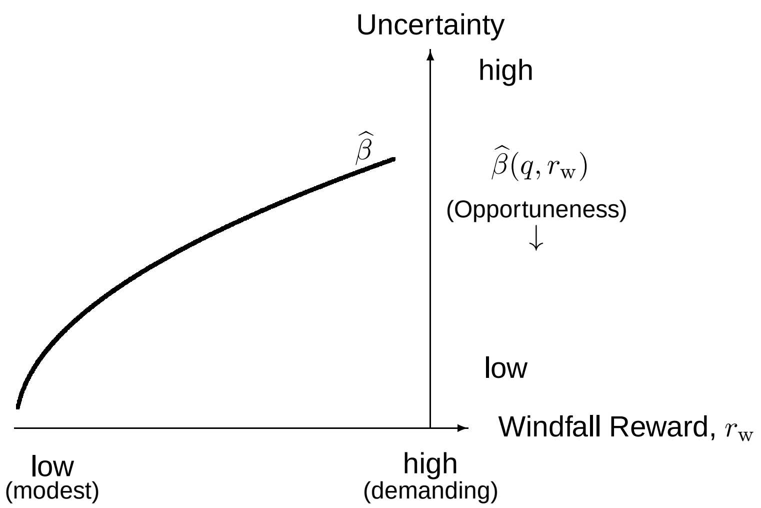

Figure 9: An opportuneness curve.

¶ The trade-off:

- High windfall reward is possible only at high ambient uncertainty.
- Low uncertainty can be bought only by giving up windfall opportunity.

¶ There is a coherence between
- robustness vs. reward trade-off

and
- certainty vs. windfall trade-off.

In both cases,
as the decision maker gives up expectation by reducing demand
(reducing $r_{\rm c}$ or $r_{\rm w}$),
both $\widehat{h}$ and $\widehat{\beta}$ show more optimistic picture.

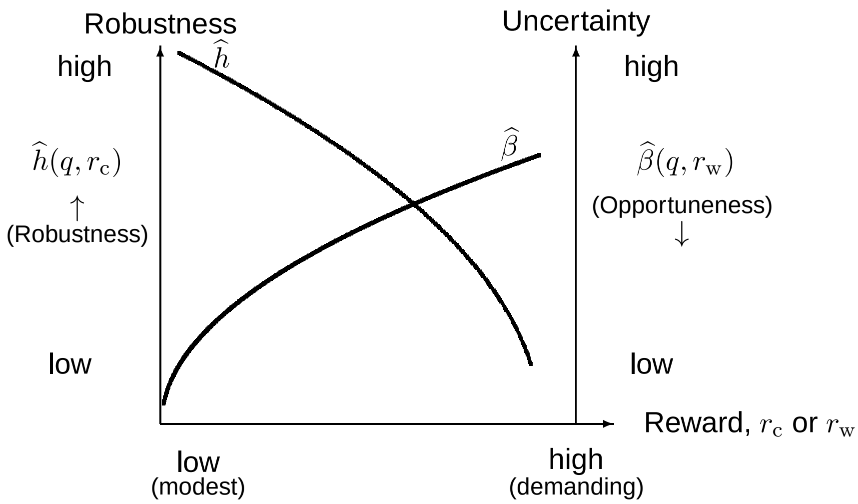

Figure 10: Robustness and opportuneness curves.

¶ Later we will explore a different type of trade-off.
We will explore the question:
- If q is changed to increase $\widehat{h}(q, r_{\rm c})$,
  will $\widehat{\beta}(q, r_{\rm w})$ get better or worse?
- That is, are robustness and opportuneness
  antagonistic or **sympathetic?**

12 Portfolio Investment

(Y. Ben-Haim, *Info-Gap Decision Theory*, section 3.2.7). See also Lecture Notes on Portfolio Management.[2]

¶ For many decision problems, the
response or reward R
is proportional to the
investment of resource q,
while the
coefficient of proportionality u is uncertain:

$$R(q,u) = \sum_{i=1}^{N} q_i u_i = q^T u \tag{155}$$

¶ The prototype is
portfolio investment q
with uncertain return u.
$q_i =$ amount invested in commodity i.
$u_i =$ dollar earned for each dollar invested in commodity i.

¶ This is also typical of many other decision problems:
- Resource distribution with proportional return.
- Elastic deflection at small strain: q_i is force, u_i is strain.
- Acoustic response.
- etc.

¶ We will consider uncertain u-vectors with the following information:
- Nominal $\widetilde{u}$ is known, calculated as historical mean.
- Shape of clusters of u-vectors is roughly known.
  We have the historical covariance of u-vectors.

Thus we will adopt an ellipsoid-bound info-gap model:

$$\mathcal{U}(h,\widetilde{u}) = \left\{ u = \widetilde{u} + v : \ v^T W v \leq h^2 \right\}, \quad h \geq 0 \tag{156}$$

where W is a known, real, symmetric, positive definite matrix, chosen as the inverse of the historical covariance matrix.
- Explain intuitively why W (ellipsoidal shape matrix) is the inverse covariance matrix:
  - Shape of the uncertain cluster expresses variance and covariance.
  - Special case: diagonal $W = (1/\sigma_1^2, \ldots, 1/\sigma_n^2)$.

[2] \lectures\Econ-Dec-Mak\portfolio-mgt001.tex

12.1 Robustness Function

¶ $\widehat{h}(q, r_{\rm c}) =$ greatest uncertainty at which reward is no less than $r_{\rm c}$ for investment portfolio q:

$$\widehat{h}(q, r_{\rm c}) = \max \left\{ h : \left(\min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \right) \geq r_{\rm c} \right\} \tag{157}$$

To evaluate $\widehat{h}(q, r_{\rm c})$ we must determine:

$$\min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) = q^T \widetilde{u} + \min_{v^T W v \leq h^2} q^T v \tag{158}$$

¶ To evaluate this optimum we use **Lagrange optimization.** Define:

$$H = q^T v + \lambda \left(h^2 - v^T W v \right) \tag{159}$$

The condition for an extremum:

$$0 = \frac{\partial H}{\partial v} = q - 2\lambda W v \tag{160}$$

$$\Longrightarrow \qquad v = \frac{1}{2\lambda} W^{-1} q \tag{161}$$

Using the constraint:

$$h^2 = v^T W v = \frac{1}{4\lambda^2} q^T W^{-1} W W^{-1} q \tag{162}$$

which leads to:

$$\frac{1}{2\lambda} = \frac{\pm h}{\sqrt{q^T W^{-1} q}} \tag{163}$$

Hence:

$$v = \frac{\pm h}{\sqrt{q^T W^{-1} q}} W^{-1} q \tag{164}$$

So the minimum is:

$$\min_{v^T W v \leq h^2} q^T v = -h \sqrt{q^T W^{-1} q} \tag{165}$$

Consequently:

$$\min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) = q^T \widetilde{u} - h \sqrt{q^T W^{-1} q} \tag{166}$$

¶ To find $\widehat{h}$: Equate this minimum to $r_{\rm c}$ and solve for h:

$$\widehat{h}(q, r_{\rm c}) = \frac{q^T \widetilde{u} - r_{\rm c}}{\sqrt{q^T W^{-1} q}} \tag{167}$$

unless this is negative, in which case:

$$\widehat{h}(q, r_{\rm c}) = 0 \tag{168}$$

Note:

- Trade-off between robustness, $\widehat{h}(q, r_{\rm c})$, and satisficed return, $r_{\rm c}$.
- Zero robustness at nominal return, $q^T \widetilde{u}$.

12.2 Robust Optimal Investment

¶ Question: how to choose the investment vector q?
Strategy:

- $\widehat{h}(q, r_{\rm c})$ depends on the decision vector q.
- For $\widehat{h}$ we know that: "bigger is better".
- So, choose q to maximize $\widehat{h}(q, r_{\rm c})$ subject to budget constraint:

$$\sum_{i=1}^{N} q_i = Q = \text{total available budget (or weight)} \tag{169}$$

$q_i > 0 \implies$ buy commodity i (increase weight at point i).
$q_i < 0 \implies$ sell commodity i (decrease weight at point i).
¶ To express eq.(169) vectorially, define the N-vector:

$$\mathbf{1} = \begin{pmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{pmatrix} \tag{170}$$

Thus:

$$\sum_{i=1}^{N} q_i = q^T \mathbf{1} \tag{171}$$

So the constraint is:

$$q^T \mathbf{1} = Q \tag{172}$$

¶ Consider a special case:

$$\widetilde{u}_i = u_o \quad \text{for all} \quad i \tag{173}$$

That is: all commodities have the same nominal value.
Of course, the uncertainties may differ between commodities.
Eq.(173) can be expressed:

$$\widetilde{u} = u_o \mathbf{1} \tag{174}$$

¶ The robustness, eq.(167), becomes:

$$\widehat{h}(q, r_{\rm c}) = \frac{u_o q^T \mathbf{1} - r_{\rm c}}{\sqrt{q^T W^{-1} q}} \tag{175}$$

$$= \frac{u_o Q - r_{\rm c}}{\sqrt{q^T W^{-1} q}} \tag{176}$$

¶ So, how to choose the investment vector q?
From eq.(176) we maximize $\widehat{h}$
by choosing q to minimize $q^T W^{-1} q$ (**meaning:** minimize impact of uncertainty)
subject to the constraint $q^T \mathbf{1} = Q$.
Note: we are **not** minimizing the uncertainty itself, rather, the **impact** of uncertainty.

¶ We again use Lagrange optimization. The optimal q is:

$$\widehat{q}_{\rm c} = \frac{Q}{\mathbf{1}^T W \mathbf{1}} W \mathbf{1} \tag{177}$$

The optimal robustness becomes:

$$\widehat{h}(\widehat{q}_{\rm c}, r_{\rm c}) = \frac{(u_o Q - r_{\rm c}) \mathbf{1}^T W \mathbf{1}}{Q} \tag{178}$$

This shows the usual trade-off between robustness vs. critical reward, as in fig.11:

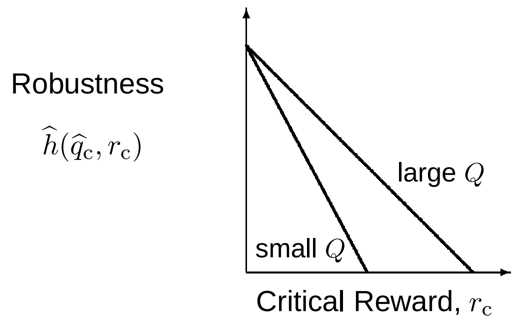

Figure 11: Robustness function vs critical reward.

Slope $\propto -\frac{1}{Q}$, where $Q =$ total investment.
Question: Are things better or worse with large investment Q?
Answers:

- Greater robustness at fixed aspiration $r_{\rm c}$, for larger Q.
- Aspiration-cost of an increment in robustness increases as Q increases.

12.3 Comparing Portfolios

¶ Consider 2 sets of investment options, each with:
- Constant nominal return, $\widetilde{u}_i = u_{o,i}\mathbf{1}$, $i = 1, 2$.
- Ellipsoid-bound info-gap models as in eq.(156) on p. 49:

$$\mathcal{U}_i(h, \widetilde{u}_i) = \left\{ u = \widetilde{u}_i + v : \ v^T W_i v \le h^2 \right\}, \quad h \ge 0, \ i = 1, 2 \tag{179}$$

Consider the following special case:

$$u_{o,1} \quad < \quad u_{o,2} \tag{180}$$

$$\mathbf{1}^T W_1 \mathbf{1} \quad > \quad \mathbf{1}^T W_2 \mathbf{1} \tag{181}$$

- Eq.(180) implies that option 1 is nominally worse than option 2.
- Eq.(181) implies that option 1 is nominally more certain than option 2. (Recall: W is **inverse** covariance matrix).
- This is characteristic of an "**innovation dilemma**".

The optimum robustness function for investment option i is, from eq.(178) on p. 52:

$$\widehat{h}_i(\widehat{q}_{ci}, r_c) = \frac{(u_{o,i}Q - r_c)\mathbf{1}^T W_i \mathbf{1}}{Q} \tag{182}$$

¶ These two optimal robustness functions appear as in fig. 12:

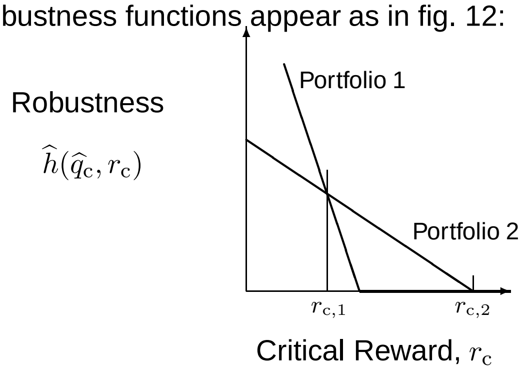

Figure 12: Robustness functions for two different portfolio investment alternatives.

Clearly:
- We prefer portfolio 1 for rewards $r_c < r_{c,1}$.
  Portfolio 2 is more risky than portfolio 1.
- We prefer portfolio 2 for rewards $r_{c,1} < r_c < r_{c,2}$.
  Portfolio 1 is more risky than portfolio 2.
- Neither portfolio is acceptable for rewards $r_{c,2} < r_c$.
  Both portfolios very risky.

¶ Robustness curves cross, as in fig. 12, if and only if there is an innovation dilemma.

12.4 Opportuneness Function

¶ We now develop the opportuneness function, $\widehat{\beta}(q, r_{\mathrm{w}})$.
$\widehat{\beta}(q, r_{\mathrm{w}}) =$ least uncertainty needed to sustain possibility of reward as large as r_{w}:

$$\widehat{\beta}(q, r_{\mathrm{w}}) = \min \left\{ h : \ \left(\max_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \right) \geq r_{\mathrm{w}} \right\} \tag{183}$$

where:

$$r_{\mathrm{w}} \gg r_{\mathrm{c}} \tag{184}$$

Compare this to the robustness function, eq.(157) on p.50:

$$\widehat{h}(q, r_{\mathrm{c}}) = \max \left\{ h : \ \left(\min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \right) \geq r_{\mathrm{c}} \right\} \tag{185}$$

¶ $\widehat{\beta}(q, r_{\mathrm{w}})$ and $\widehat{h}(q, r_{\mathrm{c}})$ are **dual functions.**

¶ Distinct decision strategies:
$\widehat{\beta}(q, r_{\mathrm{w}})$: windfalling at r_{w}.
$\widehat{h}(q, r_{\mathrm{c}})$: satisficing at r_{c}.

¶ Proceeding as before we find:

$$\max_{u \in \mathcal{U}(h, \widetilde{u})} q^T u = q^T \widetilde{u} + h \sqrt{q^T W^{-1} q} \tag{186}$$

Equate this to r_{w} and solve for h to find opportuneness function:

$$\widehat{\beta}(q, r_{\mathrm{w}}) = \frac{r_{\mathrm{w}} - q^T \widetilde{u}}{\sqrt{q^T W^{-1} q}} \tag{187}$$

Note trade-off of certainty vs. windfall reward.
¶ Impose the same budget constraint:

$$q^T \mathbf{1} = Q \tag{188}$$

Also, assume as before:

$$\widetilde{u} = u_o \mathbf{1} \tag{189}$$

The opportuneness function becomes:

$$\widehat{\beta}(q, r_{\mathrm{w}}) = \frac{r_{\mathrm{w}} - u_o Q}{\sqrt{q^T W^{-1} q}} \tag{190}$$

Recall the robustness function, eq.(178) on p. 52:

$$\widehat{h}(q, r_{\mathrm{c}}) = \frac{u_o Q - r_{\mathrm{c}}}{\sqrt{q^T W^{-1} q}} \tag{191}$$

¶ **Trade off and zeroing for robustness and opportuneness.** See fig. 13 on p.55.

¶ Recall "Bigger is better" for $\widehat{h}$
$\Longrightarrow$ choose q to maximize $\widehat{h}$.
"Big is bad" for $\widehat{\beta}$
$\Longrightarrow$ choose q to minimize $\widehat{\beta}$.

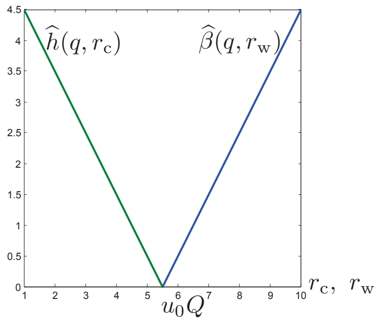

Figure 13: Trade off and zeroing for robustness and opportuneness. Eqs. (190) and (191).

¶ Can we optimize $\widehat{h}$ and $\widehat{\beta}$ with the same q?
- $\max \widehat{h}$ requires minimum $q^T W^{-1} q$.
- $\min \widehat{\beta}$ requires maximum $q^T W^{-1} q$.

So we cannot simultaneously optimize $\widehat{h}$ and $\widehat{\beta}$:
Any change in q which increases $\widehat{h}$ also increases $\widehat{\beta}$.
Any change in q which decreases $\widehat{h}$ also decreases $\widehat{\beta}$.
Thus $\widehat{h}$ and $\widehat{\beta}$ are **antagonistic.**

¶ Trade-off between robustness and opportuneness. From eqs.(190) and (191):

$$\frac{\mathrm{d}\widehat{h}(q, r_c)}{\mathrm{d}q} = -\frac{u_o Q - r_c}{q^T W^{-1} q} \underbrace{\frac{\mathrm{d}\sqrt{q^T W^{-1} q}}{\mathrm{d}q}}_{v} = -\frac{u_o Q - r_c}{q^T W^{-1} q}\, v \tag{192}$$

$$\frac{\mathrm{d}\widehat{\beta}(q, r_w)}{\mathrm{d}q} = -\frac{r_w - u_o Q}{q^T W^{-1} q} \underbrace{\frac{\mathrm{d}\sqrt{q^T W^{-1} q}}{\mathrm{d}q}}_{v} = -\frac{r_w - u_o Q}{q^T W^{-1} q}\, v \tag{193}$$

Hence:

$$\frac{\mathrm{d}\widehat{h}}{\mathrm{d}\widehat{\beta}} = \frac{u_o Q - r_c}{r_w - u_o Q} > 0 \tag{194}$$

The trade-off between robustness and opportuneness is shown schematically in fig. 14.

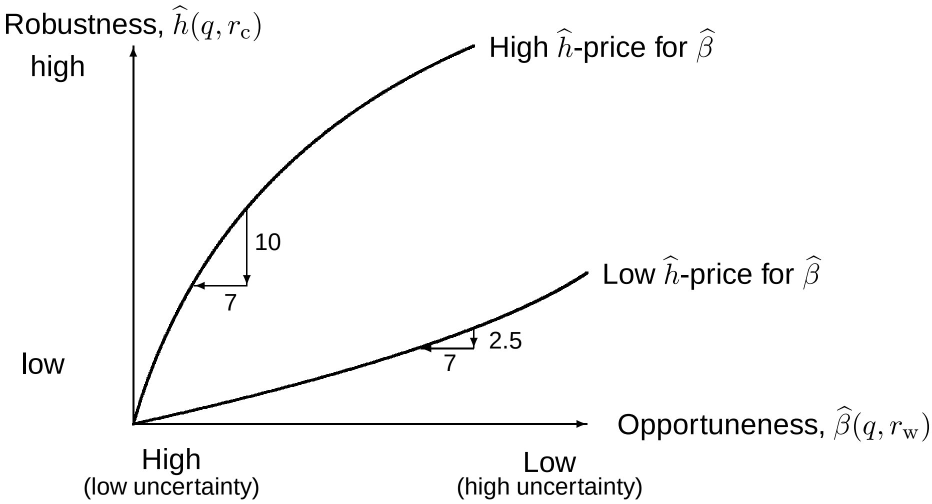

Figure 14: Trade-off between robustness and opportuneness.

¶ Does $\widehat{\beta}$ have an optimum?
Can we maximize $q^T W^{-1} q$ subject to $q^T \mathbf{1} = Q$?
No. See fig. 15.
For any constant = $q^T W^{-1} q$
There is a q which also satisfies the constraint.
However, as q moves far from the origin,
other constraints become active.

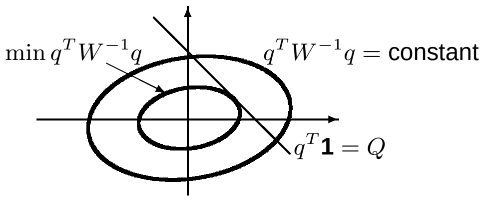

Figure 15: Schematic illustration of constrained optimization of $q^T W^{-1} q$.

13 Search and Evasion

¶ Tracking problem:

- Intelligent "hunter" tries to catch an intelligent "prey".
- Examples:
  - ○ Homing missile.
  - ○ Robotic grasping.
  - ○ Job search.

¶ Dynamics:

- Hunter and prey move on a line.
- $x(t) =$ hunter's position. $x(0) = 0$.
- $u(t) =$ prey's position. $u(0) > 0$.
- The hunter measures prey's position but
  hunter does not know prey's evasion strategy.
- Hunter moves according to:

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = q\,[u(t) - x(t)] \tag{195}$$

q = constant which hunter chooses before chase.

¶ Hunter has limited info about prey's evasive strategy:
- $\widetilde{s}$ = typical speed.
- Actual speed differs from $\widetilde{s}$ by unknown constant.
- Hunter's slope-bound info-gap model:

$$\mathcal{U}(h, \widetilde{s}) = \left\{ u(t) : \left| \frac{\mathrm{d}u(t)}{\mathrm{d}t} - \widetilde{s} \right| \leq h \right\}, \quad h \geq 0 \tag{196}$$

¶ Consider more information:
- Prey is thought to move more quickly if hunter and prey are close. E.g.:

$$\widetilde{s}(x, u) = \frac{\gamma}{(x-u)^2} \tag{197}$$

- The info-gap model in eq.(196) now becomes:

$$\mathcal{U}(h, \widetilde{s}) = \left\{ u(t) : \left| \frac{\mathrm{d}u(t)}{\mathrm{d}t} - \widetilde{s}(x, u) \right| \leq h \right\}, \quad h \geq 0 \tag{198}$$

¶ We will use the info-gap model in eq.(196).

¶ Performance requirement:
The hunter is successful if, at a specified time T,
the hunter-prey distance $\leq \Delta$:

$$|x(T) - u(T)| \leq \Delta \tag{199}$$

¶ Hunter must choose q in eq.(195) to:
- maximize robustness to uncertain prey behavior.
- satisfying performance requirement in (199).

¶ Robustness function $\widehat{h}(q, \Delta)$:

$$\widehat{h}(q, \Delta) = \max \left\{ h : \left(\max_{u \in \mathcal{U}(h, \widetilde{s})} |x(T) - u(T)| \right) \leq \Delta \right\} \tag{200}$$

¶ Dynamics again: solution of eq.(195) is:

$$x_u(t) = q \int_0^t \mathrm{e}^{-q(t-\tau)} u(\tau)\, \mathrm{d}\tau \tag{201}$$

After manipulation, including a partial integration, eq.(201) is:

$$\begin{aligned} x_u(t) - u(t) &= -\mathrm{e}^{-qt} u(t) - \mathrm{e}^{-qt} \int_0^t (\mathrm{e}^{q\tau} - 1) \left(\frac{\mathrm{d}u}{\mathrm{d}\tau} - \widetilde{s} \right) \mathrm{d}\tau \\ &\quad - \frac{\widetilde{s}}{q} \left(1 - \mathrm{e}^{-qt} \right) + \widetilde{s} t \mathrm{e}^{-qt} \end{aligned} \tag{202}$$

This is negative if the target runs quickly. Thus, for info-gap model in eq.(196), the maximum $|x-u|$ occurs for $u(t) = u(0) + (\widetilde{s} + h)t$ which becomes:

$$\max_{u \in \mathcal{U}(h, \widetilde{s})} |x_u(t) - u(t)| = \mathrm{e}^{-qt} u(0) + \frac{\widetilde{s} + h}{q} \left(1 - \mathrm{e}^{-qt} \right) \tag{203}$$

¶ Robustness: equate eq.(203) to Δ and solve for h:

$$\widehat{h}(q,\Delta) = \frac{\left(\Delta - \mathrm{e}^{-qT}u(0)\right)q}{1-\mathrm{e}^{-qT}} - \widetilde{s} \tag{204}$$

unless this is negative, in which case the robustness is zero.

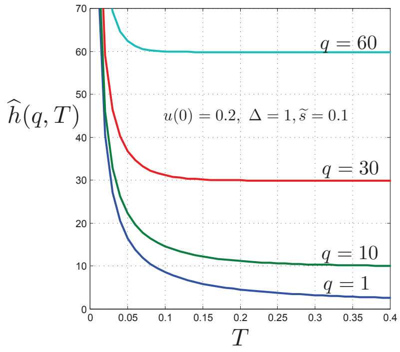

Figure 16: Robustness versus time, eq.(204), assuming $u(0) < \Delta$. The value of q increases from the bottom to the top curve.

¶ Results: eq.(204) is plotted in fig. 16, assuming $u(0) < \Delta$.
- q increases from the bottom to the top curve.
- q is a measure of hunter's effort:
  - ○ Large q implies large effort.
  - ○ Large q implies large robustness.
- $\widehat{h}(q,\Delta)$ decreases with chase time T if $u(0) < \Delta$:
  Long chase has low robustness.
- Choose q according to:
  - ○ required robustness.
  - ○ required chase duration.

¶ Return to eq.(204) on p. 59. We see that:

$$\frac{\partial\widehat{h}}{\partial T} > 0 \text{ if } u(0) > \Delta \tag{205}$$

$$\frac{\partial\widehat{h}}{\partial T} < 0 \text{ if } u(0) < \Delta \tag{206}$$

Meaning:
- Robustness increases in time, eq.(205), when chasing "distant" prey.
- Robustness decreases in time, eq.(206), in ambush.

14 Assay Design: Environmental Monitoring

14.1 Measuring Biomass

§ This section is based on section 3.2.10 in:
Yakov Ben-Haim, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* 2nd edition, Academic Press, London.

§ **The problem:**

• The local municipality will release waste into the river.

• We must design a monitoring system to detect contamination.

• The monitoring system measures local biomass at each of N locations along the river.

• We *wish* to trigger an alarm if the total biomass downstream of the release exceeds $B_{\rm c}$.

• We *will actually* trigger an alarm if the local biomass exceeds a critical value, ρ_0, at one or more measurement sites.

• The biomass density distribution, $\rho(x)$, is highly uncertain.

• *Design task:* choose N and ρ_0.

• Method: evaluate robustness to spatial uncertainty in $\rho(x)$.

§ **Information about the spatial uncertainty.**

• $\rho(x)$, varies gradually along the length of the river.

• Maximum slope of $\rho(x)$ no more extreme than $\pm s$, estimated as $\pm\widetilde{s}$.

• Actual slope is highly uncertain.

§ **The slope-bound info-gap model.**

• Include the no-alarm assay result that density is no greater than ρ_0 at all of the N test points x_i :

$$\mathcal{U}(h, \rho_0, \widetilde{s}) = \left\{ \rho(x) : \ \rho(x_i) \le \rho_0, i = 1, \ldots, N; \ \left| \frac{|\rho'(x)| - \widetilde{s}}{\widetilde{s}} \right| \le h \right\}, \ h \ge 0 \tag{207}$$

The inequality on ρ' means that, at horizon of uncertainty h, $\rho'(x)$ satisfies one of:

$$\text{positive slope:} \quad (1-h)\widetilde{s} \le \quad \rho'(x) \quad \le (1+h)\widetilde{s} \tag{208}$$

$$\text{negative slope:} \quad -(1+h)\widetilde{s} \le \quad \rho'(x) \quad \le (-1+h)\widetilde{s} \tag{209}$$

However, the horizon of uncertainty, h, is unknown.

• Note: the info-gap model depends on the design (N, ρ_0) and on the fact that the observations ($\rho(x_i)$, $i = 1, \ldots, N$) are all “okay”.

§ **Requirement:** No missed detection.
That is, if assay does not trigger an alarm, then the total biomass is actually acceptably small.

§ **Different possible requirement:** No false detection.
That is, if assay **does** trigger an alarm, then total biomass is actually **not** acceptably small.

§ **Robustness** of N measurement sites, trigger level ρ_0, with critical total mass $B_{\rm c}$:

$$\widehat{h}(N, \rho_0, B_{\rm c}) = \max \left\{ h : \ \underbrace{\left(\max_{\rho \in \mathcal{U}(h, \rho_0, \widetilde{s})} \int_0^L \rho(x) \, {\rm d}x \right)}_{M(h)} \leq B_{\rm c} \ \right\} \tag{210}$$

§ **Evaluating the robustness: conceptual.**

- $M(h)$ is defined in eq.(210).
- $M(h)$ increases monotonically as h increases.
- Hence $M(h)$ is the inverse of $\widehat{h}(N, \rho_0, B_{\rm c})$:

$$M(h) = B_{\rm c} \quad \text{implies} \quad \widehat{h}(N, \rho_0, B_{\rm c}) = h \tag{211}$$

- A plot of h (vertical) vs. $M(h)$ (horizontal)
  is the same as a plot of
  $\widehat{h}(N, \rho_0, B_{\rm c})$ (vertical) vs. $B_{\rm c}$ (horizontal).

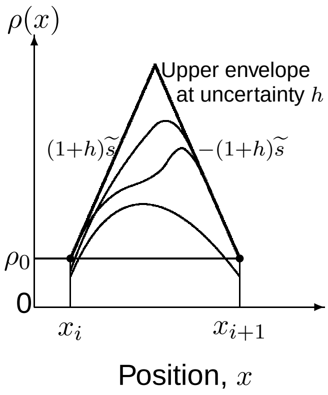

Figure 17: Evalution of $M(h)$, eq.(210), showing an upper envelope and three possible density curves.

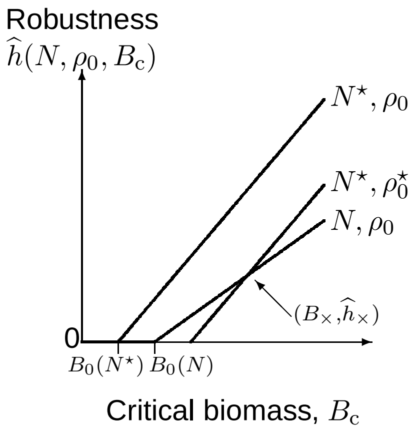

Figure 18: Robustness curves for N and $N^\star$ test points with trigger densities ρ_0 and $\rho_0^\star$, eq.(213). $N^\star > N$, $\rho_0^\star > \rho_0$.

§ **Evaluating the robustness,** (fig. 17):

- Given measured densities of ρ_0 at adjacent test points.
- Max biomass occurs at extremal slopes of $\rho(x)$.
- Max biomass at horizon of uncertainty h, in the $N-1$ equal intervals between 0 and L, is:

$$M(h) = L\rho_0 + \frac{L^2\widetilde{s}}{4(N-1)}(1+h) \tag{212}$$

Equate eq.(212) to the critical biomass B_c and solve for h yields robustness:

$$\widehat{h}(N, \rho_0, B_c) = \begin{cases} \dfrac{4(N-1)}{L^2\widetilde{s}}(B_c - L\rho_0) - 1 & \text{if } B_c \geq \underbrace{L\rho_0 + \dfrac{L^2\widetilde{s}}{4(N-1)}}_{B_0(N)} \\ 0 & \text{else} \end{cases} \tag{213}$$

where $B_0(N)$ is the nominal biomass.

§ **Trade-offs**:

- Robustness increases ($\widehat{h}$ gets larger) as the performance gets worse (B_c gets larger), fig. 18.
- Robustness increases with increase in the number of test points in the length L along the river.

And note that $B_0(N)$ decreases.

- Robustness increases as the alarm threshold, ρ_0, gets smaller.

§ **Unreliability of estimated performance,** fig. 18.

- $B_0(N)$ in eq.(213) is the biomass of a distribution whose:
  - measurements all equal ρ_0 and,
  - slope between test points equals the anticipated values of $\pm\widetilde{s}$.
- This nominal biomass has zero robustness of detection:

$$\widehat{h}(N, \rho_0, B_c) = 0 \quad \text{if} \quad B_c = B_0(N) \tag{214}$$

§ **Preference reversal.**

- Note crossing robustness curves in fig. 18 for $N < N^\star$ and $\rho_0 < \rho_0^\star$.
- That is, reducing # of measurements can be compensated for
  by reducing the trigger density,
  at constant robustness to spatial uncertainty.

§ **Demanded robustness.**

- $\widehat{h}_d$ denotes demanded robustness to slope-uncertainty.
- E.g., $\widehat{h}_d = 0.5$ implies:
  Estimated max slope, $\widetilde{s}$, can err up to 50%
  without jeopardizing missed detection of excess biomass.
- Choose N and ρ_0 to satisfy:

$$\widehat{h}(N, \rho_0, B_c) = \widehat{h}_d \tag{215}$$

14.2 Choosing Sample Size: Special Case of Small Effect Size

§ This section is a special case of a more general problem studied in:
David R. Fox, Yakov Ben-Haim, Keith R. Hayes, Michael McCarthy, Brendan Wintle, Piers Dunstan, 2007, An info-gap approach to power and sample size calculations, *Environmetrics,* vol. 18, pp.189–203.

§ **Notation:**

- $x =$ a statistic, e.g. sample mean.
- $f(x) =$ sampling distribution of x. Uncertain.
- $\widetilde{f}(x) =$ Best-estimate of the sampling distribution of x.
- $\delta =$ effect size: suspected change in the value estimated by x.

§ **Example:**

- Measurements $y_i \sim \mathcal{N}(\mu, \sigma^2)$.
- Statistic: sample mean, $\overline{x} = \frac{1}{N}\sum_{i=1}^{N} y_i$.
- Thus the sampling distribution is $\overline{x} \sim \mathcal{N}(\mu, \sigma^2/N)$.

§ **Binary decision:**

- Null hypothesis: there was no change:

$$H_0 : \quad x \sim f(x) \tag{216}$$

- Alternative hypothesis: there was a change equal to δ:

$$H_1 : \quad x \sim f(x-\delta) \tag{217}$$

- Threshold test with "critical value" C: Decide "no change" iff $x \le C$.
- $\alpha =$ Level of significance,
  = probability of falsely rejecting the null hypothesis.

$$\alpha = \int_C^{\infty} f(x)\,\mathrm{d}x \tag{218}$$

$$= 1 - \int_{-\infty}^{C} f(x)\,\mathrm{d}x \tag{219}$$

We can re-write this as follows, which will be useful later:

$$1-\alpha = \int_{-\infty}^{C} f(x)\,\mathrm{d}x \tag{220}$$

- $\beta(f) =$ 1 minus the power,
  = probability of falsely rejecting the alternative hypothesis.

$$\beta(f) = \int_{-\infty}^{C} f(x-\delta)\,\mathrm{d}x \tag{221}$$

$$= \int_{-\infty}^{C-\delta} f(x)\,\mathrm{d}x = 1-\alpha-\int_{C-\delta}^{C} f(x)\,\mathrm{d}x \tag{222}$$

§ **Power of the test:**

- Power = $1-\beta$:

$$1-\beta = \int_C^{\infty} f(x-\delta)\,\mathrm{d}x \tag{223}$$

- Power is probability of *correctly* rejecting H_0.
- Compare with Level of significance: probability of *falsely* rejecting H_0.
- We want both α and β to be *small*.
- Compare eqs.(218) and (221) to see that:

$$\frac{\partial \alpha}{\partial C} \leq 0, \quad \frac{\partial \beta}{\partial C} \geq 0 \tag{224}$$

You can't improve both α and β by changing the decision threshold C.

§ **Standard statistical approach** to determining the sample size:
- Know sampling distribution, $f(x)$.
- $f(x)$ depends on the number of measurements.
- Specify level of significance α and the effect size δ.
- Evaluate the critical value and the power from eqs.(220) and (221).
- Increase the number of measurements until the power is adequate.

§ **The problem:** $f(x)$ is highly uncertain.

§ **Fractional-error info-gap model:**

$$\mathcal{U}(h, \widetilde{f}) = \left\{ f(x) : \ f \in \mathcal{P}, \ |f(x) - \widetilde{f}(x)| \leq h\widetilde{f}(x) \right\}, \quad h \geq 0 \tag{225}$$

$\mathcal{P}$ is the set of all non-negative and normalized pdfs on the domain of x.
§ **Critical value:**
- Choose critical value based on estimated distribution:
- Let $\widetilde{C}$ denote the $1-\alpha$ quantile of the nominal distribution $\widetilde{f}(x)$:

$$1 - \alpha = \int_{-\infty}^{\widetilde{C}} \widetilde{f}(x) \, \mathrm{d}x \tag{226}$$

§ **Analyst's requirement.**
- β needs to be small.
- Let $1-\beta_{\mathrm{d}}$ be the power which is demanded by the analyst. That is, the analyst requires $\beta \leq \beta_{\mathrm{d}}$.

§ **The decision:**
- Choose sample size, N.
- Strategy: **robust-satisficing**:
  - ◦ Satisfice the power.
  - ◦ Maximize the robustness.

§ **The robustness** of N measurements, with the requirement β_{d}, is:

$$\widehat{h}(N, \beta_{\mathrm{d}}) = \max \left\{ h : \ \left(\max_{f \in \mathcal{U}(h, \widetilde{f})} \beta(f) \right) \leq \beta_{\mathrm{d}} \right\} \tag{227}$$

§ **Small Effect Size:**

$$\delta \ll 1 \tag{228}$$

Now eq.(221) can be approximated as:

$$\beta(f) = 1 - \alpha - f(C)\delta \tag{229}$$

§ **Inner max in eq.(227).**

• The pdf in $\mathcal{U}(h, \widetilde{f})$ which maximizes β is very nearly:

$$\widehat{f}(x) = \begin{cases} \widetilde{f}(x) & \text{if } x < \widetilde{C} - \delta \\ (1-h)\widetilde{f}(x) & \text{if } x \in [\widetilde{C} - \delta,\ \widetilde{C}] \\ (1+wh)\widetilde{f}(x) & \text{if } x > \widetilde{C} \end{cases} \tag{230}$$

where w is a very small positive number which normalizes $\widehat{f}(x)$. That is, w is determined so that the decrement in $\widehat{f}$ in $[\widetilde{C} - \delta,\ \widetilde{C}]$ is compensated by the increment in $(\widetilde{C}, \infty)$:

$$wh[1 - \widetilde{F}(\widetilde{C})] = h\delta\widetilde{f}(\widetilde{C}) \tag{231}$$

where $\widetilde{F}$ is the cumulative distribution function of $\widetilde{f}$.

• The inner max in eq.(227) is $\beta(\widehat{f})$ from eq.(229) and (230):

$$\beta(\widehat{f}) = 1 - \alpha - (1-h)\widetilde{f}(\widetilde{C})\delta \tag{232}$$

which is the greatest value of β at horizon of uncertainty h.

§ **Robustness.**

Equate eq.(232) to the demanded value, β_{d}, and solve for h for robustness of N measurements:

$$\widehat{h}(N, \beta_{\mathrm{d}}) = \begin{cases} 0 & \text{if } \beta_{\mathrm{d}} < 1 - \alpha - \widetilde{f}(\widetilde{C})\delta \\ \dfrac{\beta_{\mathrm{d}} - 1 + \alpha + \widetilde{f}(\widetilde{C})\delta}{\widetilde{f}(\widetilde{C})\delta} & \text{else} \end{cases} \tag{233}$$

§ **Discussion:**

• The robustness increases as β_{d} increases.

• The robustness is zero when β_{d} equals the nominal value, $\beta(\widetilde{f})$.

• This derivation is contingent on the small-effect assumption in eq.(228).

• The dependence of the robustness on the sample size arises through the nominal sampling distribution at the $1 - \alpha$ quantile, $\widetilde{f}(\widetilde{C})$.

• Note the innovation dilemma as the effect size, δ, changes, as illustrated in fig. 19, p.67:
The smaller δ is nominally preferred, but less robust at larger β_{d} values.

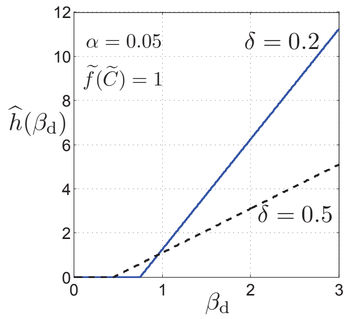

Figure 19: Innovation dilemma for choosing the effect size, δ.

15 Strategic Asset Allocation

§ This section based on section 4.4 of Yakov Ben-Haim, 2010, *Info-Gap Economics: An Operational Introduction,* Palgrave.

§ **Generic idea of an asset:**
- Energy supply to different actuators: motion on complex terrain; robotics.
- Duration and force at load points for deflection, especially in non-linear system.
- Duration at search locations (looking for treasure or enemies).
- People developing innovative ideas or projects.
- Stocks or bonds in finance: monetary return.

§ **Generic idea of strategic allocation:**
- Dynamic setting: multiple time steps.
- Allocation at each time step.
- Budget limitation.
- "Returns" or "outcomes" at each step determine resources for next step.

§ **Basic idea of asset allocation ("financial" model):**
- Choose an allocation of resources (e.g. budget) between different assets.
- The future returns are random and the pdf is uncertain.
- You require high probability that the future balance is acceptable.
  That is, the future **capital reserve** (or profit) must be adequate with high probability.

15.1 Budget Constraint

Basic variables:

x_{it} is the **quantity of the ith asset which is purchased** at time t. x_{it} can be either positive or negative. The allocation vector is $x_t = (x_{1t}, \ldots, x_{Nt})^T$. This is **chosen at time t.**

p_{it} is the **ex-dividend price of the ith asset** for purchase at time t. The vector of prices is $p_t = (p_{1t}, \ldots, p_{Nt})^T$. **Known at time t.**

y_{it} is the **payoff of the ith asset** at time $t+1$. The vector of payoffs is $y_t = (y_{1t}, \ldots, y_{Nt})^T$. **Not known at time t.**

c_t is the **capital reserve** of the financial institution[3] at time $t+1$. **Not known at time t.**

The **budget constraint:**

$$c_t + p_t^T x_t = y_t^T x_{t-1} \tag{234}$$

[3] For an individual investor c_t could be thought of as consumption.

15.2 Uncertainty

§ **Moderate uncertainty:**

- y_t, is random and known to be normally distributed.
- Moments are estimated but uncertain:
  - ○ Estimated mean of the payoff vector is μ_{yt}.
  - ○ Estimated covariance matrix of the payoff is Σ_{yt}.

§ Thus, from the budget constraint in eq.(234), the capital reserve is a normal random variable with estimated mean and variance:

$$\widetilde{\mu}_{ct} = -p_t^T x_t + \mu_{yt}^T x_{t-1} \tag{235}$$

$$\widetilde{\sigma}_{ct}^2 = x_{t-1}^T \Sigma_{yt} x_{t-1} \tag{236}$$

§ Error values of the estimated mean and standard deviation, $\widetilde{\mu}_{ct}$ and $\widetilde{\sigma}_{ct}$, are ε_μ and ε_σ.

§ Info-gap model for uncertainty in the distribution of the capital reserve, c_t:

$$\mathcal{U}(h) = \left\{ f(c_t) \sim N(\mu_{ct}, \sigma_{ct}^2) : \ \left| \frac{\mu_{ct} - \widetilde{\mu}_{ct}}{\varepsilon_\mu} \right| \le h, \ \left| \frac{\sigma_{ct} - \widetilde{\sigma}_{ct}}{\varepsilon_\sigma} \right| \le h, \ \sigma_{ct} \ge 0 \right\}, \quad h \ge 0 \tag{237}$$

15.3 Performance and Robustness

Performance requirement.

The α **quantile** of the distribution $f(c_t)$, denoted $q(\alpha, f)$, is the value of c_t for which the probability of being less than this value equals α. This quantile is defined in:

$$\alpha = \int_{-\infty}^{q(\alpha,f)} f(c_t)\,\mathrm{d}c_t \tag{238}$$

α is typically small so $q(\alpha, f)$ may be negative.

§ The **performance requirement** is:

$$q(\alpha, f) \geq r_{\mathrm{c}} \tag{239}$$

We will use the robustness function to evaluate the confidence in satisfying this requirement for chosen investment, x_t.

Robustness function:

$$\widehat{h}(x_t, r_{\mathrm{c}}) = \max\left\{h : \left(\min_{f\in\mathcal{U}(h)} q(\alpha, f)\right) \geq r_{\mathrm{c}}\right\} \tag{240}$$

§ z_α is the α quantile of the standard normal distribution.

- Assume: $\alpha < 1/2$ so that $z_\alpha < 0$.
- Typically α around 0.01.

§ One can show:

$$\widehat{h}(x_t, r_{\mathrm{c}}) = \frac{r_{\mathrm{c}} - q(\alpha, \widetilde{f})}{\varepsilon_\sigma z_\alpha - \varepsilon_\mu} \tag{241}$$

or zero if this is negative.

- The numerator and denominator are both negative, so the robustness decreases as r_{c} increases towards $q(\alpha, \widetilde{f})$.

15.4 Opportuneness Function

§ **Windfall aspiration is:**

$$q(\alpha, f) \geq r_{\rm w} > r_{\rm c} \tag{242}$$

§ **Opportuneness:**

$$\widehat{\beta}(x_t, r_{\rm w}) = \min\left\{h : \left(\max_{f\in\mathcal{U}(h)} q(\alpha, f)\right) \geq r_{\rm w}\right\} \tag{243}$$

§ Inverse of opportuneness:

- $M(h)$ denotes the **inner maximum** in eq.(243).
- $M(h)$ is the **inverse of the opportuneness.**
- That is, a plot of $M(h)$ vs. h is the same as a plot of $r_{\rm w}$ vs. $\widehat{\beta}(x_t, r_{\rm w})$.
- We will derive an explicit expression from which to evaluate $M(h)$.

§ **Ramp function:** $r(x) = 0$ if $x < 0$ and $r(x) = x$ if $x \geq 0$.

§ **One assumption:**

- z_α is the α quantile of the standard normal distribution.
- We assume that $\alpha < 1/2$, so that $z_\alpha < 0$.

§ **One can show:**

$$q(\alpha, f) = \sigma_{ct} z_\alpha + \mu_{ct} \tag{244}$$

Proof:

$$\begin{aligned}
\alpha &= \text{Prob}\,(x \leq q(\alpha, f)) && (245)\\
&= \text{Prob}\left(\frac{x-\mu_{ct}}{\sigma_{ct}} \leq \frac{q(\alpha, f)-\mu_{ct}}{\sigma_{ct}}\right) && (246)
\end{aligned}$$

Note that:

$$\begin{aligned}
z &= \frac{x-\mu_{ct}}{\sigma_{ct}} \sim \mathcal{N}(\mu_{ct}, \sigma_{ct}) && (247)\\
z_\alpha &= \frac{q(\alpha, f)-\mu_{ct}}{\sigma_{ct}} && (248)
\end{aligned}$$

Re-arranging eq.(248) leads to eq.(244).

§ **Inverse of opportuneness function:**

$$M(h) = r(\widetilde{\sigma}_{ct} - \varepsilon_\sigma h) z_\alpha + \widetilde{\mu}_{ct} + \varepsilon_\mu h \tag{249}$$

15.5 Policy Exploration

§ **Example:**

- One risk-free asset, $i = 1$, and a one uncorrelated risky asset, $i = 2$.
- Select the allocation.
- Price vector is $p_t = (7, 10)$.
- The level of confidence of the quantile is $\alpha = 0.01$.
- The standard deviation of the payoff of the risky asset is 5% of its estimated mean unless indicated otherwise.
- Thus $(\Sigma_{yt})_{22} = (0.05\mu_{yt,2})^2$. The other elements of the 2×2 covariance matrix Σ_{yt} are zero.

§ **Trade-offs and zeroing** (fig. 20):

- Robustness vs critical reserve.
- Opportuneness vs windfall reserve.

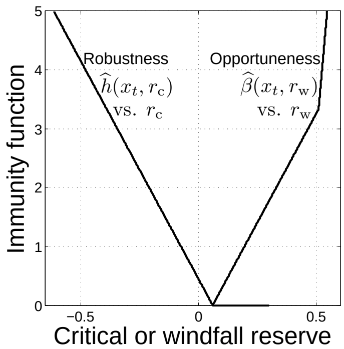

Figure 20: Robustness and opportuneness curves. $x_{t-1} = x_t = (0.7, 0.3)^T$. $\mu_{yt} = (1.04p_{1t}, 1.08p_{2t})^T$. $\varepsilon_\mu = 0.05\widetilde{\mu}_{ct}$. $\varepsilon_\sigma = 0.3\widetilde{\mu}_{ct}$.

Portfolio	$\mu_{yt,1}/p_{1t}$	$\mu_{yt,2}/p_{2t}$	$\widetilde{\mu}_{ct}$	$\widetilde{\sigma}_{ct}$	$\varepsilon_\mu/\widetilde{\mu}_{ct}$	$\varepsilon_\sigma/\widetilde{\sigma}_{ct}$
1	0.04	0.08	0.436	0.162	0.05	0.1
2	0.036	0.076	0.404	0.161	0.035	0.075

Table 1: Parameters of two portfolios. Robustness curves in fig. 21.

Choose between two portfolios, table 1.

- First portfolio has higher estimated mean payoffs and higher errors.
- Classical dilemma: portfolio 1 is better on average, but more uncertain.

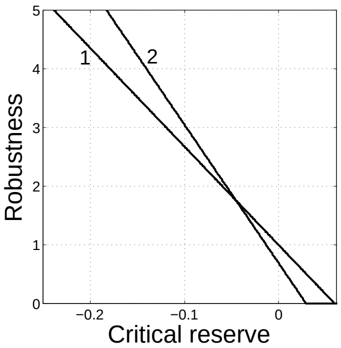

Figure 21: Robustness curves. $x_{t-1} = x_t = (0.7, 0.3)^T$. See table 1.

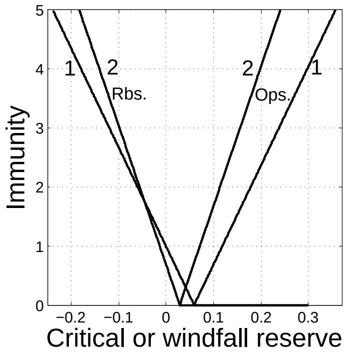

Figure 22: Robustness and opportuneness curves for portfolios in fig. 21.

§ **Preference reversal,** fig. 21.

§ **Robustness and opportuneness,** fig. 22.

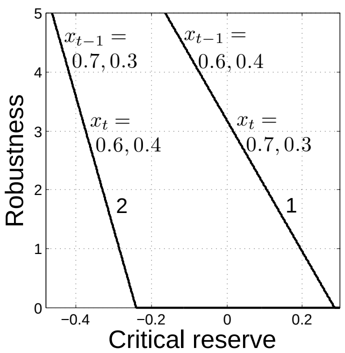

Figure 23: Robustness curves for two sequences of investments.

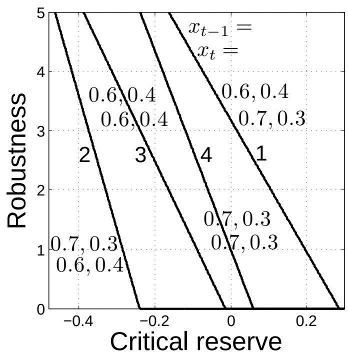

Figure 24: Robustness curves for 4 sequences of investments. Curves 1 and 2 reproduced from fig. 23.

§ **Sequence matters,** fig. 23.

- Sequence of investment vectors are reversed between the two portfolios.
- Two differences between outcomes:
  - ○ Portfolio 1 has much higher nominal α quantile (horizontal intercept).
  - ○ Portfolio 2 has steeper slope, which implies lower cost of robustness.

§ **Sequence matters,** fig. 24.

- Portfolios 1 and 2 same as fig. 23.
- Portfolio 3 and 4 are similar, and without investment change over time.

16 Military Effectiveness: Net Assessment with WEI-WUV

§ This section draws on problem 88 in ps2-02.tex: Evaluating a complex system with sub-systems of uncertain importance.

§ WEI-WUV: Weapon Effectiveness Index-Weapon Unit Value.

16.1 Problem Formulation

§ We consider the design of a complex system with sub-systems and sub-sub-systems. We evaluate the overall system with a quadratic function expressing the importance of the sub- and sub-sub-systems. This evaluation is uncertain, so the design is uncertain. We evaluate the robustness to this uncertainty, as the basis for design decisions.

§ Consider N different sub-systems, where each sub-system has J sub-sub-systems. Let q_{nj} denote the quantity of resources devoted to sub-sub-system j in sub-system n. Q is the $N \times J$ matrix of quantities q_{nj}. The overall effectiveness of the system is evaluated as:

$$E = \sum_{n=1}^{N} v_n \sum_{j=1}^{J} q_{nj} w_{nj} \tag{250}$$

where $v \in \Re^N$ is the vector of "values" of the sub-systems, and $w \in \Re^J$ is the vector of "worths" of the sub-sub-systems. We would like to choose the quantities, Q, so that the effectiveness if large.

§ The values and worths are uncertain according to a fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ v, W: \ v_n \ge 0, \ \left| \frac{v_n - \widetilde{v}_n}{s_n} \right| \le h, \ \forall\, n. \ w_{nj} \ge 0, \ \left| \frac{w_{nj} - \widetilde{w}_{nj}}{t_{nj}} \right| \le h, \ \forall\, j, n \right\}, \quad h \ge 0 \tag{251}$$

where the s_n's and t_{jn}'s are known and positive.

§ We will also sometimes consider uncertainty in the quantities q_{nj}, in which case the info-gap model of eq.(251) becomes modified as:

$$\mathcal{U}(h) = \left\{ v, W, Q: \quad v_n \ge 0, \ \left| \frac{v_n - \widetilde{v}_n}{s_n} \right| \le h, \ \forall\, n. \ \ w_{nj} \ge 0, \ \left| \frac{w_{nj} - \widetilde{w}_{nj}}{t_{nj}} \right| \le h, \ \forall\, j, n. \right.$$
$$\left. q_{nj} \ge 0, \ \left| \frac{q_{nj} - \widetilde{q}_{nj}}{u_{nj}} \right| \le h, \ \forall\, j, n \ \right\}, \quad h \ge 0 \tag{252}$$

where the s_n's, t_{jn}'s and u_{nj}'s are known and positive. Uncertainty in v and W reflects uncertainty in assessing the importance of various sub-systems. Uncertainty in Q reflects uncertainty in the actual quantities that will be produced and available for use. This production uncertainty is particularly relevant for new technologies whose production may entail unknown development challenges.

16.2 WEI-WUV Data

Consider a numerical implementation based on the WEI-WUV data in fig. 25. There are 9 weapon categories (tank, attack helicopters, etc.), so $N = 9$. Each category has either 1, 2 or 3 weapon

CHAPTER II ASSESSING THE BALANCE OF NATO AND PACT GROUND FORCES 15

TABLE 3. SAMPLE WEI/WUV CALCULATION OF A COMBAT DIVISION

Type of Weapon	Number in Unit	Weapon Effective-ness Index (WEI)	Product (Number x WEI)	Weighted Unit Value (WUV)	Total Score (Total product x WUV)
Tanks					
M60A3	150	1.11	166		
M1	150	1.31	197		
Total			363	94	34,122
Attack Helicopters					
AH-1S	21	1.00	21		
AH-64	18	1.77	32		
Total			53	109	5,777
Air Defense Weapons					
Vulcan	24	1.00	24	56	1,344
Infantry Fighting Vehicles					
Bradley fighting vehicle	228	1.00	228	71	16,188
Antitank Weapons					
TOW missile launcher	150	0.79	119		
Dragon launcher	240	0.69	166		
LAW	300	0.20	60		
Total			344	73	25,112
Artillery					
155mm howitzer	72	1.02	73		
8-inch howitzer	12	0.98	12		
MLRS	9	1.16	10		
Total			96	99	9,504
Mortars					
81mm	45	0.97	44		
107mm	50	1.00	50		
Total			94	55	5,170
Armored Personnel Carriers					
M113	500	1.00	500	30	15,000
Small Arms					
M16 rifle	2,000	1.00	2,000		
Machine guns	295	1.77	522		
Total			2,522	4	10,088
Division Total					122,305

The division's score in terms of ADEs = division score/norm for U.S. armored division. For this example, the division score = 122,305. When it is divided by the norm for a U.S. armored division--130,458--it is converted into ADEs. In this case, the illustrative division would be worth 0.94 ADEs.

SOURCE: Compiled by Congressional Budget Office from data in Department of the Army, U.S. Army Concepts Analysis Agency, *Weapon Effectiveness Indices/Weighted Unit Values III (WEI/WUV III)* (November 1979).

NOTES: TOW = tube-launched, optically tracked, wire-guided; LAW = light antitank weapon; MLRS = multiple launch rocket system; ADE = armored division equivalent.

Figure 25: U.S. Congressional Budget Office, *U.S. Ground Forces and the Conventional Balance in Europe,* U.S. Government Printing Office, June 1988, p.15. https://www.cbo.gov/sites/default/files/100th-congress-1987-1988/reports/doc01b-entire.pdf, accessed 9.2.2016. See also fig.6.4 on p.143 in Andrew F. Krepinevich and Barry D. Watts, 2015, *The Last Warrior: Andrew Marshall and the Shaping of Modern American Defense Strategy*, Basic Books, New York.

types. Thus choose $J = 3$ and specify $w_{nj} = q_{nj} = 0$ when j exceeds the number of weapon types in category n. The category values v_n (called category weights in fig. 25) are:

$$v^T = (94,\ 109,\ 56,\ 71,\ 73,\ 99,\ 55,\ 30,\ 4) \tag{253}$$

The Weight Effectiveness Indices (WEI's) from fig. 25 are:

$$W = \begin{pmatrix} 1.11 & 1.31 & 0 \\ 1.00 & 1.77 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0.79 & 0.69 & 0.20 \\ 1.02 & 0.98 & 1.16 \\ 0.97 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1.77 & 0 \end{pmatrix} \tag{254}$$

The quantities of weapon types specified in fig. 25 are:

$$Q = \begin{pmatrix} 150 & 150 & 0 \\ 21 & 18 & 0 \\ 24 & 0 & 0 \\ 228 & 0 & 0 \\ 150 & 240 & 300 \\ 72 & 12 & 9 \\ 45 & 50 & 0 \\ 500 & 0 & 0 \\ 2,000 & 295 & 0 \end{pmatrix} \tag{255}$$

16.3 Deriving the Robustness with Uncertain v and W

The robustness is defined as:

$$\widehat{h}(Q, E_{\rm c}) = \max \left\{ h : \ \left(\min_{v,W \in \mathcal{U}(h)} E(v, W) \right) \geq E_{\rm c} \right\} \tag{256}$$

Denote the inner minimum $m(h)$. Because the elements of Q are non-negative by definition, and the elements of v and W are non-negative according to the info-gap model of eq.(251), p.75, the inner minimum occurs for:

$$v_n = (\widetilde{v}_n - s_n h)^+, \ \ w_{nj} = (\widetilde{w}_{nj} - t_{nj} h)^+ \tag{257}$$

where $x^+ = x$ if $x > 0$ and equals 0 otherwise. Thus the inverse of the robustness function is:

$$m(h) = \sum_{n=1}^{N} (\widetilde{v}_n - s_n h)^+ \sum_{j=1}^{J} q_{nj} (\widetilde{w}_{nj} - t_{nj} h)^+ \tag{258}$$

Let us define the sets of indices, $J(n)$, $n = 1, \ldots, N$, for which $q_{nj} > 0$:

$$J(n) = \{ j : \ q_{nj} > 0 \} \tag{259}$$

Now we can re-write eq.(258) as:

$$m(h) = \sum_{n=1}^{N} (\widetilde{v}_n - s_n h)^+ \sum_{j \in J(n)} q_{nj} (\widetilde{w}_{nj} - t_{nj} h)^+ \tag{260}$$

A plot of h vs. $m(h)$ is equivalent to a plot of $\widehat{h}(E_{\rm c})$ vs. $E_{\rm c}$. The horizontal intercept (on the $E_{\rm c}$ axis) occurs when $E_{\rm c} = E(\widetilde{v}, \widetilde{W})$, defined in eq.(250), p.75. The vertical intercept occurs at the value of h for which $m(h) = 0$. We now show that the vertical intercept does not depend on the magnitudes of the non-zero elements of Q.

It is evident from eq.(260) that:

$$m(h) > 0 \quad \text{iff} \quad \exists\, n \text{ s.t. } h < \frac{\widetilde{v}_n}{s_n} \text{ and } h < \max_{j \in J(n)} \frac{\widetilde{w}_{nj}}{t_{nj}} \tag{261}$$

$$\text{iff} \quad h < \max_{1 \leq n \leq N} \ \min \left[\frac{\widetilde{v}_n}{s_n}, \ \max_{j \in J(n)} \frac{\widetilde{w}_{nj}}{t_{nj}} \right] \tag{262}$$

The vertical intercept of the robustness curve is the least upper bound of the h values that satisfy eq.(262). Denote this value $h_{\max}$. This value does not depend on the magnitudes of the non-zero elements of Q.

16.4 Robustness to Uncertain v and W with Constant Fractional Errors

§ Consider a special numerical case in which the fractional errors are the same for all terms:

$$\frac{s_n}{\widetilde{v}_n} = \nu \text{ for all } n \quad \text{and} \quad \frac{t_{nj}}{\widetilde{w}_{nj}} = \varepsilon \text{ for all } n,\, j \tag{263}$$

§ Thus, from eq.(262):

$$h_{\max} = \min\left[\frac{1}{\nu},\ \frac{1}{\varepsilon}\right] \tag{264}$$

§ A robustness curve for this special case is shown in fig. 26.[4]

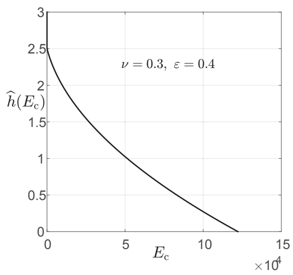

Figure 26: Robustness curve for the special case in eq.(264), with $\widetilde{v}$, $\widetilde{W}$ and Q in eqs.(253)–(255).

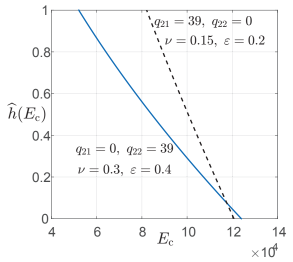

Figure 27: Robustness curves for the special case in eq.(264), with $\widetilde{v}$ and $\widetilde{W}$ in eqs.(253) and (254), and Q in eq.(255) modified as shown in the figure.

§ Fig. 27 shows robustness curves for two different configurations. The solid curve has 39 AH-64 and no AH-1S attack helicopters, while the dashed curve has no AH-64 and 39 AH-1S. The nominal performance is better in the first case (solid) than in the second case (dashed). However, the uncertainty weights are greater for the solid than for the dashed case, so, from eq.(264), we see that $h_{\max}$ is less in the solid case. This causes the cost of robustness to be greater in the solid case. Consequently, the robustness curves cross one another, resulting in the potential for preference reversal.

§ The rationale for the different fractional errors in the two cases, as specified in fig. 27, is based on two considerations. First, we are supposing that the AH-1S is less familiar and hence more uncertain than the AH-64. Second, we are supposing an inter-connectedness of the sub-sub-systems which causes a propagation of uncertainty. Thus the greater uncertainty of the AH-1S induces greater uncertainty of the other elements as well.

16.5 Deriving the Robustness with Uncertain v, W and Q

The robustness is defined as:

$$\widehat{h}(\widetilde{Q}, E_c) = \max\left\{h:\ \left(\min_{v,W,Q\in\mathcal{U}(h)} E(v,W,Q)\right) \ge E_c\right\} \tag{265}$$

[4]Calculations for figs. 26 and 27 done with matlab program c:/Ben-Haim/LECTURES/Info-Gap-Methods/Homework/weiwuv001.m

Denote the inner minimum $m(h)$. Because the elements of v, W and Q are non-negative according to the info-gap model of eq.(252), p.75, the inner minimum occurs for:

$$v_n = (\widetilde{v}_n - s_n h)^+, \quad w_{nj} = (\widetilde{w}_{nj} - t_{nj} h)^+, \quad q_{nj} = (\widetilde{q}_{nj} - u_{nj} h)^+ \tag{266}$$

where $x^+ = x$ if $x > 0$ and equals 0 otherwise. Thus the inverse of the robustness function is:

$$m(h) = \sum_{n=1}^{N} (\widetilde{v}_n - s_n h)^+ \sum_{j=1}^{J} (\widetilde{q}_{nj} - u_{nj} h)^+ (\widetilde{w}_{nj} - t_{nj} h)^+ \tag{267}$$

In analogy to eq.(261), we see that the vertical intercept of the robustness curve is the least upper bound of the set of h values for which:

$$m(h) > 0 \quad \text{iff} \quad \exists\, n \ \text{ s.t. } h < \frac{\widetilde{v}_n}{s_n} \text{ and s.t. } \left(\exists\, j \text{ s.t. } h < \frac{\widetilde{q}_{nj}}{u_{nj}} \text{ and } h < \frac{\widetilde{w}_{nj}}{t_{nj}} \right) \tag{268}$$

16.6 Comparing Two Configurations

§ Calculations done with matlab problem c:/Ben-Haim/LECTURES/Info-Gap-Methods/Homework/weiwuv002.m
Let's compare two alternative systems structures. In the first option the estimated values and quantities are $\widetilde{v}^{(1)}$, $\widetilde{W}^{(1)}$ and $\widetilde{Q}^{(1)}$ in eqs.(253)–(255). The number of weapon categories is $N^{(1)} = 9$. The second option includes a new weapons system, so now $N^{(2)} = 10$ and the estimated quantities are as follows.
The value vector v compares the alternative weapons systems. In order for the comparison of the two options to be fair, we require the nominal value vectors to have the same sum:

$$\sum_{n=1}^{N^{(1)}} \widetilde{v}_n^{(1)} = \sum_{n=1}^{N^{(2)}} \widetilde{v}_n^{(2)} \tag{269}$$

Thus we define $\widetilde{v}^{(2)}$ by appending a new element, $v_\star$, and normalizing. First define V_1 as the left sum in eq.(269). Now define $\widetilde{v}^{(2)}$:

$$\widetilde{v}^{(2)} = \frac{V_1}{V_1 + v_\star} [\widetilde{v}^{(1)},\ v_\star] \tag{270}$$

We choose $v_\star = 150$, so we find:

$$\widetilde{v}^{(2)T} \approx (75,\ 87,\ 45,\ 57,\ 58,\ 79,\ 44,\ 24,\ 3.2,\ \mathbf{120}) \tag{271}$$

The matrix of estimated WEI's, $\widetilde{W}^{(2)}$, is obtained by adding a 10th row to W in eq.(254), where the new system is estimated to have an effectiveness weight of 2:

$$\widetilde{W}^{(2)} = \begin{pmatrix} 1.11 & 1.31 & 0 \\ 1.00 & 1.77 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0.79 & 0.69 & 0.2 \\ 1.02 & 0.98 & 1.16 \\ 0.97 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1.77 & 0 \\ \mathbf{2} & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{272}$$

The matrix of estimated production quantities, $\widetilde{Q}^{(2)}$, is obtained by adding a 10th row to Q in eq.(255), where 100 units of the new weapon are expected to be produced:

$$\widetilde{Q}^{(2)} = \begin{pmatrix} 150 & 150 & 0 \\ 21 & 18 & 0 \\ 24 & 0 & 0 \\ 228 & 0 & 0 \\ 150 & 240 & 300 \\ 72 & 12 & 9 \\ 45 & 50 & 0 \\ 500 & 0 & 0 \\ 2,000 & 295 & 0 \\ \mathbf{100} & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{273}$$

Consider a special numerical case. For option 1:

$$\frac{s_n^{(1)}}{\widetilde{v}_n^{(1)}} = \nu \text{ for all } n. \quad \frac{t_{nj}^{(1)}}{\widetilde{w}_{nj}^{(1)}} = \varepsilon \text{ for all } n, j. \quad \frac{u_{nj}^{(1)}}{\widetilde{q}_{nj}^{(1)}} = \phi \text{ for all } n, j \tag{274}$$

From eq.(268) we see that the vertical intercept of the robustness curve for option 1 is the least upper bound of the set of h values for which:

$$m(h) > 0 \quad \text{iff} \quad h < \frac{1}{\nu} \text{ and } \left(h < \frac{1}{\varepsilon} \text{ and } h < \frac{1}{\phi} \right) \tag{275}$$

$$\text{iff} \quad h < \min\left[\frac{1}{\nu}, \frac{1}{\varepsilon}, \frac{1}{\phi}\right] \tag{276}$$

Thus, for option 1, the vertical intercept of the robustness curve is:

$$h_{\max}^{(1)} = \min\left[\frac{1}{\nu}, \frac{1}{\varepsilon}, \frac{1}{\phi}\right] \tag{277}$$

This does not depend on the anticipated production quantities, $\widetilde{Q}^{(1)}$.
Now consider option 2. The new innovative option does not have any systemic effect, so we have the same uncertainty weights as for option 1 except for the new weapon system, which may have different uncertainty:

$$\frac{s_n^{(2)}}{\widetilde{v}_n^{(2)}} = \begin{cases} \nu & n < N^{(2)} \\ \xi\nu & n = N^{(2)} \end{cases} \tag{278}$$

$$\frac{t_{nj}^{(2)}}{\widetilde{w}_{nj}^{(2)}} = \begin{cases} \varepsilon & \text{for all } j \text{ when } n < N^{(2)} \\ \xi\varepsilon & \text{for all } j \text{ when } n = N^{(2)} \end{cases} \tag{279}$$

$$\frac{u_{nj}^{(2)}}{\widetilde{q}_{nj}^{(2)}} = \begin{cases} \phi & \text{for all } j \text{ when } n < N^{(2)} \\ \xi\phi & \text{for all } j \text{ when } n = N^{(2)} \end{cases} \tag{280}$$

A value $\xi > 1$ implies greater uncertainty for production of the 10th weapon system. If $\xi > 1$, then we see from eq.(268) that the condition '$\exists\ n$ s.t.' holds for $n < N^{(2)}$ for a larger value of h than for $n = N^{(2)}$. Thus the vertical intercept of the robustness curve—when $\xi > 1$—is the same as for option 1:

$$h_{\max}^{(2)} = h_{\max}^{(1)} \tag{281}$$

Note that if the improved innovative system was **less** uncertain, so $\xi < 1$, then the vertical intercept of option 2 would be greater than for option 1.

In summary, we see that the vertical intercept does not change between the two systems when the innovative system is more uncertain. However, the horizontal intercept, $E(\tilde{v}, \widetilde{W}, \widetilde{Q})$ is greater for option 2 than for option one. Thus option 2 robust-dominates option 1. This is illustrated in fig. 28.

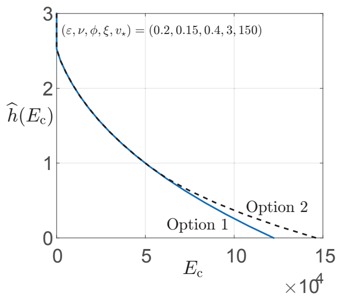

Figure 28: Robustness curves for the two options in eqs.(271)–(280).

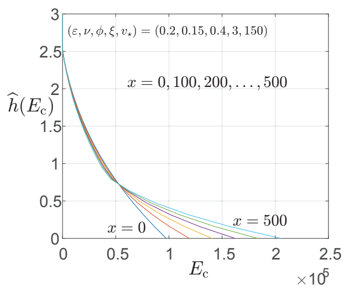

Figure 29: Robustness curves for the second example, with Q in eq.(282).

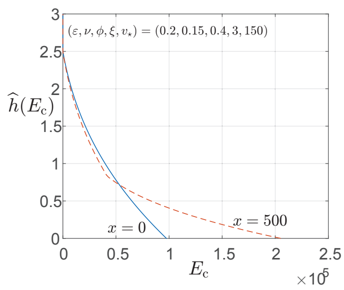

Figure 30: Robustness curves for the second example, with Q in eq.(282).

16.7 Comparing Two Configurations with Quantity Limitation

§ Calculations done with c:/Ben-Haim/LECTURES/Info-Gap-Methods/Homework/weiwuv003.m

§ Now let us suppose that the new system (item $n = 10$ in the previous example) is a new type of APC. Furthermore, it can be introduced only at the expense of item $n = 9$, the standard M113 APC. Thus the quantity options are:

$$\widetilde{Q} = \begin{pmatrix} 150 & 150 & 0 \\ 21 & 18 & 0 \\ 24 & 0 & 0 \\ 228 & 0 & 0 \\ 150 & 240 & 300 \\ 72 & 12 & 9 \\ 45 & 50 & 0 \\ 500 - x & 0 & 0 \\ 2,000 & 295 & 0 \\ x & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad (282)$$

where x is an integer between 0 and 500. Thus $N = 10$, $J = 3$, and $\tilde{v}$ and $\widetilde{W}$ are specified in eqs.(270) and (272). The uncertainty weights are specified as before, by eqs.(278)–(280) where $N^{(2)} = N = 10$.

Robustness curves are shown in fig. 29. The two extreme curves in this figure are reproduced in fig. 30. The estimated WEI-WUV index increases as the number of new systems increases because of their greater estimated quality. The horizontal intercept equals the estimated WEI-WUV index. Thus the robustness curves stretch to the right as the number of new systems increases. However, the vertical intercept is constant, as explained previously. Nonetheless, there is some weak intermediate crossing of robustness curves.

16.8 Comparing Two APC's

[5] We now consider an innovation dilemma expressed by focussing exclusively on the trade off between the standard APC, the M113, and a hypothetical innovative APC. The relevant matrices are derived from eqs.(271), (272) and (282) as follows, with $N = 2$ and $J = 1$. From the vector in eq.(271) we take elements 8 and 10:

$$\widetilde{v}^T = (24,\ 120) \tag{283}$$

From the matrix in eq.(272) we take elements (8,1) and (10,1):

$$\widetilde{W}^{(2)} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \tag{284}$$

From the matrix in eq.(282) we take elements (8,1) and (10,1):

$$\widetilde{Q} = \begin{pmatrix} 500 - x \\ x \end{pmatrix} \tag{285}$$

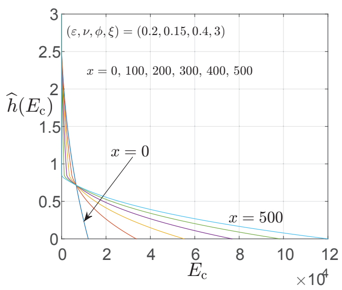

Figure 31: Robustness curves for the two APC options in eqs.(283)–(285).

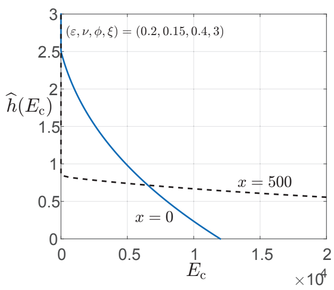

Figure 32: Robustness curves for the two APC options in eqs.(283)–(285).

Robustness curves are shown in fig. 31, with the two extreme curves reproduced, in part, in fig. 32. Notice the strong innovation dilemma and potential for preference reversal between the case of no innovative APC's ($x = 0$, solid) and 500 innovative APC's ($x = 500$, dashed).

16.9 Robustness of Decision Stability

16.9.1 Formulation

§ Consider the **choice between two alternatives,** specified by quantity matrices Q_1 and Q_2, where the overall effectiveness of each alternative is specified by eq.(250), p.75.

§ **Info-gap model:** Consider the uncertainty in the info-gap model of eq.(252), p.75.

§ Suppose that **alternative 1 is nominally preferred:**

$$E(\widetilde{v}, \widetilde{W}, \widetilde{Q}_1) > E(\widetilde{v}, \widetilde{W}, \widetilde{Q}_2) \tag{286}$$

[5] Calculations done with weiwuv004.m

§ **The robustness question is:** what is the greatest horizon of uncertainty, h, up to which this nominal robustness preference does not change?

• More precisely, what is the maximum h up to which alternative 1 is preferred over alternative 2 by a margin no less than Δ?

• Formally, the robustness is defined as:

$$\widehat{h}(\Delta) = \max \left\{ h : \left(\min_{v,W,Q \in \mathcal{U}(h)} [E(v,W,Q_1) - E(v,W,Q_2)] \right) \geq \Delta \right\} \tag{287}$$

§ Let $m(h)$ denote the **inner minimum** of eq.(287), which is the **inverse of the robustness function, $\widehat{h}(\Delta)$**.

§ **The system model is:**

$$E(A_1) - E(A_2) = \sum_{n=1}^{N} v_n \sum_{j=1}^{J} \left(q_{nj}^{(1)} - q_{nj}^{(2)} \right) w_{nj} \tag{288}$$

where the elements of Q_i are denoted $q_{nj}^{(i)}$.

§ **Evaluating the inverse of the robustness,** $m(h)$.

• From the info-gap model, eq.(252), p.75, v_{nj} and w_{nj} are non-negative. Hence the inner minimum occurs for $q_{nj}^{(1)} - q_{nj}^{(2)}$ as small as possible:

$$q_{nj}^{(1)} = \left(\widetilde{q}_{nj}^{(1)} - u_{nj} h \right)^{+} \tag{289}$$

$$q_{nj}^{(2)} = \widetilde{q}_{nj}^{(2)} + u_{nj} h \tag{290}$$

• w_{nj} is extremal, either minimal or maximal, depending on the sign of $q_{nj}^{(1)} - q_{nj}^{(2)}$ from eqs.(289) and (290):

$$w_{nj} = \begin{cases} (\widetilde{w}_{nj} - t_{nj} h)^{+} & \text{if } q_{nj}^{(1)} - q_{nj}^{(2)} \geq 0 \\ \widetilde{w}_{nj} + t_{nj} h & \text{else} \end{cases} \tag{291}$$

• v_{nj} is extremal, either minimal or maximal, depending on the sign of the sum on j from eq.(291):

$$v_n = \begin{cases} (\widetilde{v}_n - s_n h)^{+} & \text{if } \sum_{j=1}^{J} \left(q_{nj}^{(1)} - q_{nj}^{(2)} \right) w_{nj} \geq 0 \\ \widetilde{v}_n + s_n h & \text{else} \end{cases} \tag{292}$$

• Finally, $m(h)$ is obtained from eq.(288) with eqs.(289)–(292).

16.9.2 Example 1: Parameter Uncertainty

§ We will compare the robustness of the full system in three configurations: standard, innovative and conservative.[6]

§ The three configurations are distinguished in the acquisition values for tanks, attack helicopters and antitank weapons and in the uncertainties of the associated WEI values.

• Tanks: M60A1 is standard, while M1 is an advanced innovative model as reflected in the greater WEI value for the M1 (1.31 vs. 1.11).

• Attack helicopters: AH-1S is standard, while AH-64 is an advanced innovative model as reflected in the greater WEI value for AH-64 (1.77 vs. 1.00).

• Antitank weapons: LAW is standard, while Dragon and TOW are advanced innovative models as reflected in the greater WEI values for Dragon and TOW (0.69 and 0.79 vs. 0.20).

§ The 3 nominal acquisition quantities are $\widetilde{Q}_1$ (standard), $\widetilde{Q}_2$ (innovative) and $\widetilde{Q}_3$ (conservative):

$$\widetilde{Q}_1 = \begin{pmatrix} 150 & 150 & 0 \\ 21 & 18 & 0 \\ 24 & 0 & 0 \\ 228 & 0 & 0 \\ 150 & 240 & 300 \\ 72 & 12 & 9 \\ 45 & 50 & 0 \\ 500 & 0 & 0 \\ 2,000 & 295 & 0 \end{pmatrix}, \quad \widetilde{Q}_2 = \begin{pmatrix} \mathbf{0} & \mathbf{300} & \mathbf{0} \\ \mathbf{0} & \mathbf{39} & \mathbf{0} \\ 24 & 0 & 0 \\ 228 & 0 & 0 \\ \mathbf{540} & \mathbf{150} & \mathbf{0} \\ 72 & 12 & 9 \\ 45 & 50 & 0 \\ 500 & 0 & 0 \\ 2,000 & 295 & 0 \end{pmatrix}, \quad \widetilde{Q}_3 = \begin{pmatrix} \mathbf{300} & \mathbf{0} & \mathbf{0} \\ \mathbf{39} & \mathbf{0} & \mathbf{0} \\ 24 & 0 & 0 \\ 228 & 0 & 0 \\ \mathbf{0} & \mathbf{0} & \mathbf{690} \\ 72 & 12 & 9 \\ 45 & 50 & 0 \\ 500 & 0 & 0 \\ 2,000 & 295 & 0 \end{pmatrix} \tag{293}$$

§ Using the robustness of eq.(287), we will compare:

• Standard vs. Innovative: configurations 1 and 2.
• Standard vs. Conservative: configurations 1 and 3.
• Innovative vs. Conservative: configurations 2 and 3.

§ We will use the info-gap model of eq.(252), p.75. Uncertainty in v, W and Q.

§ The uncertainty weights for acquisitions are:

$$U_i = \nu \widetilde{Q}_i, \ i = 1, \ 2, \ 3 \tag{294}$$

§ The nominal values $\widetilde{v}$ and $\widetilde{W}$ are eqs.(253) and (254).

• The uncertainty weights for v are:

$$s = \nu \widetilde{v} \tag{295}$$

• The uncertainty weights for W are:

$$T_{nj} = \begin{cases} \varepsilon \nu \widetilde{W}_{nj} & \text{for } (n,j) = (1,2), (2,2), (5,1), (5,2) \\ \nu \widetilde{W}_{nj} & \text{else} \end{cases} \tag{296}$$

Thus $\widetilde{W}$ has enhanced uncertainty for the innovative models: M1, AH-64, TOW and Dragon.

[6] Computations with \LECTURES\Info-Gap-Methods\Lectures\decstab001.m.

§ The nominal estimates of the effectiveness, eq.(250), p.75, of the 3 options are:

$$E(\tilde{v}, \widetilde{W}, \widetilde{Q}_1) = 1.2224 \times 10^5 \quad \text{(standard)} \tag{297}$$
$$E(\tilde{v}, \widetilde{W}, \widetilde{Q}_2) = 1.4040 \times 10^5 \quad \text{(innovative)} \tag{298}$$
$$E(\tilde{v}, \widetilde{W}, \widetilde{Q}_3) = 1.0287 \times 10^5 \quad \text{(conservative)} \tag{299}$$

§ Thus the nominal effectiveness differences are:

$$E(\tilde{v}, \widetilde{W}, \widetilde{Q}_2) - E(\tilde{v}, \widetilde{W}, \widetilde{Q}_1) = 1.8161 \times 10^4 \quad \text{(innvovative vs standard)} \tag{300}$$
$$E(\tilde{v}, \widetilde{W}, \widetilde{Q}_1) - E(\tilde{v}, \widetilde{W}, \widetilde{Q}_3) = 1.9376 \times 10^4 \quad \text{(standard vs conservative)} \tag{301}$$
$$E(\tilde{v}, \widetilde{W}, \widetilde{Q}_2) - E(\tilde{v}, \widetilde{W}, \widetilde{Q}_3) = 3.7537 \times 10^4 \quad \text{(innovative vs conservative)} \tag{302}$$

§ Thus the **nominal preferences** are:

$$\widetilde{Q}_2 \succ \widetilde{Q}_1 \succ \widetilde{Q}_3 \tag{303}$$
$$\text{(innovative)} \succ \text{(standard)} \succ \text{(conservative)} \tag{304}$$

- Comparing eqs.(300) and (301): innov. $\succ$ stand. about as much as stand. $\succ$ conserv.
- Comparing eqs.(302) and (300): innov. $\succ$ conserv. about twice as much as innov. $\succ$ stand.

§ The nominal effectiveness differences seem substantial, compared to the average effectivenesses:

$$\frac{E(\tilde{v}, \widetilde{W}, \widetilde{Q}_2) - E(\tilde{v}, \widetilde{W}, \widetilde{Q}_1)}{[E(\tilde{v}, \widetilde{W}, \widetilde{Q}_2) + E(\tilde{v}, \widetilde{W}, \widetilde{Q}_1)]/2} = 0.1383 \quad \text{(innvovative vs standard)} \tag{305}$$
$$\frac{E(\tilde{v}, \widetilde{W}, \widetilde{Q}_1) - E(\tilde{v}, \widetilde{W}, \widetilde{Q}_3)}{[E(\tilde{v}, \widetilde{W}, \widetilde{Q}_1) + E(\tilde{v}, \widetilde{W}, \widetilde{Q}_3)]/2} = 0.1721 \quad \text{(standard vs conservative)} \tag{306}$$
$$\frac{E(\tilde{v}, \widetilde{W}, \widetilde{Q}_2) - E(\tilde{v}, \widetilde{W}, \widetilde{Q}_3)}{[E(\tilde{v}, \widetilde{W}, \widetilde{Q}_2) + E(\tilde{v}, \widetilde{W}, \widetilde{Q}_3)]/2} = 0.3086 \quad \text{(innovative vs conservative)} \tag{307}$$

§ **Robustness question:** How robust are these preferences to uncertainty in the WEI's W, WUV's v, and production quantities Q?

§ **Robustness curves** in fig. 33, based on eq.(287), for small uncertainty weights:
- **Zeroing** at nominal comparison values in eqs.(300)–(302).
- Innov.–Conserv. (2–3) most robust at $\Delta > 0$.
- However, **strong robustness trade off** as seen by low robustness at $\Delta = 0$.
- Conclusion: **Weak robustness preferences** in all three full-system comparisons.
  The 3 sub-system innovations don't strongly impact the full-system effectiveness preferences when full-system robustness is considered. This motivates example in next sub-subsection.

§ **Robustness curves** in figs. 34–35 for larger uncertainty weights:
- Similar conclusions.
- Much stronger robustness trade off: note larger scale on Δ axis.

§ **Robustness curves** in figs. 36–38 for uniform uncertainty weights: Similar conclusions.

§ **General conclusions:**
- **Nominal preferences seem substantial:** eqs.(300)–(307).
- **These preferences are not robust to uncertainty** in WEI-WUV's and production quantities.

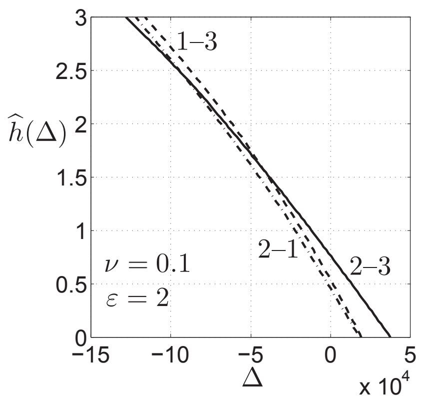

Figure 33: Robustness curves for comparing three options in eq.(293).

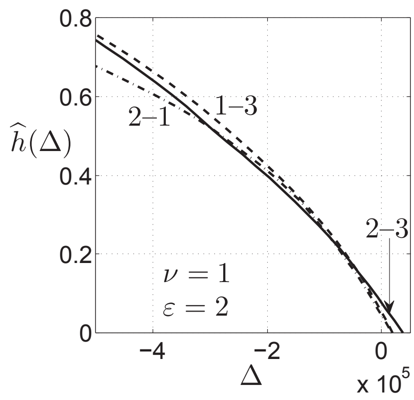

Figure 34: Robustness curves for comparing three options in eq.(293). Larger uncertainty weights.

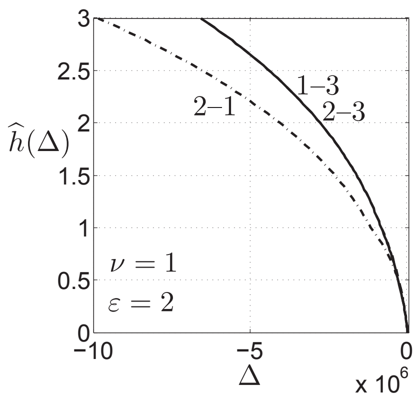

Figure 35: Same as fig. 34, reduced scale.

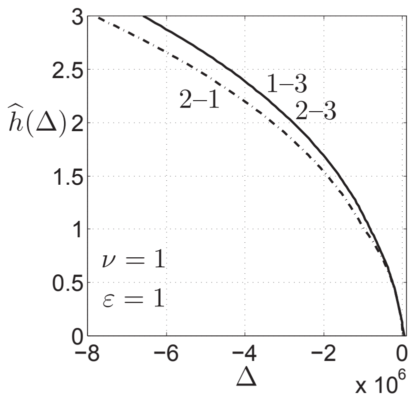

Figure 36: Robustness curves for comparing three options in eq.(293).

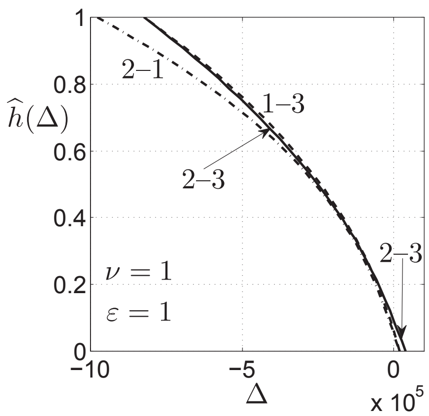

Figure 37: Same as fig. 36, expanded scale.

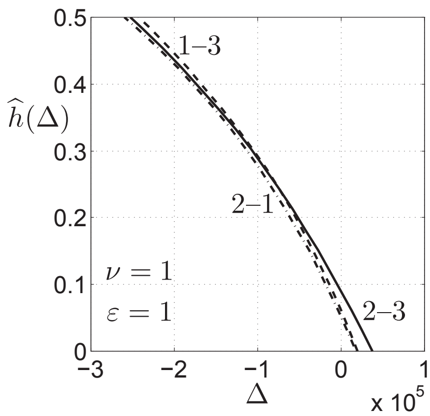

Figure 38: Same as fig. 36, expanded scale.

§ **Robustness with uncertainty only in WEI-WUV's, figs.39–41:**[7]
- s from eq.(295). T from eq.(296).
- $U = 0$, not eq.(294), so **no production uncertainty.**

§ **General conclusions:**
- Basically same as before.
- **Nominal preferences seem substantial:** eqs.(300)–(307).

[7]Computations with \LECTURES\Info-Gap-Methods\Lectures\decstab002.m.

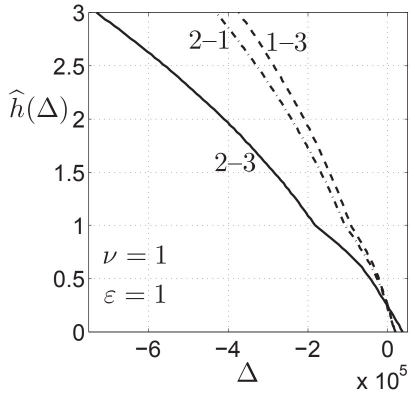

Figure 39: Robustness curves for comparing three options in eq.(293). Only WEI-WUV uncertainty.

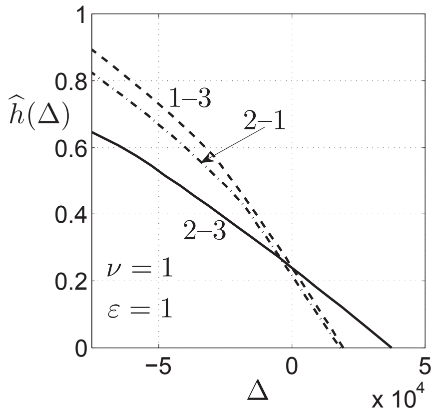

Figure 40: Same as fig. 39, expanded scale.

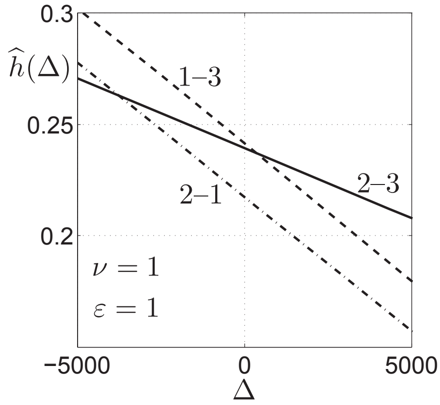

Figure 41: Same as fig. 39, expanded scale.

- Option 2 (innov) most robustly preferred over option 3 (conserv) for $\Delta > 0$, but only at low $\widehat{h}$.
- Option 1 (stand) next most robustly preferred over option 3 (conserv) for $\Delta > 0$, at low $\widehat{h}$.
- Option 2 (innov) least robustly preferred over option 1 (stand) for $\Delta > 0$, at low $\widehat{h}$.
- **These preferences are not robust to uncertainty** in WEI-WUV's, v and W.
- Kink in robustness curves at $\widehat{h} = 1$: due to zeroing of some elements of W and v.
  See eqs.(291), (292).

§ **Consider uncertainty only in WEI's of innovative systems, fig.42.**[8]

- No uncertainty in WUV's, v, so $s = 0$.
- No uncertainty in production quantities, Q, so $U = 0$.
- Uncertainty in WEI's of innovative systems only, so:

$$T_{nj} = \begin{cases} \varepsilon \widetilde{W}_{nj} & \text{for } (n,j) = (1,2),(2,2),(5,1),(5,2) \\ 0 & \text{else} \end{cases} \tag{308}$$

Thus $\widetilde{W}$ has uncertainty only for the innovative models: M1, AH-64, TOW and Dragon.

[8]Computations with \LECTURES\Info-Gap-Methods\Lectures\decstab003.m.

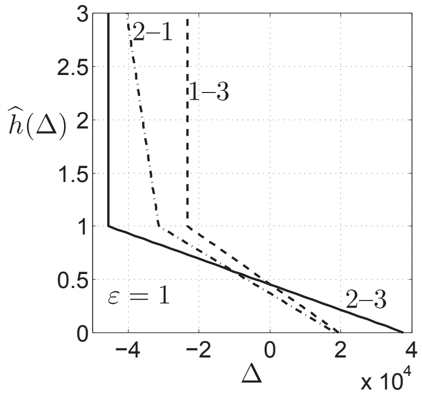

Figure 42: Robustness curves for comparing three options in eq.(293). Only WEI uncertainty and only for innovative sub-systems.

16.9.3 Example 2: Model-Structure Uncertainty

§ The effectiveness function for acquisition Q is modified from eq.(250), p.75:

$$E(Q,f) = \sum_{n=1}^{N} v_n \sum_{j=1}^{J} q_{nj} w_{nj} + f(Q) \tag{309}$$

where the function $f(Q)$ is uncertain.

§ The uncertain function $f(Q)$ may be:
• Quadratic:

$$f(Q) = q^T C q \tag{310}$$

where q is a vector form of Q and C is a symmetric matrix. $c_{nj} > 0$ reflects positive synergistic interaction between systems n and j. Conversely, $c_{nj} < 0$ reflects negative competitive interaction between systems n and j.
• Other non-linear form, containing higher-order powers.
• Discontinuous function to reflect abrupt changes in effectiveness as the force structure changes.

§ We consider decision stability, where option $\widetilde{Q}_i$ is nominally preferred over option $\widetilde{Q}_j$ as in eq.(286), p.82:

$$E(Q_i, 0) > E(Q_j, 0) \tag{311}$$

§ Define the nominal effectiveness: $\widetilde{E}_i = E(Q_i, 0)$, and the average nominal effectiveness: $\overline{E}_{ij} = (\widetilde{E}_i + \widetilde{E}_j)/2$.

§ The info-gap model for model-structure uncertainty, in considering decision stability of preference for Q_i over Q_j, is:

$$\mathcal{F}(h) = \left\{ f(Q) : \ \left| \frac{f(Q)}{\overline{E}_{ij}} \right| \le h, \ \forall \ Q \right\}, \quad h \ge 0 \tag{312}$$

• Meaning: The fractional contribution of the unknown term, $f(Q)$, relative to the average nominal effectiveness, $\overline{E}_{ij}$, is bounded but unknown.

§ The robustness for preferring Q_i over Q_j is defined as in eq.(287), p.83:

$$\widehat{h}(\Delta) = \max \left\{ h : \ \left(\min_{f(Q) \in \mathcal{F}(h)} [E(Q_i, f) - E(Q_j, f)] \right) \ge \Delta \right\} \tag{313}$$

§ Deriving the robustness:
• Let $m(h)$ denote the inner minimum in eq.(313). This is the inverse of $\widehat{h}(\Delta)$.
• $m(h)$ occurs for:

$$f(Q_i) = -h\overline{E}_{ij}, \quad f(Q_j) = +h\overline{E}_{ij} \implies m(h) = \widetilde{E}_i - \widetilde{E}_j - 2h\overline{E}_{ij} \le \Delta \implies \boxed{\widehat{h}(\Delta) = \frac{\widetilde{E}_i - \widetilde{E}_j - \Delta}{\widetilde{E}_i + \widetilde{E}_j}} \tag{314}$$

or zero if this is negative.

§ Robustness curves are shown in fig. 43.[9]

- Note zeroing at nominal effectiveness margin, $\widetilde{E}_i - \widetilde{E}_j$.
- Note trade off inversely proportional to average effectiveness: Slope = $-1/(\widetilde{E}_i + \widetilde{E}_j)$.
- Innovative-Conservative (2–3):
  - ○ Nominal effectiveness margin for innovative over conservative: $\widetilde{E}_2 - \widetilde{E}_3 = 3.7 \times 10^4$.
  - ○ Average effectiveness of innovative and conservative: $\overline{E}_{23} = 1.22 \times 10^5$.
  - ○ $\widehat{h}(\Delta = 0) = 0.15$. Decision stable up to 15% model-form error.
- Standard-Conservative (1–3):
  - ○ Nominal effectiveness margin for standard over conservative: $\widetilde{E}_1 - \widetilde{E}_3 = 1.9 \times 10^4$.
  - ○ Average effectiveness of standard and conservative: $\overline{E}_{13} = 1.13 \times 10^5$.
  - ○ $\widehat{h}(\Delta = 0) = 0.086$. Decision stable up to 8.6% model-form error.
- Innovative-Standard (2–1):
  - ○ Nominal effectiveness margin for innovative over standard: $\widetilde{E}_2 - \widetilde{E}_1 = 1.8 \times 10^4$.
  - ○ Average effectiveness of innovative and standard: $\overline{E}_{21} = 1.31 \times 10^5$.
  - ○ $\widehat{h}(\Delta = 0) = 0.069$. Decision stable up to 6.9% model-form error.

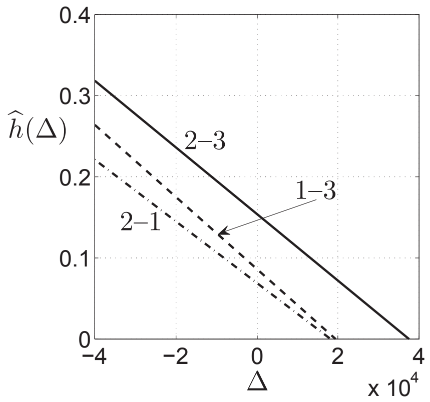

Figure 43: Robustness curves for comparing three options in eq.(293) with model uncertainty.

[9] Computations with \LECTURES\Info-Gap-Methods\Lectures\decstab004.m.

17 Behavioral Response to Feedback

17.1 Introduction

§ The Israel Electric Corporation has adopted the practice of reporting to consumers their level of energy consumption compared to a local mean. The IEC's goal, of course, is to encourage energy conservation, but the outcome may be different in the long run. Consider the following:

1. The Lo family gets feedback indicating that their energy consumption is below the average, and the Hi family's feedback shows their consumption is above the average.
2. One might expect that the Lo family will tend to increase their consumption since they are already relatively conservative. Likewise, one might expect a tendency of the Hi family to reduce consumption.
3. In the spirit of Kahneman-Tversky, let's invoke an asymmetry between positive and negative reward as in fig. 44. The Lo family gets positive reward, U, by increasing consumption, while the Hi family gets negative reward, D. The Kahneman-Tversky asymmetry would suggest that U will tend to be greater than D.

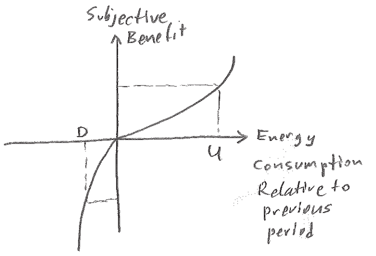

Figure 44: Kahneman-Tversky's asymmetric subjective utility function.

4. Consequently, the average will tend to drift upward over time. In other words, the IEC feedback may have the opposite effect from what was intended.
5. The behavior of the Lo and Hi families demonstrates a "reversion to the mean", as one might expect. However, the Kahneman-Tversky asymmetry implies that this reversion is asymmetric and may cause a long-range upward drift of the mean.
6. This is somewhat similar to the Lucas critique: populations tend to act, inadvertently and without coordination, to contravene long-range policy goals.
7. This "story" must be treated with caution. Life, and people, are more complicated. Nonetheless, treated as an hypothesis, it might be worth exploring, because if it is true then the IEC's feedback policy is misguided (or maybe intentional? Noooo. :)
8. The asymmetry can, however, be manipulated by changing the reference point with respect to which high and low consumption are defined. Suppose that comparison with the mean or the median causes long-term upward drift of the mean. In this case, comparison with a lower value, say the 30th percentile, could cause long-term drift downward because now fewer people feel they are conserving. Of course, predicting what reference point will cause stability, or drift up or down at a particular rate, is highly uncertain. One can then, of course, do an info-gap robustness analysis to manage this uncertainty.

9. This problem can be generalized from the specific case of energy conservation. One can think of savings vs. consumption, or risky vs. risk-free investment, or consumption of domestic vs. foreign products, etc. In some cases one may want to decrease consumption (e.g. of energy), and in others one may want to increase consumption (e.g. of domestic products).

17.2 Formulation

§ **Definitions:**

$\rho =$ a reference consumption (of energy) in time interval 1. This value is revealed to the consumers at the end of the time interval. This is the feedback to which consumers respond.

$c_1 =$ the consumption of energy (kW hr) in time interval 1, which varies from consumer to consumer.

$n(c_1)\,\mathrm{d}(c_1) =$ number of consumers whose consumption in time interval 1 was in the interval $[c_1,\ c_1 + \mathrm{d}c_1]$. Thus $n(c_1)$ is a number density, 1/kW hr. This function is known from historical data. Or, it is known at the end of time interval 1 because the consumptions of all consumers are observed.

$\Gamma_1 =$ the total consumption in time interval 1, which equals:

$$\Gamma_1 = \int_0^\infty c_1 n(c_1)\,\mathrm{d}c_1 \tag{315}$$

$f(c_1, \rho) =$ consumption in the next time interval of a consumer whose prior consumption was c_1. This function depends on ρ because the consumer's behavior responds to this feedback. $f(c_1, \rho)$ is non-negative.

$\widetilde{f}(c_1, \rho) =$ the putative consumer response function, which is known and non-negative.

$\mathcal{U}(h) =$ an info-gap model for uncertainty in the function $f(c_1, \rho)$.

$\Gamma_2 =$ the total consumption in time interval 2, which equals:

$$\Gamma_2 = \int_0^\infty f(c_1, \rho) n(c_1)\,\mathrm{d}c_1 \tag{316}$$

§ **Asymmetry.** $f(c_1, \rho)$ might have the asymmetry properties referred to in item 3 and fig. 44, p.91. Specifically, it might be that the increase in consumption by conservative consumers exceeds the decrease in consumption by excessive consumers. For any positive change in consumption, δ, define:

$\rho + \delta =$ excessive consumption in the 1st period.
$f(\rho + \delta, \rho) =$ that consumer's reduced consumption in the 2nd period: $f(\rho + \delta, \rho) < \rho + \delta$.
$\rho - \delta =$ under-consumption in the 1st period.
$f(\rho - \delta, \rho) =$ that consumer's enhanced consumption in the 2nd period: $f(\rho - \delta, \rho) > \rho - \delta$.

That is, defining U and D as in item 3 and fig. 44, p.91, for any positive increment of consumption, δ:

$$\underbrace{\rho + \delta - f(\rho + \delta, \rho)}_{D>0} < \underbrace{f(\rho - \delta, \rho) - (\rho - \delta)}_{U>0} \tag{317}$$

This implies:

$$\frac{f(\rho+\delta,\rho)+f(\rho-\delta,\rho)}{2} > \rho \tag{318}$$

Thus $f(c,\rho)$ vs. c is upward-concave.

We might expect that, when $\delta = 0$, the consumption does not change as a result of the feedback:

$$f(\rho,\rho) = \rho \tag{319}$$

§ **Performance requirement.** In general, there are two possibilities: we want total consumption to either decrease or increase by a non-negative quantity ε.
The total consumption must **decrease** by at least ε:

$$\Gamma_1 - \Gamma_2 \geq \varepsilon \tag{320}$$

The total consumption must **increase** by at least ε:

$$\Gamma_2 - \Gamma_1 \geq \varepsilon \tag{321}$$

§ **Definition of the robustness for decreasing consumption** by at least ε, from eq.(320):

$$\widehat{h}(\varepsilon,\rho) = \max\left\{h:\ \left(\min_{f\in\mathcal{U}(h)} [\Gamma_1 - \Gamma_2]\right) \geq \varepsilon\right\} \tag{322}$$

§ **Definition of the robustness for increasing consumption** by at least ε, from eq.(321):

$$\widehat{h}(\varepsilon,\rho) = \max\left\{h:\ \left(\min_{f\in\mathcal{U}(h)} [\Gamma_2 - \Gamma_1]\right) \geq \varepsilon\right\} \tag{323}$$

17.3 Robustness for Decreasing Consumption; Fractional Error Info-Gap Model

§ The info-gap model for uncertainty in the consumers' responses is:

$$\mathcal{U}(h) = \left\{f(c_1,\rho):\ f(c_1,\rho)\geq 0,\ \left|\frac{f(c_1,\rho)-\widetilde{f}(c_1,\rho)}{\widetilde{f}(c_1,\rho)}\right| \leq h\right\},\quad h\geq 0 \tag{324}$$

Note that we do not require the consumption functions to obey the conditions in eqs.(318) and (319).

§ Let $m(h)$ denote the inner minimum in the definition of the robustness, eq.(322). Note that:

$$\Gamma_1 - \Gamma_2 = \int_0^\infty [c_1 - f(c_1,\rho)]\, n(c_1)\, \mathrm{d}c_1 \tag{325}$$

§ From eq.(325) we see that $m(h)$ occurs when $f(c_1,\rho)$ is as large as possible at horizon of uncertainty h, namely:

$$f(c_1,\rho) = (1+h)\widetilde{f}(c_1,\rho) \tag{326}$$

§ We now find the inner minimum in the robustness to be:

$$\begin{aligned} m(h) &= \int_0^\infty \left[c_1 - (1+h)\widetilde{f}(c_1,\rho)\right] n(c_1)\, \mathrm{d}c_1 && (327)\\ &= \Gamma_1 - (1+h)\widetilde{\Gamma}_2(\rho) && (328)\end{aligned}$$

where $\widetilde{\Gamma}_2(\rho)$ is the putative value of the total consumption in the 2nd time interval, and it depends on the reference consumption, ρ.

§ The performance requirement is $m(h) \geq \varepsilon$, where $\varepsilon > 0$, namely:

$$\Gamma_1 - (1+h)\widetilde{\Gamma}_2 \geq \varepsilon \tag{329}$$

§ Solving for h in eq.(329) at equality yields the robustness:

$$\frac{\Gamma_1 - \varepsilon}{\widetilde{\Gamma}_2} = (1+h) \quad \Longrightarrow \quad \boxed{\widehat{h}(\varepsilon, \rho) = \begin{cases} \frac{\Gamma_1 - \varepsilon}{\widetilde{\Gamma}_2(\rho)} - 1 & \text{if } \varepsilon \leq \Gamma_1 - \widetilde{\Gamma}_2(\rho) \\ 0 & \text{else} \end{cases}} \tag{330}$$

§ ε is the required positive decrement in total consumption. Thus, if the putative 2nd-period total consumption, $\widetilde{\Gamma}_2(\rho)$, exceeds the 1st period total consumption, Γ_1, then the robustness in eq.(330) is zero.

§ The robustness function in eq.(330) is shown schematically in fig. 45, p.95, demonstrating the properties of trade off and zeroing.

§ Fig. 46, p.95, shows robustness curves for two different values of the reference consumption. Reference value ρ_2 is putatively better than reference value ρ_1 because ρ_2 results in a greater putative reduction in consumption:

$$\Gamma_1 - \widetilde{\Gamma}_2(\rho_2) > \Gamma_1 - \widetilde{\Gamma}_2(\rho_1) \tag{331}$$

However, the putative consumptions have zero robustness and therefore are not a good basis for comparing these alternatives.

§ Nonetheless, fig. 46 shows that reference value ρ_2 is more robust than ρ_1 for all values at which ρ_2 has postive robustness. Thus ρ_2 is preferred over ρ_1 based on robustness. Whether ρ_2 is actually acceptable depends on judgment of whether its robustness is great enough at an acceptable reduction of consumption.

§ Summarizing fig. 46, we see that a change in the reference consumption, ρ, that causes a **decrease** in total putative consumption, $\widetilde{\Gamma}_2(\rho_2) < \widetilde{\Gamma}_2(\rho_1)$, also causes a **decrease** in the cost of robustness: the robustness curve for ρ_2 is steeper than for ρ_1.

§ The previous observation implies a **re-enforcing impact on the robustness** of the two aspects. Lower $\widetilde{\Gamma}_2(\rho_2)$ shifts the robustness curve to the right, and lower cost of robustness makes the ρ_2 robustness curve steeper. Hence, the robustness curves do not cross one another, as we see in fig. 46.

17.4 Robustness for Increasing Consumption; Fractional Error Info-Gap Model

§ The info-gap model for uncertainty in the consumers' responses is eq.(324), as in section 17.3.

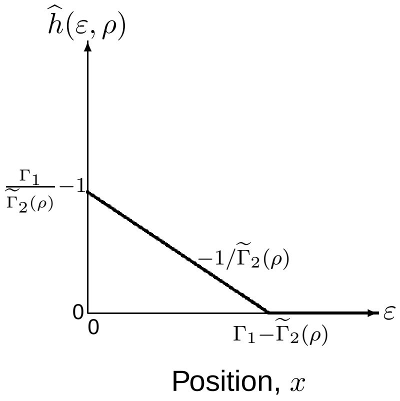

Figure 45: Robustness curve for decreasing the consumption, eq.(330), showing zeroing and trade off.

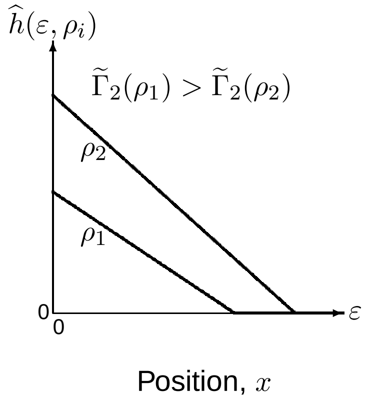

Figure 46: Two robustness curves for decreasing the consumption, with different values of the reference consumption.

§ Let $m(h)$ denote the inner minimum in the definition of the robustness, eq.(323). Note that:

$$\Gamma_2-\Gamma_1=\int_0^\infty [f(c_1,\rho)-c_1]\, n(c_1)\,\mathrm{d}c_1 \tag{332}$$

§ From eq.(332) we see that $m(h)$ occurs when $f(c_1,\rho)$ is as small as possible at horizon of uncertainty h, namely:

$$f(c_1,\rho)=(1-h)^+\widetilde{f}(c_1,\rho) \tag{333}$$

where $x^+=x$ if $x>0$ and equals 0 otherwise.

§ We now find the inner minimum in the robustness to be:

$$\begin{aligned} m(h) &= \int_0^\infty \left[(1-h)^+\widetilde{f}(c_1,\rho)-c_1\right] n(c_1)\,\mathrm{d}c_1 && (334)\\ &= (1-h)^+\widetilde{\Gamma}_2(\rho)-\Gamma_1 && (335)\end{aligned}$$

where $\widetilde{\Gamma}_2(\rho)$ is the putative value of the total consumption in the 2nd time interval, and it depends on the reference consumption, ρ.

§ The performance requirement is $m(h)\ge\varepsilon$, where $\varepsilon>0$, namely:

$$(1-h)^+\widetilde{\Gamma}_2(\rho)-\Gamma_1\ge\varepsilon \tag{336}$$

§ Solving for h in eq.(336) at equality yields the robustness:

$$\frac{\Gamma_1+\varepsilon}{\widetilde{\Gamma}_2}=(1-h)^+ \quad\Longrightarrow\quad \boxed{\widehat{h}(\varepsilon,\rho)=\begin{cases}1-\frac{\Gamma_1+\varepsilon}{\widetilde{\Gamma}_2(\rho)} & \text{if } \varepsilon\le\widetilde{\Gamma}_2(\rho)-\Gamma_1\\ 0 & \text{else}\end{cases}} \tag{337}$$

§ The robustness function in eq.(337) is shown schematically in fig. 47, demonstrating the properties of trade off and zeroing.

§ Fig. 48 shows robustness curves for two different values of the reference consumption, demonstrating that their robustness curves will cross if their putative total consumptions are different. This implies the potential for a reversal of robust preference between these alternatives.

§ Summarizing fig. 48, we see that a change in the reference consumption, ρ, that causes a **decrease** in total putative consumption, $\widetilde{\Gamma}_2(\rho_2) < \widetilde{\Gamma}_2(\rho_1)$, also causes a **decrease** in the cost of robustness: the robustness curve for ρ_2 is steeper than for ρ_1.

§ This is the same as observed with respect to fig. 46.

§ However, unlike the case of fig. 46, the previous observation implies a **conflicting, not re-enforcing, impact on the robustness** of the two aspects. Lower $\widetilde{\Gamma}_2(\rho_2)$ now shifts the robustness curve to the left (not to the right), and lower cost of robustness makes the ρ_2 robustness curve steeper. The result is crossing robustness curves and the potential for reversal of preference.

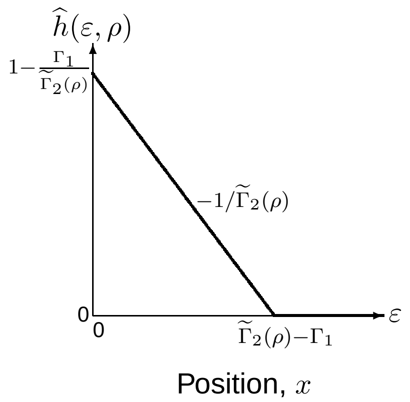

Figure 47: Robustness curve, eq.(337), showing zeroing and trade off.

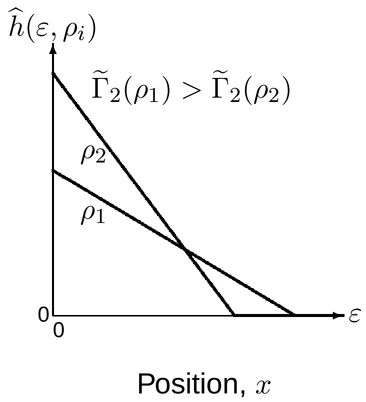

Figure 48: Two robustness curves for different values of the reference consumption.

18 Review Exercises

§ The exercises in this section are not homework problems, and they do not entitle the student to credit. They will assist the student to master the material in the lecture and are highly recommended for review and self-study.

1. Derive eq.(4) on p.4.

2. Inner extrema of robustness functions as in eq.(10) on p.5. Given an info-gap model:

$$\mathcal{U}(h) = \left\{ u(x) : \left| \frac{u(x) - \cos x}{\cos x} \right| \leq h \right\}, \quad h \geq 0 \tag{338}$$

   Find the elements of $\mathcal{U}(h)$ that maximize and minimize:

$$f(u) = \int_0^{2\pi} u(x) \sin x \, \mathrm{d}x \tag{339}$$

   What are the minimum and maximum values of $f(u)$?

3. Trade off and zeroing on p.6. Consider the following two robustness curves, corresponding to 2 different designs:

$$\widehat{h}_1(M_\mathrm{c}) = M_\mathrm{c} \tag{340}$$

$$\widehat{h}_2(M_\mathrm{c}) = 3M_\mathrm{c} - 1 \tag{341}$$

   More robustness is better than less robustness, if all else is the same. For what values of M_c is design 1 preferred over design 2? Explain this in terms of the zeroing and cost of robustness of these designs.

4. Derive eqs.(21) and (23) on p.8.

5. Explain the relation between eqs.(32) and (33) on p.9, and eq.(9) on p.4.

6. Using the method discussed in section 2.2, p.10, derive the Fourier representation of the function:

$$f(x) = \cos 3x, \quad x \in [-2,\ 2] \tag{342}$$

7. Unlike the case of eq.(53), p.12, explain why the following is **not** an ellipsoid:

$$h^2 = c_1^2 + 4c_1c_2 + c_2^2, \quad W = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \tag{343}$$

   For example, consider the case $h = 0$ and let $c_1 = bc_2$. What shapes are implied by eq.(343)? What property does the matrix W have, and how/why does this prevent c_1 vs. c_2 from being an ellipsoid?

8. As a simple case of eqs.(60) and (61) on p.13, consider the matrix:

$$W = \begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} \tag{344}$$

Show that its eigenvectors and eigenvalues are:

$$\nu_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \; \mu_1 = 3. \quad \nu_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \; \mu_2 = 2 \tag{345}$$

Show that W can be represented as:

$$W = \mu_1 \nu_1 \nu_1^T + \mu_2 \nu_2 \nu_2^T \tag{346}$$

Now, for an arbitrary real, positive definite $N \times N$ matrix W, with eigenvectors and eigenvalues, ν_i and μ_i, $i = 1, \ldots, N$, show that:

$$W = \sum_{i=1}^{N} \mu_i \nu_i \nu_i^T \tag{347}$$

9. Explain eqs.(79) and (80) on p.16 in terms of the definition of the robustness function, eq.(9) on p.4.
10. Considering eqs.(85) and (86) on p.17, provide an intuitive engineering explanation for the added robustness that results from large n_1.
11. Regarding the info-gap model of eq.(108), p.30: show that $\mathcal{U}(h)$ contains unbounded load functions for any $h > 0$. In what sense are the elements of $\mathcal{U}(h)$ transients?
12. Demonstrate that eq.(117) on p.31 is correct.
13. Explain by eq.(118) on p.31 is correct.
14. Derive eq.(119) on p.31.
15. What is the physical interpretation of negativity of the numerator of eq.(120) on p.31, and why should the robustness be zero in that case?
16. Explain the intuitive meaning of the opportuneness function in eq.(126) on p.34. In particular, compare the opportuneness with the robustness function in eq.(111) on p. 30. Explain the meaning entailed in changing the ‘max’ to ‘min’ operators.
17. Derive eq.(127) on p.34.
18. Derive eq.(130) on p.35.
19. Suppose that the last term on the right of eq.(130), p.35, does not depend on the decision, q. In that case, are robustness and opportuneness sympathetic, or antagonistic, or is this indeterminate?