Back to the Promised Land (Mathematical Analysis)

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1 *Highlights*

§ Binary decisions, expensive tests:

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§ How good is:

- Our knowledge?
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- Our intuition about our ignorance?
- Our ability to use knowledge and manage ignorance?

Highlights

§ **2 Systems, 1 Test: Probabilistic Alg.**

§ **Info-gap uncertainty on pdf: Robustify.**

§ n **Systems,** m **Tests.**

§ **Source: http://info-gap.com**

2 *Two Systems, One Test*

[0] lectures\talks\lib\two-systems02.tex, 20.1.2016. See 'Problem Set on Info-Gap Uncertainty', \lectures\risk\homework\ps1_rk.tex, #10. Yakov Ben-Haim, 2011, Two for the price of one: Info-gap robustness of the 1-test algorithm, ISIPTA2011, 25–28 July 2011, Innsbruck, Austria.

§ **Two systems, with qualities $x_1 \neq x_2 \in \Re$.**

- **Choose one system.**
- **Bigger is better.**

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- **Flip a fair coin.**
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- Flip a fair coin.
- 50/50 chance of success.

§ **One system tested:** quality x_r.

- Enhanced chance of success?
- Which system to use?
- It looks like 1 measurement can't help.

§ Algorithm for choosing a system:

- $q(y)$ **is any pdf:** $q(y) > 0$ **for all** $y \in \Re$**.**
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Probability of choosing larger x_i.

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§ Probability of success, $P_{\rm s}(q)$:

Probability of choosing larger x_i.

§ Theorem (Thomas Cover, 1987):[1]

If tested system chosen with probability 0.5,

then $P_{\rm s}(q) > 0.5$.

[1] Cover, Thomas M., 1987, Pick the largest number, chapter 5.1 in T. Cover and B. Gopinath, 1987, Open Problems in Communication and Computation, Springer-Verlag, Berlin.

3 Two Systems, One Test, CDF Known

§ $F(x)$ is known cdf.

§

§ **$F(x)$ is known cdf.**

§ **Algorithm for choosing a system:**

- **If $F(x_{\mathrm{r}}) < \frac{1}{2}$, choose un-tested system.**
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§ **Theorem:** $P_{\mathrm{s}} = \frac{3}{4}$

Proof: Robert R. Snapp, 2005.[2]

[2] Robert R. Snapp, 2005, U of Vermont, \papers\2-systems-1test\isipta2011\covers-problem.pdf

4 *Robustness of Two Systems, One Test*

§ Recall no-knowledge algorithm:

- $q(y)$ **is any pdf:** $q(y) > 0$ **for all** $y \in \Re$.
- **Draw** y **from** $q(y)$.
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§ **How to choose** $q(y)$**?**

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§ Cover's Theorem: $P_{\mathrm{s}}(q) > 0.5$.

§ How to choose $q(y)$?

Can we beat $P_{\mathrm{s}}(q) > 0.5$**?**

§ If we know $p(x_i)$ then $P_{\mathrm{s}} = 0.75$.

Can we achieve $P_{\mathrm{s}}(q) = 0.75$ **w/o knowing** $p(x_i)$**?**

§ Info-gap robust-satisficing:

- **Our guess:** $x \sim \widetilde{p}(x)$.
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- **Choose** $q(y)$ **to robust satisfice:**
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§ Info-gap robust-satisficing:

- **Our guess:** $x \sim \widetilde{p}(x)$.
- $\widetilde{p}(x)$ **highly uncertain.**
- **Choose** $q(y)$ **to robust satisfice:**
  - **Satisfy** $P_{\rm s} \geq P_{\rm c}$.
  - **Maximize robustness to uncertain** $\widetilde{p}$.

§ **Info-gap model for uncertain $\widetilde{p}(x)$: $\mathcal{U}(h)$.**

- **Nesting:** $h < h' \implies \mathcal{U}(h) \subseteq \mathcal{U}(h')$.
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§ **Info-gap model for uncertain $\widetilde{p}(x)$: $\mathcal{U}(h)$.**

- **Nesting:** $h < h' \implies \mathcal{U}(h) \subseteq \mathcal{U}(h')$.
- **Contraction:** $\mathcal{U}(0) = \{\widetilde{p}\}$.
- h **is unbounded horizon of uncertainty.**

§ Robustness, $\widehat{h}(q, P_\mathrm{c})$:

Maximum tolerable uncertainty.

$$\widehat{h}(q, P_\mathrm{c}) = \max\left\{h : \left(\min_{p \in \mathcal{U}(h)} P_\mathrm{s}(q|p)\right) \geq P_\mathrm{c}\right\}$$

§ Example:

- **Estimated pdf:** $\widetilde{p}(x) = \widetilde{\lambda}\mathrm{e}^{-\tilde{\lambda}x}$.
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- **Prob of success:** $P_{\mathrm{s}}(q|\widetilde{p}) > 0.5$
- **Putative optimal choice:**

$$\begin{aligned}\gamma^\star &= \arg\max_\gamma P_{\mathrm{s}}(q|\widetilde{p}) \\ &= \widetilde{\lambda}\sqrt{2}\end{aligned}$$

- **E.g.,** $\widetilde{\lambda} = 1$**:** $P_{\mathrm{s}}(q|\widetilde{p}) = 0.67 \gg 0.5$
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- **Estimated pdf:** $\widetilde{p}(x) = \widetilde{\lambda} \mathrm{e}^{-\widetilde{\lambda} x}$.
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- **E.g.,** $\widetilde{\lambda} = 1$**:** $P_{\mathrm{s}}(q|\widetilde{p}) = 0.67 \gg 0.5$
- **Robust to uncertainty in** $\widetilde{p}(x)$**???**

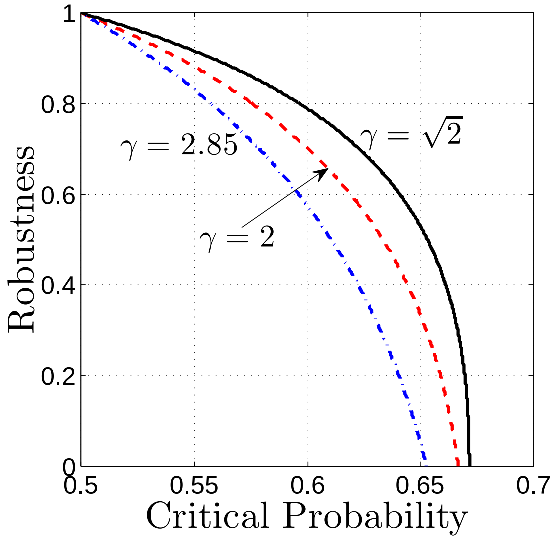

Figure 1: Robustness curves with $\widetilde{\lambda} = 1$.

§ Zeroing:

Estimated prob of success: no robustness.

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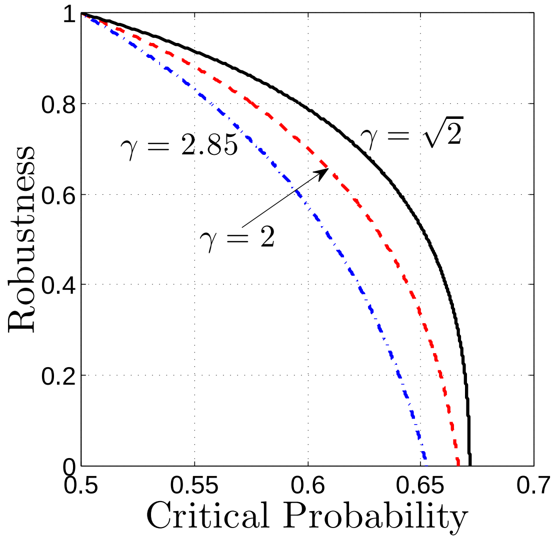

Figure 2: Robustness curves with $\widetilde{\lambda} = 1$.

§ **Zeroing:**

Estimated prob of success: no robustness.

§ **Trade off: robustness vs prob. of success.**

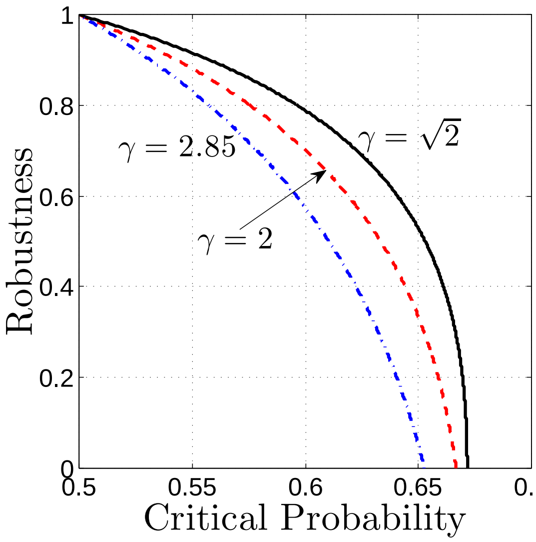

Figure 3: Robustness curves with $\widetilde{\lambda} = 1$.

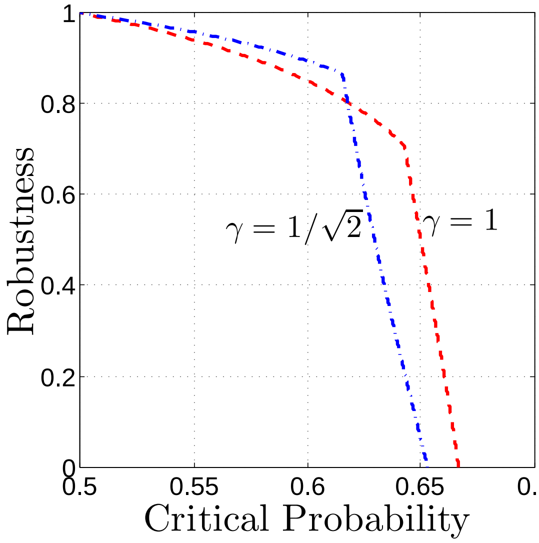

Figure 4: Robustness curves with $\widetilde{\lambda} = 1$.

§ Preference reversal.

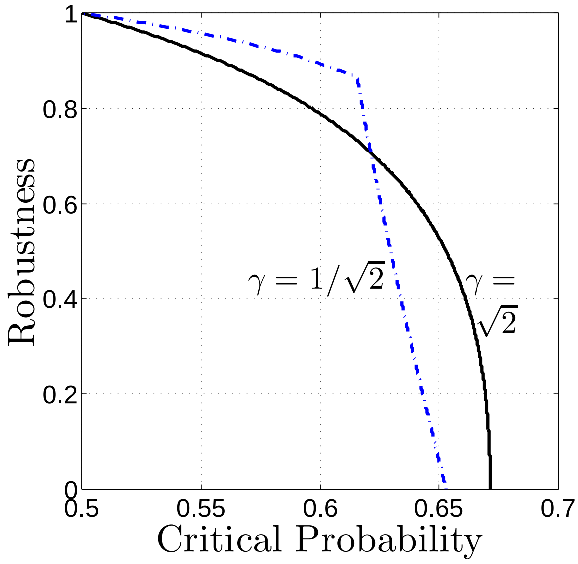

Figure 5: Robustness curves with $\widetilde{\lambda} = 1$.

§ Zeroing: no robustness of estimate.

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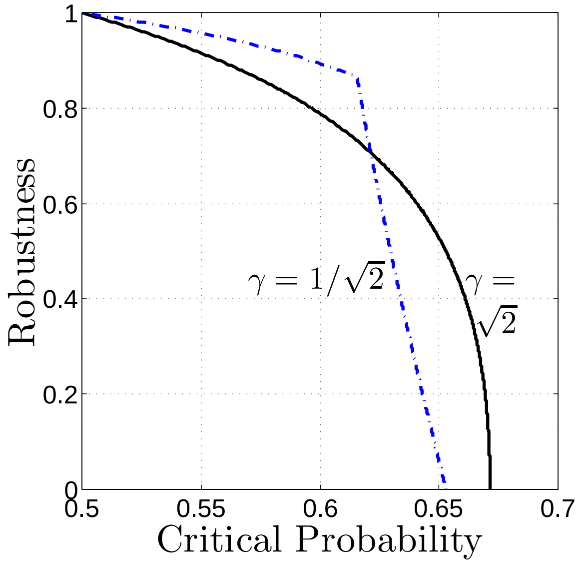

Figure 6: Robustness curves with $\widetilde{\lambda} = 1$.

§ **Zeroing:** **no robustness of estimate.**

§ **Trade off:** **robustness vs prob. of success.**

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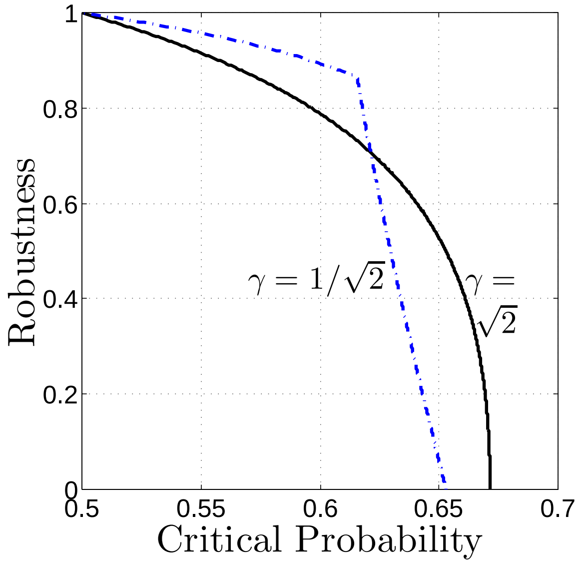

Figure 7: Robustness curves with $\widetilde{\lambda} = 1$.

§ **Zeroing: no robustness of estimate.**

§ **Trade off: robustness vs prob. of success.**

§ **Preference reversal.**

- $\gamma = \sqrt{2}$ **more robust for** $P_c > 0.62$.
- $\gamma = 1/\sqrt{2}$ **more robust for** $P_c < 0.62$.

5 *Three Systems, Two Tests*

§ 3 Systems with qualities:

$$x_1 < x_2 < x_3$$

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§ **3 Systems with qualities:**

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§ **Test two systems with revealed attributes:**

$$r_1 < r_2$$

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§ **Goal: Exclude worst system.**

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§ **Blind probability of success:** $\frac{1}{3}$

§ Algorithm:

- $q(y)$ **any non-zero pdf on** $\Re$**.**
- **Draw** y **from** $q(q)$**.**
- **If** $y < r_1$ **choose 2 tested systems.**
- **If** $r_1 \le y$ **choose** r_2 **and untested system.**

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§ Algorithm:

- $q(y)$ **any** non-zero pdf on $\Re$.
- Draw y from $q(q)$.
- If $y < r_1$ choose 2 tested systems.
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§ Theorem:

If tested systems chosen with equal prob.

then $P_{\mathrm{s}}(q) > \frac{1}{3}$.

6 *Three Systems, One Test*

§ 3 Systems with qualities:

$$x_1 < x_2 < x_3$$

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§ **3 Systems with qualities:**

$$x_1 < x_2 < x_3$$

§ **Test one system with revealed attribute r.**

§

§ **3 Systems with qualities:**

$$x_1 < x_2 < x_3$$

§ **Test one system with revealed attribute r.**

§ **Goal:** **Select best system.**

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§ **3 Systems with qualities:**

$$x_1 < x_2 < x_3$$

§ **Test one system with revealed attribute r.**

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§ **Blind probability of success:** $\frac{1}{3}$

§ Algorithm:

- $q(y)$ **any** non-zero pdf on $\Re$.
- Draw y from $q(q)$.
- If $y \leq r$ choose tested system.
- If $r < y$ choose equi-prob from untested.

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§ Algorithm:

- $q(y)$ **any non-zero pdf on** $\Re$.
- **Draw** y **from** $q(q)$.
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§ Theorem:

If tested system chosen with equal prob.

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7 *n Systems, m Tests*

§ Hypothesized generalization to n systems, m tests.

8 *Extensions*

§ **Multiple attributes.**

§ **Adaptive testing.**

§ **Best possible probability of success.**

9 *Final Thoughts*

§ **We began by asking the following questions.**

How good is:

- **Our knowledge?**
- **Our knowledge about our knowledge?**
- **Our intuition about our ignorance?**
- **Our ability to use knowledge and manage ignorance?**

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How good is:

- **Our knowledge?**
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- **Our ability to use knowledge and manage ignorance?**

§ **The 2-system 1-test example showed that:**

- **We are sometimes wrong about the answers.**
- **We should be ready for surprises.**

§ A final thought on Optimism:

- **Scientific optimism:** We're approaching the truth.
-

§ A final thought on Optimism:

- **Scientific optimism:** We're approaching the truth.
- **My optimism:**
  - We will always be **surprised.**
  - **Science** will always continue.
  - **Uncertainty** will never disappear.