Lecture Notes on Gambling and Risk-Sensitivity

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Source material: Yakov Ben-Haim, 2001, 2nd edition, 2006, *Information-Gap Decision Theory: Decisions Under Severe Uncertainty,* Academic Press. Chapter 6.

A Note to the Student: These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

Contents

1 Introduction

¶ **The immunity functions,**

$$\widehat{\alpha}(q, r_{\mathrm{c}}) \quad \widehat{\beta}(q, r_{\mathrm{w}}) \tag{1}$$

are the basic decision functions.

However, they:

- Do not determine a decision maker's choice.
- Do not determine the degree of riskiness of a contemplated action.

¶ **Risk** (Webster's):

- "Possibility of loss or injury.
- "Peril"
- "Dangerous element or factor"
- Possible etymology: cliff, rock or submarine hill (Weekley).
- Risk can be quantified in many ways.
- We will consider two approaches:
  - Assess riskiness of a system.
  - Assess risk-sensitivity of a decision maker.

2 Expected-Utility Risk Aversion

¶ Consider prizes of value

$$w_1 > w_2 > w_3 \tag{2}$$

where

$$w_2 = Pw_1 + (1-P)w_3 \tag{3}$$

and $0 < P < 1$. These prizes can be won in either of the lotteries:

$$L: \quad \widetilde{p} \;=\; (P,\, 0,\, 1-P)^T \tag{4}$$

$$L': \quad \widetilde{p}' \;=\; (0,\, 1,\, 0)^T \tag{5}$$

P and $1-P$ are the probabilities of winning w_1 and w_3 respectively.

L is a gamble between high and low gain.

L' is a sure bet on an outcome which is the mean of the extremes.

¶ **Utility function:** $\widetilde{u}(w_i)$ = decision maker's personal utility from prize w_i.

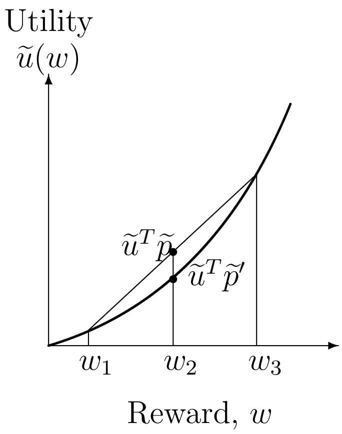

Figure 1: Convex estimated utility function. Risk proclivity.

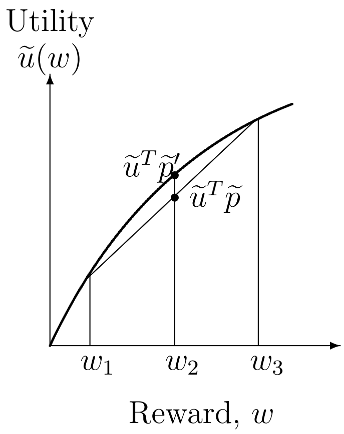

Figure 2: Concave estimated utility function. Risk aversion.

¶ **Expected utility of L and L':**

$$
\begin{aligned}
\mathrm{E}(L) &= P\tilde{u}(w_1) + (1-P)\tilde{u}(w_3) = \tilde{u}^T\tilde{p} && (6)\\
\mathrm{E}(L') &= \tilde{u}(w_2) && (7)\\
&= u[Pw_1 + (1-P)w_3] = \tilde{u}^T\tilde{p}' && (8)
\end{aligned}
$$

¶ **Expected-utility preference:**

$$
L \succ L' \quad \text{if and only if} \quad P\tilde{u}(w_1) + (1-P)\tilde{u}(w_3) > \tilde{u}(w_2) \tag{9}
$$

¶ **Risk aversion:**

- Prefer certainty-equivalent sure thing, w_2, over gamble between w_1 and w_3.
- Prefer L' over L.
- Fig. 2: Concave utility function.
- Degree of risk aversion: curvature of utility function.

¶ **Risk proclivity:**

- Prefer gamble between w_1 and w_3 over certainty-equivalent sure thing, w_2.
- Prefer L over L'.
- Fig. 1: Convex utility function.
- Degree of risk proclivity: curvature of utility function.

¶ **Expected-utility risk aversion:**

- Depends on knowing utilities and probabilities.
- Not usually suited for severe uncertainty.

¶ We will study risk sensitivity from an **info-gap perspective.**

3 Preview

(Section 6.1)

¶ Tentative and preliminary ideas of **info-gap risk:**

• Low immunity to failure.

or:

• Limited opportunity for windfall.

¶ **Interdependence of risk sensitivity and preferences:**

• Risk sensitivity of a decision maker is expressed by **choices among options.**

• Decision-maker's choice among options is assisted by interpreting the options in terms of **perceived riskiness.**

¶ **Risk sensitivity** is evaluated by comparing:

what **is chosen**

against

what **could have been chosen**

and by evaluating those choices in terms of

robustness and opportuneness.

¶ We will consider **3 types of choices.**

1. Given 1 robustness curve, $\widehat{\alpha}$ vs. r_c,

where along the curves does the decision maker choose to operate?

This will focus on the **robustness premium.**

2. Given 2 robustness curves,

• Which is preferred?

• How much reward would the decision maker willingly relinquish to move from one curve to the other?

This will focus on the **robustness premium** and the **reward premium.**

3. Given the alternative between

• Robustness strategy

• Opportuneness strategy

which is preferred?

This will focus on the **robustness premium** and the **opportuneness premium.**

¶ **Facets of risk sensitivity.**

- The profile of:
  - riskiness
  - risk sensitivity

  will not necessarily be consistent between the 3 approaches.
- They represent different facets of human response to uncertainty, danger, opportunity.

4 Risk Sensitivity and the Robustness Curve

(Section 6.2)

¶ We are familiar with the usual trade-offs:

- $\widehat{\alpha}(q, r_{\rm c})$ vs. $r_{\rm c}$.
- $\widehat{\alpha}(\widehat{q}_{\rm c}(r_{\rm c}), r_{\rm c})$ vs. $r_{\rm c}$.

¶ In the single-robustness-curve context, we assess risk sensitivity in terms of **where on the curve the decision maker chooses to operate,** fig. 3.

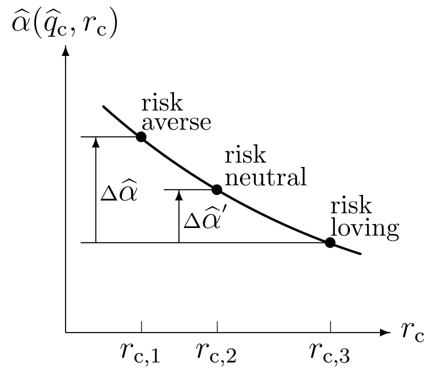

Figure 3: A robustness curve: robustness versus demanded reward. Illustrating robustness premia, and risk aversion, neutrality and proclivity.

¶ Suppose the decision maker can choose $r_{\rm c}$:

$$\text{low reward} = r_{\rm c,1} \le r_{\rm c} \le r_{\rm c,3} = \text{high reward} \tag{10}$$

- The decision maker is **risk loving** if he chooses $\widehat{q}_{\rm c}(r_{\rm c,3})$:
  - Demanding maximum available reward.
  - Relinquishing greatest amount of immunity to uncertainty.
- The decision maker is **risk averse** if he chooses $\widehat{q}_{\rm c}(r_{\rm c,1})$:
  - Demanding maximum available robustness.
  - Relinquishing greatest amount of reward.
- The decision maker is **risk neutral** if he chooses $\widehat{q}_{\rm c}(r_{\rm c,2})$.

¶ What distinguishes these 3 situations is the **robustness premium** chosen by the DM.

The risk-averse decision maker selects max robustness premium:

$$\Delta\widehat{\alpha} = \widehat{\alpha}(\widehat{q}_{\mathrm{c}}(r_{\mathrm{c},1}), r_{\mathrm{c},1}) - \widehat{\alpha}(\widehat{q}_{\mathrm{c}}(r_{\mathrm{c},3}), r_{\mathrm{c},3}) \tag{11}$$

in exchange for minimum reward, $r_{\mathrm{c},1}$.

¶ These concepts of

risk aversion, neutrality, proclivity

make some sense, but they refer to a limited context:

A single robustness curve.

We must consider additional contexts.

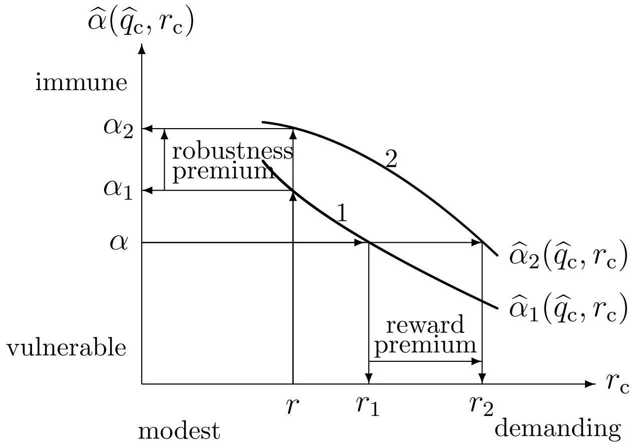

Figure 4: Maximal robustness versus demanded reward for two alternative options. Illustrating robustness and reward premia.

5 Risk Sensitivity and Two Robustness Curves

(Section 6.3)

¶ Assess risk-sensitivity wrt 2 strategy options, each with its own optimal robustness curve, fig. 4.

¶ Consider the arrows rising from $r_{\rm c} = r$:

$$\widehat{\alpha}_2 > \widehat{\alpha}_1 \tag{12}$$

There is a **robustness premium:**

$$\Delta\widehat{\alpha} = \widehat{\alpha}_2(\widehat{q}_{{\rm c},2}(r), r) - \widehat{\alpha}_1(\widehat{q}_{{\rm c},1}(r), r) \tag{13}$$

for strategy 2 over strategy 1.

A risk-averse DM will tend to prefer strategy 2 over strategy 1.

This is similar to the 1-curve analysis except:

Now the $\Delta\widehat{\alpha}$ is between two curves at fixed reward.

¶ These 2 strategies each have their own robust-satisficing action:

$$\widehat{q}_{c,1}(r), \quad \widehat{q}_{c,2}(r) \tag{14}$$

which may differ.

- The risk-averse DM may be willing to invest resources in order to
  implement $\widehat{q}_{c,2}(r)$
  rather than $\widehat{q}_{c,1}(r)$.
- For instance, the risk-averse DM may be willing to
  relinquish some or all of the reward premium accruing to strategy 2.

¶ To define the **reward premium** consider the arrows at constant α in fig. 4:

$$\Delta r(\alpha) = r_2 - r_1 \tag{15}$$

where:

$$\widehat{\alpha}_1(\widehat{q}_{c,1}(r_1), r_1) = \alpha = \widehat{\alpha}_2(\widehat{q}_{c,2}(r_2), r_2) \tag{16}$$

- Risk averse DM: willing to forfeit $\Delta r(\alpha)$.
- Risk loving DM: unwilling to forfeit $\Delta r(\alpha)$.
- Note:
  - $\Delta r(\alpha)$ depends on horizon of uncertainty, α.
  - $\widehat{\alpha}(r)$ depends on demanded reward, r.

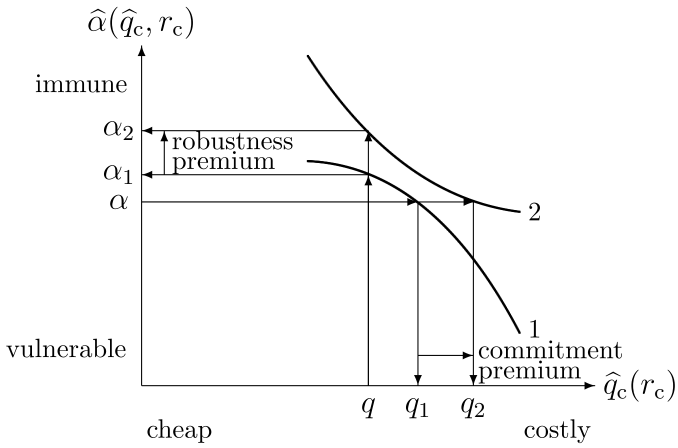

Figure 5: Maximal robustness versus robust-satisficing action for two alternative options. Illustrating robustness and commitment premia.

¶ Suppose q = **scalar,** so we can plot $\widehat{\alpha}(\widehat{q}_c(r_c), r_c)$ vs. $\widehat{q}_c(r_c)$, as in fig. 5.

- **Robustness premium** at fixed action $\widehat{q}_c$, which may correspond to different r_c's:

$$\Delta\alpha = \widehat{\alpha}_2(\widehat{q}_c, r_{c,2}) - \widehat{\alpha}_1(\widehat{q}_c, r_{c,1}) \tag{17}$$

- **Commitment premium** at fixed horizon of uncertainty, α:

$$\Delta q(\alpha) = q_2 - q_1 \tag{18}$$

¶ **Summary.** We evaluated risk-sensitivity in terms of:

- **Robustness premium**, as in section 4.
- **Reward premium.**
- **Commitment premium.**

6 Initial Commitment and Uncertain Future

(Section 6.4)

¶ We now consider an **example** before considering opportuneness.

¶ Development project:

q = size of plant or investment; decision variable.

$L(q)$ = lead time: construction time of plant, or maturation time of investment.

¶ Nominal profit, discounted for interest on investment:

$$R(q) = q\rho \mathrm{e}^{-\delta L(q)} - c(q) \tag{19}$$

ρ = revenue per unit of plant.

δ = discount (or interest) rate on investment.

$c(q)$ = cost of plant of size q.

¶ Many things are uncertain. We consider uncertain **interest** and **revenue.**

- Known nominal discounted unit revenue:

$$\widetilde{u}(q) = \rho \mathrm{e}^{-\delta L(q)} \tag{20}$$

- Unknown actual discounted unit revenue: $u(q)$.
- Uncertain profit function:

$$R(q, u) = qu(q) - c(q) \tag{21}$$

- Info-gap model for uncertain $u(q)$:

$$\mathcal{U}(\alpha, \widetilde{u}) = \left\{ u(q) : \ \left| u(q) - \rho \mathrm{e}^{-\delta L(q)} \right| \le \alpha \right\}, \quad \alpha \ge 0 \tag{22}$$

¶ This info-gap model, eq.(22), contains u-functions with **unbounded variation.**

- This may be unrealistic.
- We may have spectral information constraining variation of $u(q)$:

$$u(q) = \widetilde{u}(q) + \sum_{n=n_1}^{n_2} [x_n \sin n\pi q + y_n \cos n\pi q] \tag{23}$$

where the Fourier coefficients x_n and y_n are uncertain.

- Define:

  $N = n_2 - n_1 + 1$ be the number of modes in the expansion of $u(q)$.

  $\sigma(q)$ and $\gamma(q)$ be column N-vectors of the sines and cosines.

  x and y be the column N-vectors of Fourier coefficients.

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \eta(q) = \begin{pmatrix} \sigma(q) \\ \gamma(q) \end{pmatrix} \tag{24}$$

- Thus we can write eq.(23) more succinctly:

$$u(q) = \widetilde{u}(q) + z^T \eta(q) \tag{25}$$

- A Fourier ellipsoid-bound info-gap model for uncertainty in the discounted revenue is:

$$\mathcal{U}(\alpha, \widetilde{u}) = \left\{ u(q) = \widetilde{u}(q) + z^T \eta(q) : \ z^T W z \le \alpha^2 \right\}, \quad \alpha \ge 0 \tag{26}$$

where W is a known, real, symmetric, positive definite matrix and, as in eq.(22), $\widetilde{u}(q) = \rho \mathrm{e}^{-\delta L(q)}$.

¶ We study risk-sensitivity of both info-gap models, eq.(22) and (26).

6.1 Uniformly Bounded Uncertainty

(Section 6.4.1)

¶ The robustness function for the uniform-bound info-gap model, eq.(22) on p.12, is:

$$\widehat{\alpha}(q, r_{\rm c}) = \rho \mathrm{e}^{-\delta L(q)} - \frac{r_{\rm c} + c(q)}{q} \tag{27}$$

Usual trade-off: $\widehat{\alpha}$ vs. $r_{\rm c}$, at fixed plant size q.

¶ Consider the robust-satisficing plant size, $\widehat{q}_{\rm c}(r_{\rm c})$. Special case:

$c(q)$ = cost of plant = $a\sqrt{q}$.

$L(q)$ = construction time = $b_1 q + b_0$.

δ = discount rate = 0.08.

$\rho \mathrm{e}^{-\delta b_0} = 3a$.

Now we can evaluate:

$$\widehat{q}_{\rm c}(r_{\rm c}) \quad \text{and} \quad \widehat{\alpha}(\widehat{q}_{\rm c}(r_{\rm c}), r_{\rm c}) \tag{28}$$

as in fig. 6.

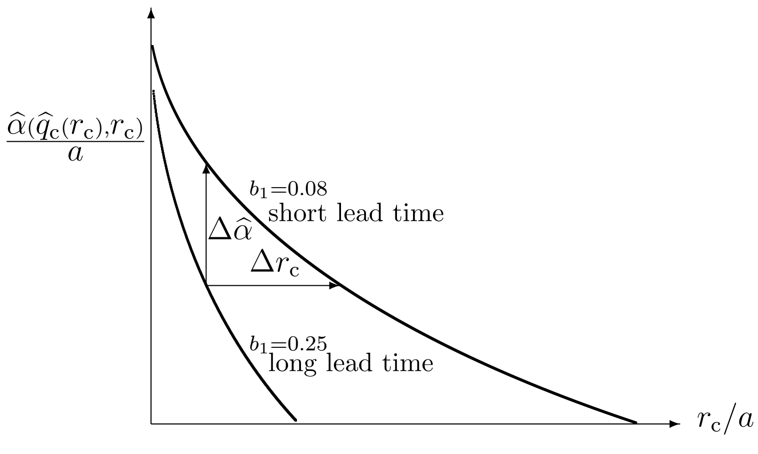

Figure 6: Maximal robustness curves for two values of the construction duration.

¶ From fig. 6 we note:

- Long construction time: lower acceptable uncertainty.
- Short construction time: higher acceptable uncertainty.
- Large robustness premium, $\Delta\alpha$, for short lead time.
- Large reward premium, Δr_{c}, for short lead time.
- Premia vary with r_{c} and α.

¶ Sensitivity to risk exposed by response to the question:

How much of Δr_{c} would the DM willingly forfeit in order to move from long to short lead time?

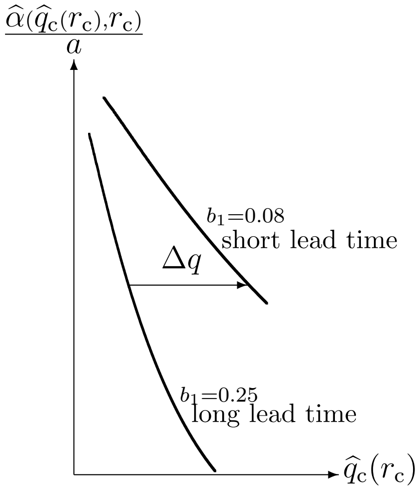

Figure 7: Maximal robustness versus the robust-satisficing plant size, for two values of the construction duration.

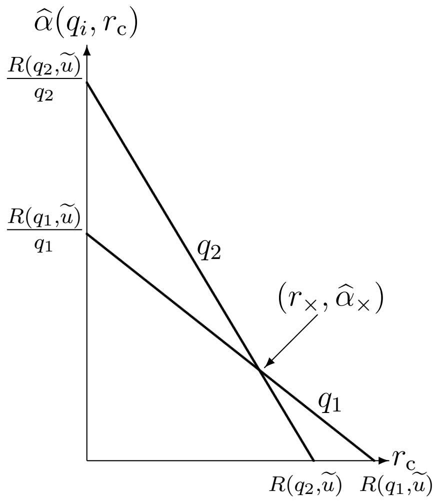

Figure 8: Reversal of preference between two plant sizes.

¶ For each r_{c} there is a unique $\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}})$.

- Thus the r_{c}-axis of fig. 6 can be transformed to a $\widehat{q}_{\mathrm{c}}$-axis, as in fig. 7.
- We see a substantial commitment premium, Δq, for short over long lead time.

¶ Fig. 8 shows a preference reversal.

- The robustness can be written:

$$\widehat{\alpha}(q, r_{\mathrm{c}}) = \frac{R(q, \widetilde{u}) - r_{\mathrm{c}}}{q} \tag{29}$$

- Consider the choice between two plant sizes, q_1 and q_2, where:

$$R(q_1, \widetilde{u}) > R(q_2, \widetilde{u}) \quad \text{and} \quad \frac{R(q_1, \widetilde{u})}{q_1} < \frac{R(q_2, \widetilde{u})}{q_2} \tag{30}$$

  ◦ The left relation implies nominal preference for q_1 over q_2.

  ◦ The right relation implies crossing of robustness curves as in fig. 8.

  ◦ Crossing of robustness curves implies reversal of preference between q_1 and q_2, depending on:

    — Required reward.

    — Required robustness.

6.2 Bounded Fourier Uncertainty

¶ We now find:

$$\widehat{\alpha}_f(q, r_\mathrm{c}) = \frac{1}{\sqrt{\eta(q)^T W^{-1} \eta(q)}} \left(\rho \mathrm{e}^{-\delta L(q)} - \frac{r_\mathrm{c} + c(q)}{q} \right) \tag{31}$$

where the term in parentheses equals the robustness of the uniform-bound case, $\widehat{\alpha}_u(q, r_\mathrm{c})$, eq.(27) on p.14.

¶ Special case:

$$W = \mathrm{diag}(w_1, \ldots, w_N, w_1, \ldots, w_N) \tag{32}$$

Now:

$$\eta(q)^T W^{-1} \eta(q) = \sum_{i=1}^{N} \frac{1}{w_i} \tag{33}$$

Thus:

$$\widehat{\alpha}_f(q, r_\mathrm{c}) = c \widehat{\alpha}_u(q, r_\mathrm{c}) \tag{34}$$

where c is a constant, independent of q.

Thus $\widehat{q}_\mathrm{c}(r_\mathrm{c})$ is the same for uniform and Fourier info-gap models.

However, the magnitude of $\widehat{\alpha}$ may differ, causing different preferences for r_c.

7 Risk-Sensitivity, Robustness and Opportuneness

(Section 6.5)

¶ We have previously studied risk sensitivity by examining a DM's choices on:

- A single robustness curve (section 4).
- Two robustness curves (section 5).

We now consider choices between robustness and opportuneness strategies.

¶ The DM could choose:

- Robustness strategy:
  - Action $\widehat{q}_{\rm c}(r_{\rm c})$ to maximize $\widehat{\alpha}(q, r_{\rm c})$:
  - Guarantee survival.
  - Characteristic of risk aversion.
- Opportuneness strategy:
  - Action $\widehat{q}_{\rm w}(r_{\rm w})$ to minimize $\widehat{\beta}(q, r_{\rm w})$:
  - Facilitate windfall.
  - Characteristic of risk proclivity.
- We must also consider whether these immunities are **antagonistic** or **sympathetic.**

¶ Consider two levels of reward:

$r_{\rm c}$ = critical survival level of reward.

$r_{\rm w}$ = larger windfall level of reward.

$$r_{\rm w} > r_{\rm c} \tag{35}$$

The DM could choose:

$\widehat{q}_{\rm c}(r_{\rm c})$ suggesting risk aversion.

$\widehat{q}_{\rm w}(r_{\rm w})$ suggesting risk proclivity.

However, consider the corresponding robustnesses: $\widehat{\alpha}(\widehat{q}_{\rm c}(r_{\rm c}), r_{\rm c})$ and $\widehat{\alpha}(\widehat{q}_{\rm w}(r_{\rm w}), r_{\rm c})$.

By definition of $\widehat{q}_{\rm c}(r_{\rm c})$:

$$\widehat{\alpha}(\widehat{q}_{\rm c}(r_{\rm c}), r_{\rm c}) \geq \widehat{\alpha}(\widehat{q}_{\rm w}(r_{\rm w}), r_{\rm c}) \tag{36}$$

There there is a non-negative robustness premium for $\widehat{q}_{\rm c}(r_{\rm c})$ over $\widehat{q}_{\rm w}(r_{\rm w})$:

$$\Delta\widehat{\alpha} = \widehat{\alpha}(\widehat{q}_{\rm c}(r_{\rm c}), r_{\rm c}) - \widehat{\alpha}(\widehat{q}_{\rm w}(r_{\rm w}), r_{\rm c}) \geq 0 \tag{37}$$

¶ More specifically,

If DM chooses $\widehat{q}_{\rm c}(r_{\rm c})$ over $\widehat{q}_{\rm w}(r_{\rm w})$ and if $\Delta\widehat{\alpha} \gg 0$,

Then DM shows **great risk aversion** since he chose

great robustness, $\widehat{\alpha}(\widehat{q}_{\rm c}(r_{\rm c}),\ r_{\rm c})$, and limited reward, $r_{\rm c}$,

rather than

great reward, $r_{\rm w}$, and limited robustness, $\widehat{\alpha}(\widehat{q}_{\rm w}(r_{\rm w}),\ r_{\rm c})$.

¶ Likewise, if DM chooses $\widehat{q}_{\rm c}(r_{\rm c})$ over $\widehat{q}_{\rm w}(r_{\rm w})$ and if $\Delta\widehat{\alpha} \sim 0$,

Then DM shows **slight risk aversion** since:

- Robustness strategy preferred, but
- small alterations could cause the DM to change the choice.

¶ Now we interpret a choice of $\widehat{q}_{\rm c}(r_{\rm w})$ over $\widehat{q}_{\rm c}(r_{\rm c})$, which indicates a **proclivity for risk.**

However, we have yet to consider the implications of antagonism or sympathy of the immunity functions.

¶ Consider the opportuneness functions for each choice: $\widehat{\beta}(\widehat{q}_{\rm w}(r_{\rm w}),\, r_{\rm w})$ and $\widehat{\beta}(\widehat{q}_{\rm c}(r_{\rm c}),\, r_{\rm w})$.

By definition, there is an opportuneness premium for $\widehat{q}_{\rm w}(r_{\rm w})$ over $\widehat{q}_{\rm c}(r_{\rm c})$:

$$\Delta\widehat{\beta} = \widehat{\beta}(\widehat{q}_{\rm c}(r_{\rm c}), r_{\rm w}) - \widehat{\beta}(\widehat{q}_{\rm w}(r_{\rm w}), r_{\rm w}) \geq 0 \tag{38}$$

A large value of $\Delta\widehat{\beta}$ attracts the **risk loving DM.**

¶ More specifically,

If DM chooses $\widehat{q}_{\rm w}(r_{\rm w})$ over $\widehat{q}_{\rm c}(r_{\rm c})$ and if $\Delta\widehat{\beta} \gg 0$,

Then DM shows **great risk proclivity** since he chose

great opportuneness, $\widehat{\beta}(\widehat{q}_{\rm w}(r_{\rm w}),\, r_{\rm w})$, and large reward, $r_{\rm w}$,

rather than

great robustness, $\widehat{\alpha}(\widehat{q}_{\rm c}(r_{\rm c}),\, r_{\rm c})$, and limited reward, $r_{\rm c}$.

¶ Likewise, if DM chooses $\widehat{q}_{\rm w}(r_{\rm w})$ over $\widehat{q}_{\rm c}(r_{\rm c})$ and if $\Delta\widehat{\beta} \sim 0$,

Then DM shows **slight risk proclivity** since:

- Opportuneness strategy preferred, but
- small alterations could cause the DM to change the choice.

¶ We must now consider **antagonism** and **sympathy** of the immunity functions.

• Sympathetic immunities: robustness and opportuneness can be enhanced together.

• Antagonistic immunities: either immunity can be improved only at the expense of the other.

• If the immunities are **sympathetic**
and if the DM chooses $\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}})$ over $\widehat{q}_{\mathrm{w}}(r_{\mathrm{w}})$,
he may not be risk averse at all.
It is possible that a highly robust strategy is also highly opportune.

• If the immunities are **antagonistic**
and if the DM chooses $\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}})$ over $\widehat{q}_{\mathrm{w}}(r_{\mathrm{w}})$,
then he is strongly risk averse
since robustness and opportuneness cannot be improved together.

• Analogous considerations apply to choice of $\widehat{q}_{\mathrm{w}}(r_{\mathrm{w}})$ over $\widehat{q}_{\mathrm{c}}(r_{\mathrm{c}})$.

8 Risk-Neutral Line

¶ In the previous section we considered **risk aversion** and **risk proclivity** in terms of the DM's choice between robust-satisficing, $\widehat{q}_{\rm c}(r_{\rm c})$, and opportune-windfalling, $\widehat{q}_{\rm w}(r_{\rm w})$.

- We could not make any prediction.
- We now consider the question: **what is risk-neutrality?**
- We will make a prediction.

¶ For any pair of rewards, $(r_{\rm c},\ r_{\rm w})$, we could evaluate the DM's risk-sensitivity as in the previous section:

- By asking the DM his choice between $\widehat{q}_{\rm c}(r_{\rm c})$ and $\widehat{q}_{\rm w}(r_{\rm w})$.
- We can identify regions of risk-aversion and risk-sensitivity in the $r_{\rm c}$-vs.-$r_{\rm w}$ plane.
- This would be an observation, not a prediction.

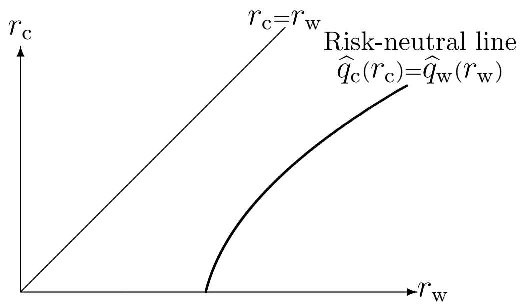

Figure 9: r_c-vs.-r_w plane showing the risk-neutral line.

¶ There is a curve in the $(r_c,\ r_w)$ plane, shown in fig. 9, at which:

$$\widehat{q}_c(r_c) = \widehat{q}_w(r_w) \tag{39}$$

• Along this curve the DM is behaviorally indifferent between robustness and opportuneness: they are identical.

• There is no robustness premium for $\widehat{q}_c(r_c)$ over $\widehat{q}_w(r_w)$:

$$\Delta\widehat{\alpha} = \widehat{\alpha}(\widehat{q}_c(r_c), r_c) - \widehat{\alpha}(\widehat{q}_w(r_w), r_c) = 0 \tag{40}$$

• Likewise there is no opportuneness premium for $\widehat{q}_w(r_w)$ over $\widehat{q}_c(r_c)$:

$$\Delta\widehat{\beta} = \widehat{\beta}(\widehat{q}_c(r_c), r_w) - \widehat{\beta}(\widehat{q}_w(r_w), r_w) = 0 \tag{41}$$

• Along the curve in eq.(39) and fig. 9 the DM is **operationally risk neutral.**

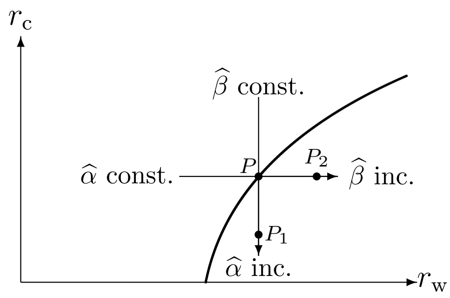

Figure 10: Variation of the immunity functions around the risk-neutral line.

¶ Consider a point P on the risk-neutral line, as in fig. 10.

- Along a vertical line through P:
  - ∘ r_w is constant so $\widehat{\beta}(\widehat{q}_w(r_w), r_w)$ is constant.
  - ∘ r_c is decreasing so $\widehat{\alpha}(\widehat{q}_c(r_c), r_c)$ is increasing.
- So there is a positive robustness premium for points below P compared to P.
- We would expect a risk-averse DM to prefer P_1 over P:

$$P_1 \succ_{av} P \tag{42}$$

¶ Now consider a horizontal line through P as in fig. 10.

- Along a horizontal line through P:
  - ∘ r_c is constant so $\widehat{\alpha}(\widehat{q}_c(r_c), r_c)$ is constant.
  - ∘ r_w is decreasing so $\widehat{\beta}(\widehat{q}_w(r_w), r_w)$ is increasing.

• So moving to the right on the horizontal line through P is like moving up the opportuneness curve of fig. 11 on p.25 to:

- ∘ Greater windfall reward.
- ∘ Lower ambient uncertainty.

• The risk-loving DM is drawn to the right by the possibility of large gain in exchange for low certainty.

- Thus we expect a risk-loving DM to prefer P_2 over P.
- Conversely, a risk-averse DM will prefer P over P_2:

$$P \succ_{av} P_2 \tag{43}$$

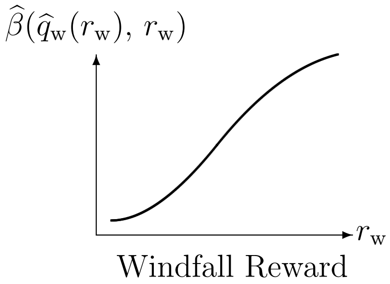

Figure 11: Optimal opportuneness function versus windfall reward.

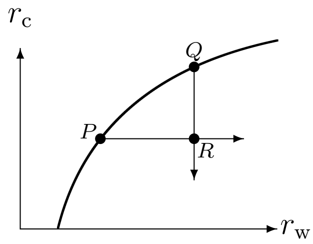

Figure 12: Possible non-transitivity of preferences.

¶ We con summarize this discussion as follows, see fig. 12.

• For a risk-averse DM:

$$P \succ_{\rm av} R \quad \text{and} \quad R \succ_{\rm av} Q \tag{44}$$

• However, we are not able to deduce the DM's choice between P and Q. These points lie on different robustness curves.

• We could explore the preference between P and Q by examining these different robustness curves and the corresponding opportuneness curves.

• We cannot conclude that the DM has transitive preferences.

¶ In fact, non-transitive preferences are not rare.

• A very old example (Condorcet, 19th c.).

• Consider 3 options: A, B and C (e.g. apartments).

  ◦ Each option has 3 features: X, Y and Z, (e.g. location, size and price).

  ◦ The DM can rank each option according to each feature. E.g. table 1.

		Options		
		A	B	C
Features	X	low	med	high
	Y	med	high	low
	Z	high	low	med

Table 1: Preference ranks of the options.

• If we compare options by voting on the features we see:

$$B \succ A, \qquad C \succ B, \qquad A \succ C \tag{45}$$

• The DM's preferences among the options are non-transitive:

$$B \succ A \succ C \succ B \tag{46}$$

9 Pure Competition with Uncertain Cost

(Section 6.7)

¶ We will illustrate the risk-neutral line with a simple example.

¶ Production.

q = number of items to produce, which the DM must choose.

$p(q)$ = known the sale price per item.

$u(q)$ = uncertain cost of producing q items.

Assume all items are sold.

Profit is:

$$R(q) = qp(q) - u(q) \tag{47}$$

¶ The info-gap model for uncertain manufacturing cost is:

$$\mathcal{U}(\alpha, \widetilde{u}) = \left\{ u(q): \ |u(q) - \widetilde{u}(q)| \le \alpha \psi(q) \right\}, \quad \alpha \ge 0 \tag{48}$$

$\widetilde{u}(q)$ and $\psi(q)$ are known.

¶ The robustness and opportuneness functions for production volume q are:

$$\widehat{\alpha}(q, r_{\mathrm{c}}) = \frac{qp(q) - r_{\mathrm{c}} - \widetilde{u}(q)}{\psi(q)} \tag{49}$$

$$\widehat{\beta}(q, r_{\mathrm{w}}) = \frac{r_{\mathrm{w}} - qp(q) + \widetilde{u}(q)}{\psi(q)} \tag{50}$$

¶ Special case:

$$\psi(q) = q \quad \text{and} \quad \widetilde{u}(q) = u_1 q - u_2 q^{\xi} \tag{51}$$

where $0 < \xi < 1$, $u_1 > 0$, $u_2 > 0$ and $\widetilde{u}(q) > 0$.
$\dfrac{\widetilde{u}(q)}{q}$ increases with q: diseconomy of scale:

$$\frac{\mathrm{d}[\widetilde{u}(q)/q]}{\mathrm{d}q} = (1 - \xi) u_2 q^{\xi - 2} > 0 \tag{52}$$

Also:

$$p(q) = p_0 = \text{constant} \tag{53}$$

Pure competition: the firm's production volume does not influence the market price.

¶ The immunity functions are:

$$\widehat{\alpha}(q, r_{\rm c}) = p_0 - u_1 - \frac{r_{\rm c}}{q} + u_2 q^{\xi-1} \tag{54}$$

$$\widehat{\beta}(q, r_{\rm w}) = u_1 - p_0 + \frac{r_{\rm w}}{q} - u_2 q^{\xi-1} \tag{55}$$

¶ The robust-satisficing production volume is:

$$\widehat{q}_{\rm c}(r_{\rm c}) = \left(\frac{r_{\rm c}}{(1-\xi)u_2} \right)^{1/\xi} \tag{56}$$

which maximizes $\widehat{\alpha}(q, r_{\rm c})$.

¶ The opportune-windfalling production volume, $\widehat{q}_{\rm w}(r_{\rm w})$, minimizes $\widehat{\beta}(q, r_{\rm w})$. $\widehat{\beta}(q, r_{\rm w})$ cannot be negative, so $\widehat{q}_{\rm w}(r_{\rm w})$ is the solution for q of:

$$\widehat{\beta}(q, r_{\rm w}) = 0 \tag{57}$$

or:

$$(p_0 - u_1)q + u_2 q^{\xi} = r_{\rm w} \tag{58}$$

¶ The curve of risk-neutrality is the locus of points $(r_{\rm w},\, r_{\rm c})$ at which:

$$\widehat{q}_{\rm c}(r_{\rm c}) = \widehat{q}_{\rm w}(r_{\rm w}) \tag{59}$$

To formulate the risk-neutral line we substitute $\widehat{q}_{\rm c}(r_{\rm c})$ from eq.(56) for q in eq.(58) and re-arrange to obtain:

$$r_{\rm w} = (p_0 - u_1) \left(\frac{r_{\rm c}}{(1-\xi)u_2} \right)^{1/\xi} + \frac{r_{\rm c}}{1-\xi} \tag{60}$$

¶ Fig. 13 shows risk-neutral lines for various ξ:

- ξ = 'small' $\Longrightarrow$ large dis-economy of scale.
- ξ = 'large' $\Longrightarrow$ small dis-economy of scale.
- Recall from fig. 12 on p.25:

$$P \succ_{\rm av} R \quad \text{and} \quad R \succ_{\rm av} Q \tag{61}$$

- So, in fig. 13, risk-neutral lines shifts right with decreasing dis-economy of scale:

  The DM becomes less risk-averse as the dis-economy of scale decreases.

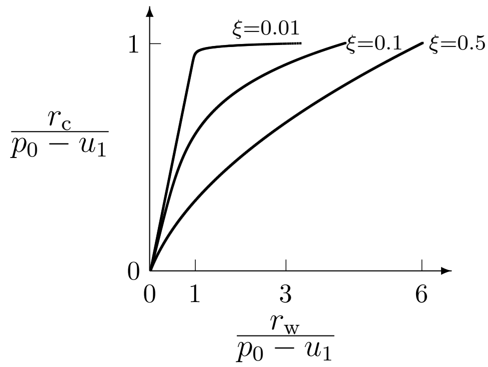

Figure 13: Risk-neutral lines for various dis-economies of scale.