Lecture Notes on Performance vs. Robustness of a Cantilever

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Source material: Yakov Ben-Haim, 2005, Info-gap Decision Theory For Engineering Design. Or: Why 'Good' is Preferable to 'Best', in *Engineering Design Reliability Handbook,* Edited by E. Nikolaides, D. Ghiocel and Surendra Singhal, CRC Press.

Notes to the Student:

• These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

• Section 6 contains review exercises that will assist the student to master the material in the lecture and are highly recommended for review and self-study. The student is directed to the review exercises at selected places in the notes. They are not homework problems, and they do not entitle the student to extra credit.

Contents

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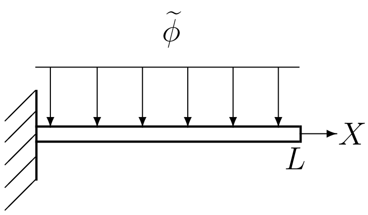
§ Overview:

- Optimal design of cantilever with uncertain load.
- Performance-optimal design has no robustness to uncertainty: Unreliable.
- Trade-off: Performance vs. robustness.
- Moral: Feasible designs are sub-optimal.
- Moral: Designers are “maximizers”, “optimizers”.

  The question is: Max-optimizers of what?
- Robustness and safety factors.

1 PERFORMANCE OPTIMIZATION



CANTILEVER WITH UNIFORM LOAD.

Figure 1: CANTILEVER WITH UNIFORM LOAD.

§ Parameters:

L = beam length (fixed).

W = beam width (uniform, fixed).

ρ = beam density (uniform, fixed).

$\widetilde{\phi}$ = uniform continuous load density (fixed).

$\sigma_T(x)$ = max bending stress in section at x.

$T(x)$ = beam thickness (height) = **design function.**

§ Performance criteria:

$$\text{Mass:} \quad \min_{T(x)>0} \int_0^L T(x)\,\mathrm{d}x \tag{1}$$

$$\text{Stress:} \quad \min_{T(x)>0} \max_{0\le x\le L} |\sigma_T(x)| \tag{2}$$

§ Can we optimize both criteria?

§ These criteria conflict. Trade-off needed.

§ **Review exercise 1 on p.14.**

§ Consider beams of mass θ:

$$\int_0^L T(x)\,\mathrm{d}x = \theta \qquad (3)$$

§ Linear-taper profile minimizes the maximum stress:

$$\widehat{T}_\theta(x) = \frac{2\theta(L-x)}{L^2} \qquad (4)$$

§ Resulting min-max bending stress:

$$\sigma_{\widehat{T}_\theta} = \frac{3\widetilde{\phi}L^4}{4W\theta^2} \qquad (5)$$

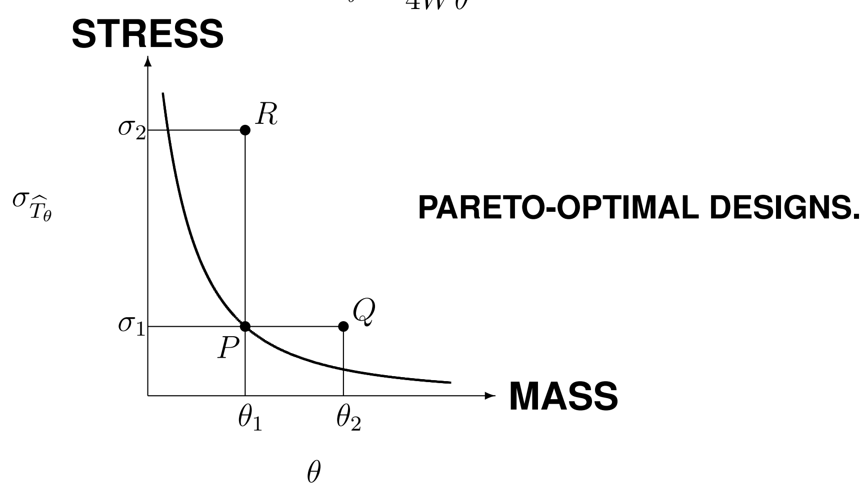


Figure 2: PARETO-OPTIMAL DESIGNS.

§ Where on the plane are the feasible designs? The optimal designs?

§ Designs on the curve are **Pareto efficient:**

- Pareto efficiency is a short-blanket idea:
  Pull up your blanket to warm your nose, your toes get cold!
- Stress can not be reduced w/o increasing mass.
- Mass can not be reduced w/o increasing stress.

§ Designs above the curve are **sub-optimal.**

§ Are the optimal designs Feasible? Reliable? Robust to uncertainty?
Nope.

§ **Review exercise 2 on p.14.**

2 Robustness to Uncertain Load

$\widetilde{\phi}(x)$ = Nominal, typical, design-base, best-estimate
of the load-density profile.

$\widetilde{\phi}(x)$ is undoubtedly **wrong.**

$\phi(x)$ = actual, unknown load-density profile.

Disparity between $\widetilde{\phi}(x)$ AND $\phi(x)$ is an **info-gap:**

Information-Gap:
Disparity between what **is known**
and what **needs to be known**
for a good decision.

§ Many info-gap models of uncertainty.

§ We will use:

$$\mathcal{U}(h) = \left\{ \phi(x) : \ \left| \phi(x) - \widetilde{\phi}(x) \right| \le h \right\}, \quad h \ge 0 \tag{6}$$

§ Info-gap model: Family of nested sets of events:

$$h = 0 \quad \Longrightarrow \quad \mathcal{U}(h, \widetilde{\phi}) \ = \ \left\{ \widetilde{\phi} \right\} \tag{7}$$

$$h \le h^{\bullet} \quad \Longrightarrow \quad \mathcal{U}(h, \widetilde{\phi}) \ \subseteq \ \mathcal{U}(h^{\bullet}, \widetilde{\phi}) \tag{8}$$

h = Horizon of uncertainty.

§ Two levels of uncertainty:

- Unknown load at horizon h.
- Unknown horizon of uncertainty: no worst case.

§ **Review exercise 3 on p.14.**

§ Performance requirement: maximum bending stress less than critical value:

$$\max_{0 \le x \le L} \ |\sigma_T(x, \phi)| \ \le \ \sigma_c \tag{9}$$

§ Robustness, $\widehat{h}(T, \sigma_c)$, of design $T(x)$:

- Max horizon of uncertainty without failure.
- How much can we err in $\phi(x)$ w/o jeopardizing the beam?

$$\widehat{h}(T,\sigma_c) = \max\left\{ h :\ \left(\max_{\phi \in \mathcal{U}(h)} \ \max_{0 \le x \le L} \ |\sigma_T(x,\phi)| \right) \le \sigma_c \right\} \tag{10}$$

§ Interpreting the robustness function:

- σ_c : max acceptable bending stress.
  : Design specification.
  : Performance-requirement.
- $T(x)$: Design.
- $\widehat{h}(T,\sigma_c)$: robustness to uncertainty of design $T(x)$ with requirement σ_c.
  : "Bigger is better".
  : Design function.
  : Preference ranking of alternative designs:

$$T \ \succ \ T^{\bullet} \quad \text{IF} \quad \widehat{h}(T,\sigma_c) \ > \ \widehat{h}(T^{\bullet},\sigma_c) \tag{11}$$

§ What θ-mass design maximizes the robustness, with design requirement σ_c?

The linear taper which min-maxes the stress:

$$\widehat{T}_{\theta}(x) = \frac{2\theta(L-x)}{L^2} \tag{12}$$

§ Good news or bad news?

3 Info-gap Robust-Optimal Design

§ Robustness of the optimal taper:

$$\widehat{h}(\widehat{T}_\theta, \sigma_c) = \frac{4W\theta^2\sigma_c}{3L^4} - \widetilde{\phi} \tag{13}$$

§ Trade-off: Robustness to uncertainty, $\widehat{h}$, vs. Performance, σ_c.

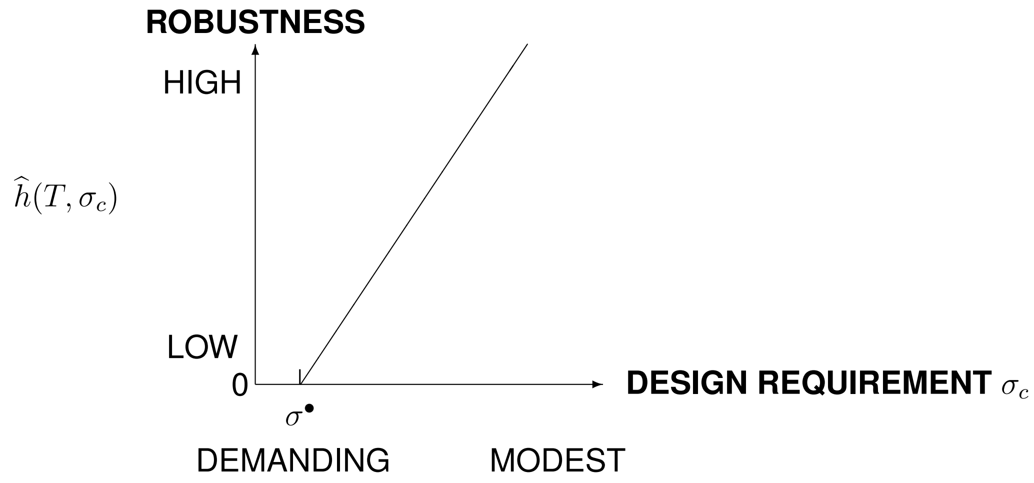


Figure 3: Optimal robustness curve, $\widehat{h}(\widehat{T}_\theta, \sigma_c)$, versus the maximum-stress design requirement σ_c.

§ **Review exercise 4 on p.14.**

§ Infeasible performance-requirement:

$$\sigma^\bullet = \frac{3\widetilde{\phi}L^4}{4W\theta^2} \tag{14}$$

§ **Review exercise 5 on p.14.**

§ Pareto-efficient min-max bending stress:

$$\sigma_{\widehat{T}_\theta} = \frac{3\widetilde{\phi}L^4}{4W\theta^2} \tag{15}$$

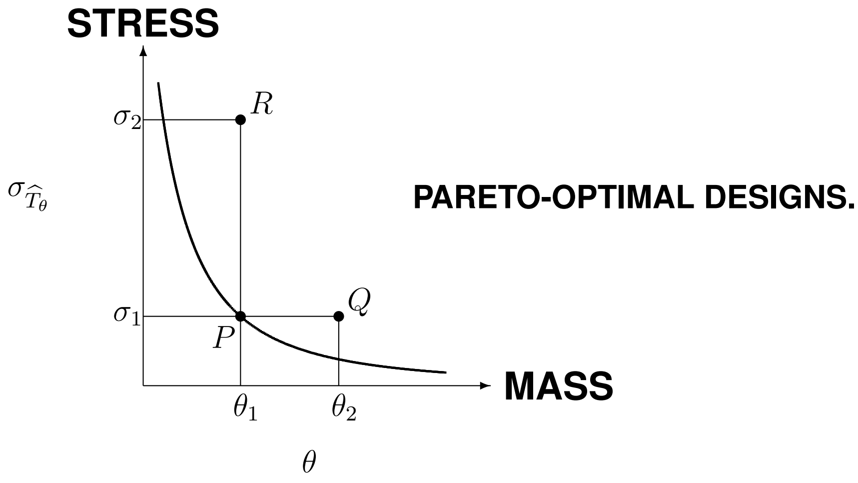


Figure 4: PARETO-OPTIMAL DESIGNS.

§ Clash of performance and robustness:

- Any design on the Pareto surface, e.g. P,
  has no immunity to load uncertainty.
- Only sub-optimal designs, e.g. Q, R, are feasible.
- ‘Good’ is preferable to ‘best’.

§ **Review exercise 6 on p.14.**

4 Robustness, Sub-optimal Design and Safety Factor

§ Sub-optimal design:

$$T^s(x) = \widehat{T}_\theta(x) + \gamma, \quad \gamma \ > \ 0 \tag{16}$$

§ **Review exercise 7 on p.14.**

§ With max-stress requirement:

$$\sigma^\bullet = \frac{3\widetilde{\phi}L^4}{4W\theta^2} \tag{17}$$

§ Robustness of sub-optimal design is:

$$\widehat{h}(T^s, \sigma^\bullet) = \frac{\widetilde{\phi}L^2}{4\theta^2}\left(\frac{2\theta}{L} + \gamma\right)^2 - \widetilde{\phi} \tag{18}$$

§ γ controls:

- Sub-optimality of design.
- Positivity of robustness.
- Related to classical "safety factor".

§ Recall:

- h = Horizon of load uncertainty.
- Info-gap model:

$$\mathcal{U}(h, \widetilde{\phi}) = \left\{\phi(x) : \ \left|\phi(x) - \widetilde{\phi}(x)\right| \le h\right\}, \quad h \ge 0 \tag{19}$$

- $\widehat{h}(T^s, \sigma^\bullet)$ = maximum h such that $\sigma \ \le \ \sigma^\bullet$ for all loads.

§ Choosing the safety factor:

F = Fraction of nominal load, $\widetilde{\phi}$.

Demanded robustness:

$\widehat{h}_D = F\widetilde{\phi}$.

Choice of γ so that $\widehat{h}(T^s, \sigma^\bullet) = \widehat{h}_D$:

$$\gamma = \frac{2\theta}{L}\left(\sqrt{1+F} - 1\right) \tag{20}$$

§ Optimal design:

$F = 0$: No safety factor.

$\widehat{h} = 0$: No robustness.

$\gamma = 0$: Pareto-efficiency.

§ Sub-optimal design:

$F > 0$: Safety.

$\widehat{h} > 0$: Robustness.

$\gamma > 0$: Material inefficiency.

5 Opportuneness Function

- Two faces of uncertainty:
  - ○ Pernicious: Potential for failure.
  - ○ Propitious: Potential for windfall.

- Two measures of performance:
  - ○ Robustness: Immunity from uncertainty.
  - ○ Opportuneness: Potential from uncertainty.

- Two strategies for decision:
  - ○ Satisficing: Enhance robustness. $(\widehat{h})$
  - ○ Windfalling: Exploit opportunity. $(\widehat{\beta})$

§ Info-gap model for uncertain load:

$$\mathcal{U}(h, \widetilde{\phi}) = \left\{ \phi(x) : \left| \phi(x) - \widetilde{\phi}(x) \right| \leq h \right\}, \quad h \geq 0 \tag{21}$$

§ Robustness of design $T(x)$:

$$\widehat{h}(T, \sigma_c) = \max \left\{ h : \left(\max_{\phi \in \mathcal{U}(h)} \max_{0 \leq x \leq L} |\sigma_{\phi,T}(x)| \right) \leq \sigma_c \right\} \tag{22}$$

Max h without failure for design requirement σ_c.

§ Design aspirations:

σ_c = Greatest acceptable stress.

σ_w = Very low and desirable stress, “windfall”, where:

$$\sigma_w \ \leq \ \max_{0 \leq x \leq L} |\sigma_{\widetilde{\phi},T}(x)| \ \leq \ \sigma_c \tag{23}$$

Usually: $\sigma_w \ \ll \ \sigma_c$.

§ Opportuneness of design $T(x)$: min h at which windfall is possible.

$$\widehat{\beta}(T, \sigma_w) = \min \left\{ h : \left(\min_{\phi \in \mathcal{U}(h)} \max_{0 \leq x \leq L} |\sigma_{\phi,T}(x)| \right) \leq \sigma_w \right\} \tag{24}$$

§ Opportuneness function, $\widehat{\beta}(T, \sigma_w)$, is an immunity:

- Immunity to windfall performance.
- **"Big is bad"** for $\widehat{\beta}(T, \sigma_w)$:

$$T \succ_O T^\bullet \quad \text{IF} \quad \widehat{\beta}(T, \sigma_w) < \widehat{\beta}(T^\bullet, \sigma_w) \tag{25}$$

§ Robustness function, $\widehat{h}(T, \sigma_c)$, is an immunity:

- Immunity to failure.
- **"Bigger is better"** for $\widehat{h}(T, \sigma_c)$:

$$T \succ_R T^\bullet \quad \text{IF} \quad \widehat{h}(T, \sigma_c) > \widehat{h}(T^\bullet, \sigma_c) \tag{26}$$

§ **Review exercise 8 on p.14.**

§ Does the robustness preference, $\succ_R$,

agree with the opportuneness preference, $\succ_O$?

Not necessarily.

The immunity functions may be either **Sympathetic** or **Antagonistic.**

§ In the present beam example:

$$\widehat{\beta}(T, \sigma_w) = -\underbrace{\frac{\sigma_w}{\sigma_c}\widehat{h}(T, \sigma_c)}_{A} + \underbrace{\left(1 - \frac{\sigma_w}{\sigma_c}\right)\widetilde{\phi}}_{B} \tag{27}$$

§ Sympathetic immunities:

- B does not depend upon design, $T(x)$
- A is positive.
- Any change in $T(x)$ which

  improves the robustness ($\widehat{h}$ grows),

  also

  improves the opportuneness ($\widehat{\beta}$ diminishes).

§ **Review exercise 9 on p.14.**

§ Many examples of antagonistic immunities.

E.g., when "B" **does** depend on decision.

§ Immunity functions may change from
antagonistic to sympathetic
in different domains of the problem.

6 Review Exercises

§ The exercises in this section are not homework problems, and they do not entitle the student to credit. They will assist the student to master the material in the lecture and are highly recommended for review and self-study.

1. Explain intuitively why the two design requirements in eqs.(1) and (2) on p.3 conflict.

2. Explain why the negative slope in fig. 2 on p.4 reflects a trade off or conflict between the two design requirements.

3. Explain why not knowing the value of the horizon of uncertainty, h, implies that a worst case is not known (p.5).

4. Explain why the positive slope in fig. 3 on p.7 represents a trade off. What trades off against what?

5. Explain why $\sigma^\bullet$ in eq.(14) on p.7 is infeasible.

6. $\sigma^\bullet$ in eq.(14) equals $R(\widehat{T}_\theta)$ in eq.(15) on p.7. Explain why this equality represents a "clash between performance and robustness."

7. Explain why '$\gamma > 0$' in eq.(16) on p.9 represents a sub-optimal design.

8. Explain why "big is bad" in eq.(25) and "bigger is better" in eq.(26) on p.12.

9. Explain why the robustness and opportuneness are sympathetic if B in eq.(27) on p.12 does not depend on the design.