Problem Set on

Robustness and Opportuneness

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101. Energy conservation by feedback (based on exam in 035018, 22.5.2019), p.123 (p.345).

102. Quality control with weighted means (based on exam in 035018, 14.7.2019), p.125 (p.347).

103. Multi-site adverse events (based on exam in 035018, 26.9.2019), p.127 (p.350).

104. Targeting an evasive object (Exam in 036057, 21.1.2020), p.129 (p.352).

Based on models developed in: Cindy E. Hauser and Michael A. McCarthy, 2009, Streamlining ‘search and destroy’: Cost-effective surveillance for invasive species management, *Ecology Letters,* 12: 683–692.

1. **Robustness and opportuneness.** Consider an uncertain scalar function $u(t)$. Adopt the following "minimal requirement" in the definition of robustness:

$$u(t) \geq r_{\rm c} \quad \text{for} \quad 0 \leq t \leq T \tag{1}$$

Likewise, the condition for "sweeping success" in the definition of the opportuneness is chosen to be:

$$u(t) \geq r_{\rm w} \quad \text{for} \quad 0 \leq t \leq T \tag{2}$$

where $r_{\rm w}$ is greater, usually much greater, than $r_{\rm c}$.

For each of the info-gap models listed below,

(a) Evaluate the robustness $\widehat{h}$ and the opportuneness $\widehat{\beta}$.

(b) Compare these two immunities by expressing one as a function of the other. Also, note their different variation with the threshold values, $r_{\rm c}$ and $r_{\rm w}$.

(c) Explain why "bigger is better" for $\widehat{h}$, while "big is bad" for $\widehat{\beta}$.

Uniform bound:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u(t) : \ |u(t) - \widetilde{u}(t)| \leq h \right\}, \quad h \geq 0 \tag{3}$$

Energy bound:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u(t) : \ \int_0^\infty [u(t) - \widetilde{u}(t)]^2 \, {\rm d}t \leq h^2 \right\}, \quad h \geq 0 \tag{4}$$

Fourier ellipsoid bound. The uncertain function is expanded in a truncated Fourier series:

$$\begin{aligned} u(t) &= \widetilde{u}(t) + \sum_{n=n_1}^{n_2} [a_n \cos n\pi t + b_n \sin n\pi t] && (5) \\ &= \widetilde{u}(t) + c^T \phi(t) && (6) \end{aligned}$$

where c is the vector of uncertain Fourier coefficients and $\phi(t)$ is the vector of corresponding trigonometric functions. The info-gap model is:

$$\mathcal{U}(h, 0) = \left\{ u(t) = \widetilde{u}(t) + c^T \phi(t) : \ c^T W c \leq h^2 \right\}, \quad h \geq 0 \tag{7}$$

where W is a known, real, symmetric, positive definite matrix.

2. **Robustness, opportuneness and reward.** Consider an uncertain scalar u for which the reward function is:

$$R(q, u) = q_1 u + q_2 \tag{8}$$

where the coefficients q_1 and q_2 can be controlled by the decision maker, so that the decision vector is $q = (q_1, q_2)^T$. Small values of $R(q, u)$ are more desirable than large values.

For each of the info-gap models listed in problem 1,

(a) Evaluate the robustness and opportuneness functions, $\widehat{h}(q, r_{\rm c})$ and $\widehat{\beta}(q, r_{\rm w})$.

(b) In each case, explore the variation of these immunity functions as the decision vector q is changed. If q is modified to improve the robustness, does the opportuneness improve or deteriorate? Do optima exist for the immunity functions? If not, impose constraints on q to allow an optimum. Do the robust-optimal and opportuneness-optimal decisions agree?

3. **Linear optimization on an ellipsoid.** (p.138) Let x be a vector in the N-dimensional ellipsoidal set:

$$\mathcal{X} = \left\{ x : \; x^T W x \le 1 \right\} \tag{9}$$

where W is a real, symmetric, positive definite matrix with eigenvectors $u_1, \ldots, u_N$ and corresponding eigenvalues $\mu_1 \le \cdots \le \mu_N$.

Let y be an N-vector of unit length:

$$y^T y = 1 \tag{10}$$

Find the vector y for which $x^T y$ is a maximum for all x in $\mathcal{X}$. We can state this more explicitly as follows. Define the function:

$$f(y) = \max_{x \in \mathcal{X}} x^T y \tag{11}$$

What we are seeking is the vector $\widehat{y}$ for which $f(y)$ is a maximum:

$$f(\widehat{y}) = \max_{y^T y = 1} f(y) \tag{12}$$

4. **Static deflection of a cantilever.** (p.140) In many problems the reward function can be expressed as a linear function of an info-gap uncertain vector. We have seen some examples already. Here is another example.

   Consider a uniform cantilever beam subject to N static point loads applied perpendicular to the beam axis. The loads all lie in a single plane. The vector of loads is denoted:

$$f = (f_1, \ldots, f_N)^T \tag{13}$$

   The deflection of the free end of the cantilever, y, is linearly related to the loads by:

$$y = k^T f \tag{14}$$

   where k is a column vector of known flexibility coefficients.

   The load vector is uncertain and belongs to the following Fourier ellipsoid bound info-gap model:

$$F(h, \widetilde{f}) = \left\{ f : \left(f - \widetilde{f} \right)^T W \left(f - \widetilde{f} \right) \leq h^2 \right\}, \quad h \geq 0 \tag{15}$$

   where W is a real, symmetric, positive definite matrix.

   (a) If h is given, find the load vector f which results in the maximum end deflection.

   (b) If h is given, find the maximum end deflection.

   (c) The beam fails if the end deflection exceeds the critical value y_{c}. What is the robustness function of the beam?

   (d) Now consider the choice between two designs with flexibility vectors k_1 and k_2 for which:

$$k_1^T \widetilde{f} \quad > \quad k_2^T \widetilde{f} \tag{16}$$

$$k_1^T W^{-1} k_1 \quad < \quad k_2^T W^{-1} k_2 \tag{17}$$

   What is the meaning of eqs.(16) and (17)? What dilemma is embedded in them? For what values of y_{c} do you prefer option 1?

   (e) Now assume that h is a random variable with a known exponential distribution. Combine this information with the info-gap uncertainty about the load vector f to evaluate a hybrid robustness.

5. **Dynamic deflection of a cantilever.** Consider a rigid cantilever of length L and mass μ. The angle $\theta(t)$ between the cantilever axis and the support oscillates with linear rotational stiffness k [Nm/radian]. An external moment of force $M(t)$ is applied to the free end of the cantilever. For the initial T seconds of oscillation the moment of force varies from the nominal value M_o in an unknown but bounded manner; after time T the moment vanishes. We can represent the uncertainty in $M(t)$ with the envelope-bound info-gap model:

$$\mathcal{U}(h, M_o) = \{M(t): \ |M(t) - M_o| \le h\psi(t)\}, \quad h \ge 0 \tag{18}$$

with the envelope function $\psi(t)$ chosen as:

$$\psi(t) = \begin{cases} 1, & 0 \le t \le T \\ 0, & t \ge T \end{cases} \tag{19}$$

The angular deflection $\theta(t)$ of the rigid beam is described by:

$$J\ddot{\theta}(t) + k\theta(t) = M(t) \tag{20}$$

where the moment of inertia of the beam is $J = \mu L^2/3$.

The system can tolerate deviation of the angular deflection of the beam by no more than $\theta_{\rm c}$. Ideally, the angular deflection is less than a much smaller threshold, $\theta_{\rm w}$.

   (a) Evaluate the robustness of the system, in terms of the amount of input uncertainty which the system can tolerate without failing. Consider durations less than T and much longer than T.

   (b) Evaluate the opportuneness of the system, in terms of the amount of input uncertainty which must be present in order for it to be possible that the beam deviates by no more than $\theta_{\rm w}$. Consider durations less than T and much longer than T.

   (c) Consider the stiffness k and the moment of inertia J as decision variables, so $q = (k, J)^T$ is the decision vector. Explore the variation of the robustness and of the opportuneness with q.

6. **Static deflection of a cantilever: continued.** (p.142) In some problems the reward function can be expressed as a quadratic function of an info-gap uncertain vector. We modify problem 4 as an example. Instead of the deflection y in eq.(14), which is a linear performance function, let us consider:

$$r = f^T V f \tag{21}$$

r is proportional to an elastic energy of deformation. V is a known, real, symmetric, positive definite matrix.

Re-examine questions 4a–4c for the following two cases: **(a)** $\widetilde{f} = 0$. **(b)** $\widetilde{f} \ne 0$.

7. **Flaw-resistant manufacture** (p.144). An automatic milling machine is equipped with a sensor system which detects flaws in the workpiece. When a flaw is detected, the cutting tool is raised above the surface of the work piece. This lifting mechanism is functional provided that the cutting tool is raised at least 1mm above the surface within 2 seconds of detecting the flaw. This is because the height of the workpiece varies by less than 1mm during 2 seconds of feed. The height of the cutting tool as a function of time develops nominally according to the function:

$$\widetilde{x}(t) = ht^2 \ \ [\mathrm{mm}] \tag{22}$$

However, there is uncertainty in the response of the lifting mechanism due to wear in the power chain. In other words, the height as a function of time does not always behave according to $\widetilde{x}(t)$. Uncertainty in the height as a function of time is described by the uniform-bound info-gap model:

$$\mathcal{U}(h, \widetilde{x}) = \{x(t) : \ |x(t) - \widetilde{x}(t)| \le h\}, \quad h \ge 0 \tag{23}$$

The uncertainty parameter h describes the inaccuracy in the tool height as a result of wear in the power chain. What is the robustness of the lifting mechanism? What is the opportuneness function? What is the interpretation of these immunity functions, and how are they used in evaluating the nominal performance of the flaw-recovery system?

8. **Quality control.** (p.142) A long, thin, cylindrically symmetrical surgical needle is finely milled to conform to its specified shape, $\widetilde{t}(x)$. The allowed deviation of the actual thickness $t(x)$ from the designed thickness $\widetilde{t}(x)$ is $\pm D$ microns throughout the length L of the needle. The manufacturer guarantees the following two conditions:

   (i) The precise dimension of the needle will be verified at each end by direct measurement. Any needle for which $t(0) \neq \widetilde{t}(0)$ or $t(L) \neq \widetilde{t}(L)$ will be rejected. This quality control measurement is completely accurate (or its accuracy is vastly greater than the allowed tolerance).

   (ii) The deviation of the slope of the surface of the needle from its specified slope is bounded, in an attempt to exclude bristles and dents. However, the value of the bound on the slope is unknown.

   (a) Formulate an info-gap model for the shape of needles produced to these specifications.

   (b) Construct the robustness and opportuneness functions for this manufacturing process.

   (c) The manufacturer now proposes to replace the first condition above with the following extended quality check:

      (i′) The precise dimension of the needle will be verified at each end *and* at N equi-distant intermediate points by direct measurement. Any needle which deviates from the specified dimension at any measured point will be rejected.

      The second condition, (ii), remains valid. Construct the immunity functions and use them to choose the number N of measurement points. In particular, study the marginal utility of the Nth measurement.

   (d) ‡ What modification of the initial information, and consequently of the info-gap model, would lead to a more meaningful opportuneness function?

   (e) ‡ Modify the quality control specification in item (i) to consider error in the quality control measurement. Specifically, any needle for which $|t(0) - \widetilde{t}(0)| > \varepsilon D$ or $|t(L) - \widetilde{t}(L)| > \varepsilon D$ will be rejected, where $0 \leq \varepsilon \leq 1$. Now repeat questions 8a and 8b for the robustness function only.

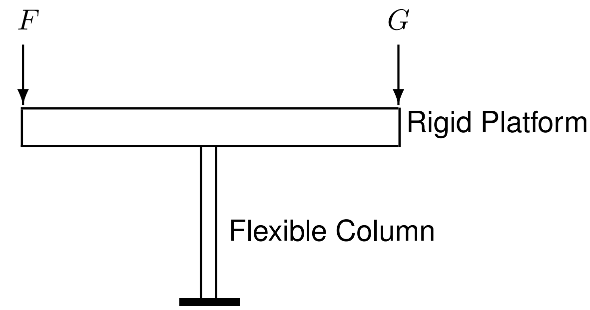

Figure 1: Platform for problem 9.

9. **Stability of a platform.** (p.145) A thin rigid beam-like platform is supported from below at its midpoint by a flexible column which is at elastic equilibrium when the platform is horizontal, as shown in fig. 1. The flexural stiffness of the elastic column is k [Nm/radian] and it applies a restoring moment of force $M = k\theta$ at the midpoint when the platform is tilted by θ radians. The width of the platform is $2L$ [m]. The platform is loaded at its two ends by static forces F and G which are uncertain but bounded. That is, forces F and G belong to the following info-gap model of uncertainty:

$$\mathcal{U}(h, 0) = \{F, G : \; |F| \; \leq \; h, \quad |G| \; \leq \; h\}, \quad h \geq 0 \tag{24}$$

The platform is satisfactorily level if the angle of tilt at static equilibrium is never greater than the critical value $\theta_{\rm c}$:

$$|\theta| \leq \theta_{\rm c} \tag{25}$$

The condition of static equilibrium requires that the moment of force at the midpoint vanish:

$$0 = FL - GL + k\theta \tag{26}$$

Determine the robustness function of the platform. Consider first the static equilibrium of the platform, and then consider the rotational vibration of the platform. The decision vector is $q = (k, L)^T$. Study the variation of the immunity functions as these design variables are changed.

10. **Stability of a platform: continued.** (p.147) We now modify problem 9 to consider uncertain distributed loads, $f(x)$ [N/m], $-L \le x \le L$, on the platform. Evaluate the robustness and opportuneness for each of the following info-gap models for uncertainty in the load.

    (a) *Uniform-bound:*

    $$\mathcal{U}(h, \widetilde{f}) = \left\{ f(x) : \ \left| f(x) - \widetilde{f} \cos \frac{\pi x}{L} \right| \le h \right\}, \quad h \ge 0 \tag{27}$$

    where $\widetilde{f}$ is a known constant.

    (b) *Fourier ellipsoid bound:* The uncertain part of the load profile is a truncated sine series:

    $$\begin{aligned} f(x) &= \widetilde{f} \cos \frac{\pi x}{L} + \sum_{n=1}^{N} c_n \sin \frac{n \pi x}{L} && (28) \\ &= \widetilde{f} \cos \frac{\pi x}{L} + c^T \sigma(x) && (29) \end{aligned}$$

    where c is the vector of uncertain Fourier coefficients and $\sigma(x)$ is the vector of sine functions. The info-gap model is:

    $$\mathcal{U}(h, \widetilde{f}) = \left\{ f(x) = \widetilde{f} \cos \frac{\pi x}{L} + c^T \sigma(x) : \ c^T W c \le h^2 \right\}, \quad h \ge 0 \tag{30}$$

    where W is a known, real, symmetric, positive definite matrix.

    (c) *Different nominal load.* How will the answers to questions 10a and 10b change if the nominal load is:

    $$\widetilde{f}(x) = \widetilde{f} \sin \frac{\pi x}{L} \tag{31}$$

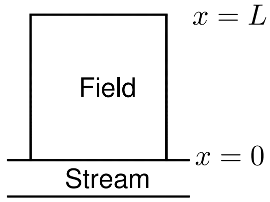

Figure 2: Illustration for problem 11.

11. **Environmental contamination.** The concentration of contaminant, $f(x)$, varies over a farmer's field as a function of distance x from a stream passing along one side of the field:

$$f(x) = \widetilde{f} + \sum_{n=1}^{N} \phi_n x^n \tag{32}$$

$$= \widetilde{f} + \phi^T \xi \tag{33}$$

where ϕ is the vector of unknown coefficients and ξ is the corresponding vector of powers of x. The nominal concentration, $\widetilde{f}$, is a known constant. The stream is located at $x = 0$ and the far edge of the field is at $x = L$.

The quantity of contaminant which reaches the stream by the end of the season is:

$$g = \mu \int_0^L \left[1 - \left(\frac{x}{L} \right)^{\nu} \right] f(x) \, \mathrm{d}x \tag{34}$$

where μ and ν are positive constants which reflect absorption and transport properties of the soil, and which can be influenced in known ways by treating the field.

The water in the stream is potable if the quantity of contaminant does not exceed the value g_{c}, but it is highly desirable that the quantity of contaminant not exceed the far lower value g_{w}.

For the Fourier ellipsoid bound info-gap model specified below, determine the robustness and opportuneness functions and study their behavior as a function of the decision variables μ and ν. Discuss the choice of these variables. Consider the antagonism and sympathy of the immunity functions, $\widehat{h}$ and $\widehat{\beta}$.

$$\mathcal{U}(h, \widetilde{f}) = \left\{ f(x) = \widetilde{f} + \phi^T \xi : \ \phi^T W \phi \le h^2 \right\}, \quad h \ge 0 \tag{35}$$

where W is a known, real, symmetric, positive definite matrix.

12. **Financial investment.** Consider an investment problem in which a vector q of investments in N different options results in a vector x of returns in M different commodities. Investments and returns are related by:

$$x = Aq \tag{36}$$

where the $M \times N$ matrix A is uncertain.

The returns are satisfactory if:

$$x_m \ge x_{m,\mathrm{c}}, \quad m = 1, \ldots, M \tag{37}$$

The investment results in windfall returns if:

$$x_m \ge x_{m,\mathrm{w}}, \quad m = 1, \ldots, M \tag{38}$$

where:

$$x_{m,\mathrm{w}} > x_{m,\mathrm{c}}, \quad m = 1, \ldots, M \tag{39}$$

Construct the robustness and opportuneness functions for each of the info-gap models listed below. Discuss the use of these immunity functions in evaluating an investment.

Interval bound:

$$\mathcal{U}(h, \widetilde{A}) = \left\{ A : \ \left| A_{mn} - \widetilde{A}_{mn} \right| \le h, \qquad \begin{array}{l} m = 1, \ldots, M, \\ n = 1, \ldots, N \end{array} \right\}, \quad h \ge 0 \tag{40}$$

Row-wise ellipsoid bound: For any matrix R, let R_m be a row vector denoting the mth row of R.

$$\mathcal{U}(h, \widetilde{A}) = \left\{ A = \widetilde{A} + R : \ R_m W R_m^T \le h^2, \ m = 1, \ldots, M \right\}, \quad h \ge 0 \tag{41}$$

where W is a known, real, symmetric, positive definite matrix.

13. (p.154) **Satellite targeting.**

   (a) A satellite is launched from point L directly at a stationary target located a distance D away at point T. The satellite moves in a single plane, but the slope of the satellite trajectory, with respect to the line LT, varies in an unknown manner during flight. The satellite carries a payload of photographic devices, which are effective only if the satellite-target distance is no greater than $r_{\rm c}$ as the satellite passes the target. The payload is highly effective if the fly-by distance is as small as $r_{\rm w}$. Evaluate the robustness and opportuneness functions.

   (b) Now consider a slightly more complex situation. The satellite is initially launched at an angle below the line from L to T and at a known slope $\widetilde{s}$ with respect to the line from L to T. The slope of the satellite trajectory deviates in flight from $\widetilde{s}$ in an unknown manner. The mission fails if the satellite passes too far from the target or too near to it. That is, failure occurs if the satellite-target distance is greater than $r_{\rm c,2}$ or less than $r_{\rm c,1}$ as the satellite passes the target, where $r_{\rm c,1} < r_{\rm c,2}$. Evaluate the robustness function.

   (c) Continue part (13b) and consider opportuneness. The mission is highly successful if the satellite passes no farther than $r_{\rm w,2}$ from the target, and no closer than $r_{\rm w,1}$. These values are related as:

$$r_{\rm c,1} < r_{\rm w,1} < r_{\rm w,2} < r_{\rm c,2} \tag{42}$$

   Evaluate the opportuneness function.

14. ‡ **Heat conduction.** (p.158) Consider an unknown one-dimensional heat source, $g(x)$ [W/m], distributed along x between $+1$ and -1. The temperature distribution is $T(x)$ degrees K. The source is thermally insulated along its length, and cooled only at each end. The heat-source density function, $g(x)$, is uncertain and belongs to an info-gap model.

Safe operation requires that the central temperature be less than a critical value:

$$T(0) \leq T_{\mathrm{c}} \tag{43}$$

We are able to control the end temperatures, $T(\pm 1)$.

The differential equation for heat conduction along the axis of the source is:

$$0 = \frac{\mathrm{d}^2 T(x)}{\mathrm{d}x^2} + \frac{g(x)}{k} \tag{44}$$

where k is the thermal conductivity, in units of W·m/K.

Consider the following two info-gap models.

Uniform bound:

$$\mathcal{U}(h, \widetilde{g}) = \{g(x) : \ |g(x) - \widetilde{g}| \leq h\}, \quad h \geq 0 \tag{45}$$

where $\widetilde{g}$ is a known constant.

Fourier ellipsoid bound:

$$\mathcal{U}(h, \widetilde{g}) = \left\{ g(x) = \widetilde{g} + c^T \gamma(x) : \ c^T W c \leq h^2 \right\}, \quad h \geq 0 \tag{46}$$

where W is a known, real, symmetric, positive definite matrix and $\gamma(x)$ is the vector:

$$\gamma(x) = (\cos \pi x, \, \cos 2\pi x, \, \ldots, \, \cos N\pi x)^T \tag{47}$$

$\widetilde{g}$ is a known constant.

(a) Study the robustness and the opportuneness as a function of surface temperature, for each of the above info-gap models of heat-source uncertainty. Discuss the meaning of these two immunity functions. Develop general expressions for the immunity functions and then consider the special case where W is the following diagonal matrix:

$$W = \mathrm{diag}\left(\frac{1}{n^2}, \, n = 1, \ldots, 6 \right) \tag{48}$$

(b) Now consider a specific numerical case. The material is steel, whose thermal conductivity is $k = 17.3$ [W·m/K]. The critical temperature is $T_{\mathrm{c}} = 400$ [K]. The nominal heat-source density is $\widetilde{g} = 250$ [W/m]. For each of the info-gap models, what range of surface temperature values are very reliable? Very unreliable? Compare the results for the two info-gap models.

15. **Simple harmonic oscillator.** (p.164) The displacement $x(t)$ of an oscillator in simple harmonic motion is:

$$m\ddot{x}(t) + kx(t) = u(t) \tag{49}$$

where the mass m and stiffness k are known but the driving force $u(t)$ is uncertain. Assume that the initial displacement and velocity of the oscillator are both zero.

The energy of the oscillator is proportional to the square of the displacement. Failure occurs if the energy at a specified time T exceeds a critical value:

$$x^2(T) \geq E_{\mathrm{c}} \tag{50}$$

Note that E_{c} is in units that are proportional (but not equal) to energy.

The uncertainty in $u(t)$ is represented by the following Fourier ellipsoid bound info-gap model:

$$\begin{aligned} u(t) &= \widetilde{u}(t) + \sum_{n=1}^{N} \phi_n \sin\frac{n\pi t}{T} && (51) \\ &= \widetilde{u}(t) + \phi^T \sigma(t) && (52) \end{aligned}$$

where ϕ is the vector of uncertain Fourier coefficients and $\sigma(t)$ is the vector of corresponding sine functions. The info-gap model is:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u(t) = \widetilde{u}(t) + \phi^T \sigma(t) :\ \phi^T W \phi \leq h^2 \right\}, \quad h \geq 0 \tag{53}$$

where W is a known, real, symmetric, positive definite matrix.

(a) Evaluate the robustness function and illustrate its use in choosing the system parameters, m and k. Assume that the initial displacement and velocity of the oscillator are both zero.

(b) Evaluate the opportuneness function and illustrate its use in choosing the system parameters, m and k.

16. ‡ **Two coupled harmonic oscillators.** (p.167) Consider two equal masses and three identical linear springs connected in sequence between two rigid walls as in fig. 3.

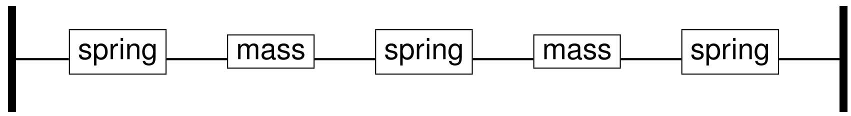

Figure 3: Mass-spring system for problem 16.

The masses are initially at rest in their equilibrium positions. An uncertain time varying force $u(t)$ is applied to the righthand mass. Let $x_1(t)$ and $x_2(t)$ denote the subsequent displacements of the left and righthand masses.

There are two natural modes of vibration of this system:

(a) In the 'coherent' mode the two masses oscillate together from right to left and back again. In this mode, the masses are always moving in the same direction.

(b) In the 'anti-coherent' mode the two masses are always moving in opposite directions: towards each other or away from each other.

Failure occurs if the amplitude of the incoherent mode exceeds the critical value $A_{\rm c}$.

Construct the robustness and opportuneness functions of the system for each of the following info-gap models of uncertainty in the load:

Uniform bound:

$$\mathcal{U}(h,0) = \{u(t): \ |u(t)| \le h\}, \quad h \ge 0 \tag{54}$$

Energy bound:

$$\mathcal{U}(h,0) = \left\{u(t): \ \frac{1}{T}\int_0^\infty u^2(t)\,{\rm d}t \le h^2\right\}, \quad h \ge 0 \tag{55}$$

where T is a characteristic time of interest.

Fourier ellipsoid bound:

$$u(t) = \sum_{n=1}^{N} c_n \sin\frac{n\pi t}{T} \tag{56}$$

$$= c^T\sigma(t) \tag{57}$$

where c is the vector of unknown Fourier coefficients, $\sigma(t)$ is the vector of corresponding sine functions, and T is a characteristic time of interest.

$$\mathcal{U}(h,0) = \left\{u(t) = c^T\sigma(t): \ c^TWc \le h^2\right\}, \quad h \ge 0 \tag{58}$$

where W is a known, real, symmetric, positive definite matrix.

Note that h has the same units as $u(t)$ in all three info-gap models.

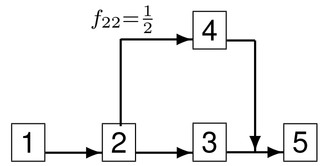

Figure 4: A 5-task project schedule for problem 17.

17. **Project management.** (p.171) Consider the 5-task project shown in fig. 4. This project has two task-paths:

Path 1: 1 $\longrightarrow$ 2 $\longrightarrow$ 3 $\longrightarrow$ 5
Path 2: 1 $\longrightarrow$ 2 $\longrightarrow$ 4 $\longrightarrow$ 5

In path 2, task 4 is initiated when task 2 is half finished, as indicated by $f_{22} = 0.5$.

The nominal task durations are:

$$\widetilde{t}_1 = \widetilde{t}_2 = \widetilde{t}_5 = 1, \quad \widetilde{t}_3 = q, \quad \widetilde{t}_4 = 1 - q \tag{59}$$

where q is a parameter which the project manager is free to choose in the interval $[0, 1]$. q represents a valuable resource which must be allocated between tasks 3 and 4.

The uncertainty in the actual task durations is represented by an interval info-gap model:

$$\mathcal{U}(h, \widetilde{t}) = \left\{ t : \ \frac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \le h, \ n = 1, \ldots, 5 \right\} \quad h \ge 0 \tag{60}$$

The project must be completed within about 4 time units. Construct the robustness and opportuneness functions and demonstrate their use in choosing q.

18. **Project management: continued.** Consider a modification of problem 17. If the project completes in a duration T, then the "reward" is $R(T)$. $R(T)$ decreases as T grows to express the penalty associated with delayed termination. The project owner would very much like to earn reward as large as r_{w}, and cannot tolerate reward less than r_{c}. Formulate and evaluate the robustness and opportuneness functions. Indicate the choice of the parameter q. Explain how the robustness and opportuneness functions can be used to choose aspirations r_{c} and r_{w}. Compare these results to the solution of problem 17.

19. ‡ **Project management: continued.** Consider the 16-task project in fig. 5.

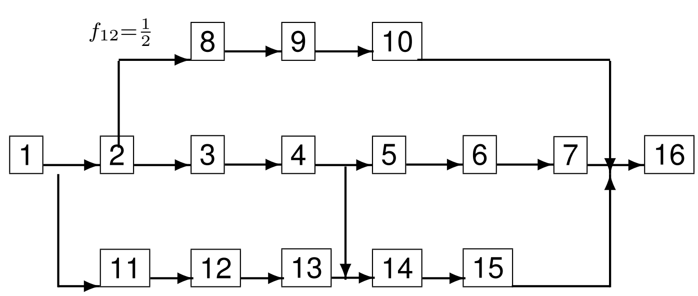

Figure 5: A 16-activity project schedule for problem 19.

t_n is the duration of the nth task, which is uncertain, and t is the vector of task durations. The project must be completed within the duration T_c.

The info-gap model for uncertainty in t is:

$$\mathcal{U}(h, \widetilde{t}) = \left\{ t : \ \frac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \le w_n h, \ n = 1, \ldots, N \right\}, \quad h \ge 0 \tag{61}$$

The nominal durations $\widetilde{t}_n$ and uncertainty weights w_n are given in table 1.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\widetilde{t}_n$	1	1	2	3	3	3	2	1	2	3	3	3	1	3	2	1
w_n	1	1	1	1	1	1	1	1	1	1	3	2	2	3	2	1

Table 1: Nominal durations and uncertainty-weights for problem 19.

(a) Evaluate the dependence of the path-robustnesses and the overall robustness upon the participation factor f_{12} and the nominal duration $\widetilde{t}_2$ of task 2. Perform numerical calculations and discuss implications for improving the reliability of the project.

(b) Construct the opportuneness function of the project and evaluate its dependence on the participation factor f_{12} and the nominal duration $\widetilde{t}_2$ of task 2. Discuss the meaning of this immunity function, and its use, in conjunction with the robustness function, in designing and managing the project.

(c) Evaluate the dependence of the path-robustnesses and the overall robustness upon the uncertainty weight w_{14} of task 14. Perform numerical calculations and discuss implications for improving the reliability of the project.

(d) We now modify the performance requirement of the project. Time over-runs are allowed but penalized, while early completion is rewarded. The reward function for completing the project in duration T is:

$$R(T) = R_0 \mathrm{e}^{-\mu(T - T_c)} \tag{62}$$

where T is the total project duration, and T_c, R_0 and μ are known positive constants.

The project manager is instructed to select an "attainable" reward goal r_c, and to choose a "realistic" project duration T_c.

Determine the robustness and opportuneness functions for completion of the project. Study the variation of these immunity functions with T_c and r_c. Explain how these results assist the project manager in choosing T_c and r_c.

20. **Control of a production system.** The performance of a production system depends on temperature T, pressure P and time θ according to the relation:

$$y = T - 2P + \theta/3 \tag{63}$$

Define the vectors $x^T = (T, P, \theta)$ and $z^T = (1, -2, 1/3)$. Eq.(63) becomes:

$$y = z^T x \tag{64}$$

The system operator chooses operating values for T, P and θ, which will be denoted $\widetilde{T}$, $\widetilde{P}$ and $\widetilde{\theta}$, or collectively: $\widetilde{x}^T = (\widetilde{T}, \widetilde{P}, \widetilde{\theta})$.

The actual implemented value of x can vary in an unknown manner from the selected operational value $\widetilde{x}$. The uncertain deviation of x from $\widetilde{x}$ is described by an info-gap model:

$$\mathcal{U}(h, \widetilde{x}) = \left\{ x : \left| \frac{x_i - \widetilde{x}_i}{\widetilde{x}_i} \right| \le h, \quad i = 1,\, 2,\, 3 \right\}, \quad h \ge 0 \tag{65}$$

The product of the system is unsatisfactory if:

$$y > y_{\rm c} \tag{66}$$

where $y_{\rm c}$ is a known value.

(a) Derive an expression for the robustness of this production system, to uncertain variation of the system parameters.

(b) Consider the choice of the operational value $\widetilde{T}$, which must be selected in the interval $69^{\rm o}$C $\le \widetilde{T} \le 85^{\rm o}$C. What value would you recommend, and why?

21. **Reliability of a milling process–1.** (p.175) An automated cutting tool moves at constant horizontal velocity across a work piece. The height $y(t)$ of the tool varies in transit. The desired height profile is $\widetilde{y}(t)$:

$$\widetilde{y}(t) = \sum_{n=n_1}^{n_2} \widetilde{b}_n \cos \frac{n\pi t}{T} = \widetilde{b}^T \gamma(t), \quad 0 \le t \le T \tag{67}$$

where $\widetilde{b}$ is the vector of known Fourier coefficients and $\gamma(t)$ is the vector of corresponding cosine functions.

The actual height profile differs from $\widetilde{y}(t)$ in an uncertain manner:

$$y(t) = \widetilde{y}(t) + \sum_{n=n_1}^{n_2} b_n \cos \frac{n\pi t}{T} = \widetilde{y}(t) + b^T \gamma(t), \quad 0 \le t \le T \tag{68}$$

where b is a vector of unknown Fourier coefficients. Uncertainty in b is represented by the following info-gap model:

$$\mathcal{U}(h, \widetilde{b}) = \left\{ b : \left| \frac{b_n}{\widetilde{b}_n} \right| \le h, \quad n = n_1,\, \ldots, n_2 \right\}, \quad h \ge 0 \tag{69}$$

The milling process fails if the cutting tool is too far above the planned height at the end of the run, $t = T$. That is, failure is defined as:

$$y(T) - \widetilde{y}(T) > D_{\rm c} \tag{70}$$

(a) Formulate an expression that defines the robustness function.

(b) Derive an explicit algebraic expression for the greatest deviation of the actual height above the planned height, up to uncertainty h.

(c) Derive an explicit algebraic expression for the robustness function.

(d) Repeat parts 21a–21c if the failure criterion is revised as follows. Failure occurs if the actual height deviates by more than $D_{\rm c}$ from the planned height at any time during the milling run. That is, failure is:

$$|y(t) - \widetilde{y}(t)| > D_{\rm c}, \quad 0 \le t \le T \tag{71}$$

22. **Reliability of robotic motion.** (p.176) The arm of a robot moves in the (x, y) plane. The trajectory of the end effector as a function of time t is specified by:

$$x(t) = c_1(t+1) \tag{72}$$
$$y(t) = \frac{c_2}{t+1} \tag{73}$$

for $t \geq 0$. The coefficients c_1 and c_2 are uncertain and the uncertainty is described by the following info-gap model:

$$\mathcal{U}(h, \widetilde{c}) = \{c: \ |c_i - \widetilde{c}_i| \leq h\widetilde{c}_i, \ \ i = 1, 2\}, \quad h \geq 0 \tag{74}$$

The values of $\widetilde{c}_i$ are known and positive.

The end effector must reach the following location at specified time $t = T$:

$$\widetilde{x} = \widetilde{c}_1(T+1) \tag{75}$$
$$\widetilde{y} = \frac{\widetilde{c}_2}{T+1} \tag{76}$$

The robot motion fails if the end-effector location at time T deviates from the desired location by more than a distance D_{c}.

(a) Derive an expression for the robustness of the robotic motion.

(b) What is the optimal value of T, from the point of view of robustness? (Note that the final coordinates, $\widetilde{x}$ and $\widetilde{y}$, change as T changes. Thus, we aren't really concerned with *where* the end effector ends up, but just that it *meet* an object whose position will be $(\widetilde{x}, \widetilde{y})$.) Does the robust-optimal duration depend on the critical distance? What is the significance of this?

(c) Now let's change this to a real tracking problem. An object moves uncertainly on the plane with $(x(t), y(t))$ coordinates given by eqs.(72) and (73). The coefficients are uncertain as represented by the info-gap model of eq.(74). The object will be tracked by the end effector which moves precisely with coordinates $(\xi(t), \eta(t))$. The goal is to satisfice the squared path error:

$$\Delta = \int_0^T \left([x(t) - \xi(t)]^2 + [y(t) - \eta(t)]^2 \right) \mathrm{d}t \tag{77}$$

We require $\Delta \leq \Delta_{\mathrm{c}}$. We want to choose the end effector's path. To make things simpler, let's suppose the end effector follows the same parametric path as the object, but with possibly different—known—coefficients:

$$\xi(t) = \gamma_1(t+1) \tag{78}$$
$$\eta(t) = \frac{\gamma_2}{t+1} \tag{79}$$

Given γ, determine the robustness of any choice of tracking error Δ_{c}. Show that $\gamma = \widetilde{c}$ is maximally robust for any Δ_{c}. (Hint: evaluate the inverse of the robustness function.)

23. **Planning machine initialization.** (p.179) A manufacturing machine runs continuously, and the longer it runs, the more it produces. However, the productivity, in terms of number of items produced per unit time, decreases the longer the machine runs. The total amount produced in time t is $g(t)$, which is an increasing function whose slope decreases in time. That is, $\dot{g}(t) > 0$ and $\ddot{g}(t) < 0$. The function $g(t)$ is imperfectly known. An info-gap model for uncertainty in $g(t)$ is:

$$\mathcal{U}(h, \widetilde{g}) = \left\{ g(t) : \ \dot{g}(t) > 0, \ \ddot{g}(t) < 0, \ |g(t) - \widetilde{g}(t)| \le h f(t) \right\}, \quad h \ge 0 \tag{80}$$

where $\widetilde{g}(t)$ and $f(t)$ are known positive functions with positive slope and negative curvature.

You must plan the production schedule for a total time of T hours. If the machine runs continuously for T hours its production will be $g(T)$. You can plan to stop and initialize the machine periodically. After each initialization the machine restarts with its initial productivity, which is high. However, each initialization requires τ hours. You must choose the number of restarts, assuming that the machine will run for the same duration after each initialization. The goal is to assure that the total production will not be less than $r_{\rm c}$. The first initialization of the machine, like all subsequent initializations, requires τ hours.

(a) Derive an explicit expression for the robustness to uncertainty in the production function, when the machine is initialized n times during the T hours, assuming that the machine operates for the same duration after each initialization.

(b) Now assume that:

$$\widetilde{g}(t) = b\sqrt{t} \tag{81}$$

$$f(t) = \sqrt{t} \tag{82}$$

What is the robust-optimal number of initializations?

24. **Efficient fuel allocation.** (p. 182) The distance which a vehicle can travel with quantity q of fuel is:

$$f(q,c) = \frac{q}{1+cq^2} \tag{83}$$

where c is an uncertain parameter described by the following info-gap model:

$$\mathcal{U}(h,\widetilde{c}) = \left\{c: \ |c-\widetilde{c}| \le h\widetilde{c}\right\}, \quad h \ge 0 \tag{84}$$

The vehicle must be able to travel at least a distance $f_{\rm c}$.

(a) Find an explicit algebraic expression for the robustness of fuel quantity q to uncertainty in the coefficient c.

(b) Find an explicit algebraic expression for the quantity of fuel quantity q which maximizes the robustness. Compare this to the value of q which maximizes the distance based on the nominal value of c and discuss the result.

(c) Show that the robustness curves for different quantities of fuel can cross. What does this imply for the choice of fuel quantity?

25. **Moments on a robotic arm.** (p.183) The angle of rotation of a robotic arm, y, varies according to the moments M_i applied at the joints according to:

$$y = \sum_{i=1}^{3} k_i M_i \tag{85}$$

where k_i is a known positive flexibility parameter. The moments M_i are poorly known. The best estimate of M_i is $\widetilde{M}_i$ which is positive. The fractional error of M_i is unknown:

$$\left| \frac{M_i - \widetilde{M}_i}{\widetilde{M}_i} \right| \le h, \quad h \ge 0 \tag{86}$$

(a) The angle of rotation must be at least y_{c}. What is the robustness, to uncertainty in the moments, of the robotic rotation?

(b) In the solution to part (a) you found that the robustness depends on the nominal (anticipated) angle of rotation, $k^T \widetilde{M}$. By considering the robustness function, discuss whether it is desirable to design the robot so that $k^T \widetilde{M}$ is large or small.

(c) The nominal moments $\widetilde{M}_i$ can be chosen subject to the constraint:

$$\sum_{i=1}^{3} \widetilde{M}_i^2 = \mu^2 \tag{87}$$

where μ is known. What choice of the nominal moments maximizes the robustness?

26. **Machine efficiency.** (p.184) The efficiency of a machine is described by:

$$f(q) = q + \frac{c}{q} \tag{88}$$

where $q > 0$, $\widetilde{c} > 0$ and c is uncertain and described by an info-gap model:

$$\mathcal{U}(h, \widetilde{c}) = \{c : \ |c - \widetilde{c}| \le h\sigma\}, \quad h \ge 0 \tag{89}$$

(a) It is required that $f(q)$ be no less than f_c. What is the robustness of the machine to uncertainty in c, for a given value of q?

(b) The designer can choose q in the interval $[q_1, \, q_2]$ What choice of q do you recommend? How does this compare with the putatively optimal choice of q?

27. **Uncertain lotteries.** (p.185) Consider a lottery with two prizes whose values are $v_\ell > v_{\rm s}$. Each participant wins either the large prize or the small prize. The probability of winning the larger prize is uncertain; the best estimate of this probability if $\widetilde{p}$; and the info-gap model for uncertainty in the probability is:

$$\mathcal{U}(h, \widetilde{p}) = \left\{ p : \ 0 \le p \le 1, \ \left| \frac{p - \widetilde{p}}{\widetilde{p}} \right| \le h \right\}, \quad h \ge 0 \tag{90}$$

(a) For any critical value of the expected reward $v_{\rm c}$, such as the cost of a lottery ticket, what is the robustness, to uncertainty in $\widetilde{p}$, of winning at least $v_{\rm c}$ on average?

(b) Now consider a different lottery with prizes $v'_\ell > v'_{\rm s}$ and estimated probability $\widetilde{p}'$ of winning v'_ℓ. Furthermore, the estimated average prize is now greater: $\widetilde{p}' v'_\ell + (1 - \widetilde{p}') v'_{\rm s} > \widetilde{p} v_\ell + (1 - \widetilde{p}) v_{\rm s}$. However, the smaller prize is now even smaller: $v'_{\rm s} < v_{\rm s}$. The uncertainty of the probability is represented with the info-gap model of eq.(90), now centered on $\widetilde{p}'$. Under what conditions (e.g., with what values of $v_{\rm c}$) will you prefer this new lottery? (Consider the crossing of the robustness curves of these two lotteries.)

28. **Braking system–1.** (p.186) Consider a braking system upon which force $f(t)$ acts and for which the stopping distance is:

$$s(g,f) = \int_0^\infty g(t)f(t)\,\mathrm{d}t \tag{91}$$

The sign of $f(t)$ can change over time. The estimated braking function is:

$$\widetilde{g}(t) = \mathrm{e}^{-\mu t}\sin\omega t \tag{92}$$

The physics of braking and energy dissipation is complex and poorly understood. An info-gap model for uncertainty in the braking function is:

$$\mathcal{U}(h,\widetilde{g}) = \left\{ g(t) : \ |g(t) - \widetilde{g}(t)| \le h \right\}, \quad h \ge 0 \tag{93}$$

(a) Given the driving function $f(t)$, and the requirement that the stopping distance not exceed s_c, derive the robustness function.

(b) We must choose between two driving functions, $f_1(t)$ and $f_2(t)$, where:

$$\int_0^\infty |f_1(t)|\,\mathrm{d}t < \int_0^\infty |f_2(t)|\,\mathrm{d}t, \quad s(\widetilde{g},f_1) > s(\widetilde{g},f_2) \tag{94}$$

Use the robustness function to specify the range of s_c-values for which each driving function is preferred.

29. **Ballistics.** (p.187) A missile is designed to follow the trajectory:

$$\widetilde{g}(x) = \frac{\theta x(D-x)}{D} \tag{95}$$

where x is the horizontal distance from the launch site and $\widetilde{g}(x)$ is the height of the trajectory.

(a) The actual trajectory is uncertain due to wind and other disturbances, and is described by an info-gap model:

$$\mathcal{U}(h, \widetilde{g}) = \left\{ g(x) : \ |g(x) - \widetilde{g}(x)| \le hx \right\}, \quad h \ge 0 \tag{96}$$

Operational considerations require that the strike distance of the missile (which happens when the height is zero) be no less than D_{c}. What is the robustness to uncertainty in the trajectory?

(b) The info-gap model of eq.(96) allows unrealistically erratic trajectories. Re-do part (a) with this info-gap model:

$$\mathcal{U}(h, \widetilde{g}) = \left\{ g(x) : \ g(0) = 0, \ |g'(x) - \widetilde{g}'(x)| \le \frac{hx}{D} \right\}, \quad h \ge 0 \tag{97}$$

where the prime implies differentiation. Do you expect the new robustness to be less or greater than the robustness in part (a)? Might the answer depend on the value of D_{c}?

30. **Strain energy.** (p.188) The strain energy E of a mechanical system is described by:

$$E = x^T x \tag{98}$$

where x is proportional to a vector of strains that result from a vector f of forces:

$$x = Vf \tag{99}$$

where V is a known real matrix.

(a) The uncertainty in the forces is described by the following info-gap model:

$$\mathcal{U}(h) = \left\{ f : \ f^T f \le h^2 \right\}, \quad h \ge 0 \tag{100}$$

We require that the strain energy not exceed E_c. Derive an explicit expression for the robustness function.

(b) Replace the info-gap model of eq.(100) by the following and derive the robustness function.

$$\mathcal{U}(h) = \left\{ f : \ f^T W f \le h^2 \right\}, \quad h \ge 0 \tag{101}$$

W is a known, real, symmetric, positive definite matrix.

(c) Continue from part (b) and suppose that we can design the system by choosing the matrix V, which we now assume to be a square matrix subject to various physical constraints. For simplicity we represent these constraints as:

$$v_i > 0, \quad \sum_{i=1}^{N} v_i = \gamma \tag{102}$$

where γ is a known positive constant and v_i is the ith eigenvalue of V. Furthermore, to make things really simple, suppose that V and W are both diagonal matrices. How should we design the system (that is, choose V) to maximize the robustness? Present an intuitive physical interpretation of the result.

31. **Non-linear force-deflection relation** (p.191). Equilibrium of a 1-dimensional system is specified by:

$$xf = k + kf^2 \tag{103}$$

where x is the deflection, f is the force, which is known to be non-negative, and k is a known positive constant. The system is safe if $x \le x_{\rm c}$. The nominal force is $\widetilde{f}$, for which the system is safe. The uncertainty of the actual force is described by an info-gap model:

$$\mathcal{U}(h, \widetilde{f}) = \left\{ f : \; f \ge 0, \; \left| \frac{f - \widetilde{f}}{\widetilde{f}} \right| \le h \right\}, \quad h \ge 0 \tag{104}$$

(a) Derive the robustness function if $\widetilde{f} = 1$.

(b) Derive the robustness function if $\widetilde{f} > 1$. Discuss the qualitative difference between the two solutions.

32. **Embedded expected utility.**(p.193) Consider uncertain location along a water pipeline at which contaminant enters through a crack. Or, uncertain coastal location where an invasive species enters and endangers the local habitat. Or, uncertain point on a perimeter where a terrorist attack could occur. We formalize these situations as follows. A random variable x takes values in the interval $0 \le x \le 1$. For any realization of x, the associated loss is $L(x)$. The probability density function (pdf) for x is $p(x)$. The expected losses are:

$$\mathrm{E}(L|p) = \int_0^1 L(x)p(x)\,\mathrm{d}x \tag{105}$$

It is required that the expected losses not exceed the critical value E_{c}:

$$\mathrm{E}(L|p) \le E_{\mathrm{c}} \tag{106}$$

The pdf is uncertain, and its estimated form is $\widetilde{p}(x)$. The info-gap model for the pdf is:

$$\mathcal{U}(h, \widetilde{p}) = \left\{ p(x):\ p(x) \ge 0,\ \int_0^1 p(x)\,\mathrm{d}x = 1,\ |p(x) - \widetilde{p}(x)| \le h\widetilde{p}(x) \right\}, \quad h \ge 0 \tag{107}$$

(a) Given $\widetilde{p}(x) = 2x$ and $L(x) = \lambda$ which is constant, develop an algebraic expression for the robustness function.

(b) Given $\widetilde{p}(x) = 1$ and $L(x) = x$, develop an algebraic expression for the robustness function.

(c) Now we change the story a bit. The loss-function is uncertain, its best estimate is $\widetilde{L}(x)$, and the info-gap model for $L(x)$ is:

$$\mathcal{U}(h, \widetilde{L}) = \left\{ L(x):\ |L(x) - \widetilde{L}(x)| \le h\widetilde{L}(x) \right\}, \quad h \ge 0 \tag{108}$$

where $\widetilde{L}(x) \ge 0$ but $L(x)$ can be both negative and positive, representing losses and gains. The pdf is known and equals:

$$p(x) = \frac{\widetilde{L}(x)}{\int_0^1 \widetilde{L}(x)\,\mathrm{d}x} \tag{109}$$

The requirement in eq.(106) still holds. Develop an algebraic expression for the robustness function. Sketch the robustness vs. the critical expected loss, E_{c}. What do these graphs indicate about ones' preferences regarding the loss function?

(d) In continuation to part (c), let us suppose that we can plan the estimated loss function, $\widetilde{L}(x)$. The total estimated loss, Λ, is fixed and defined as:

$$\int_0^1 \widetilde{L}(x)\,\mathrm{d}x = \Lambda \tag{110}$$

How should we choose the estimated loss function so as to maximize the robustness with expected loss satisficed at E_{c}?

‡**(e)** Now we return to the story in parts (a) and (b), but suppose that the pdf of x is chosen *strategically* by an adversary. Specifically, we think (though we don't know for sure) that the adversary knows the loss function, and hence knows $\widetilde{L}(x)$. Let us suppose that he chooses the pdf to be large where $\widetilde{L}(x)$ is large. That is, the estimated pdf is:

$$\widetilde{p}(x) = \frac{\widetilde{L}(x)}{\int_0^1 \widetilde{L}(x)\,\mathrm{d}x} \tag{111}$$

Optional: Use matlab program fort01.m\Lectures\WShops+ShrtCrs\Brisbane2008\fort01.m to numerically study the robustness function for different choices of the mean and mean-square.

However, the adversary may adopt a totally different strategy. The info-gap model for our uncertainty in the actual pdf is given by eq.(107). Now repeat parts (a) and (b) with this new $\widetilde{p}(x)$.

33. **Spatial monitoring, simple.** (p.197) The density $\rho(x)$ of some highly undesirable material (e.g., toxin, invasive species, chemical impurity, etc.) varies along a transect from $x = 0$ to $x = L$. The true value of the total quantity is $r(\rho) = \int_0^L \rho(x)\,\mathrm{d}x$. If r exceeds a very small critical amount, r_{c}, then remedial action will be taken. You will perform N measurements to verify that *none* of this material is present at positions $x_1, \ldots, x_N$. The density tends to be constant along the transect, but the actual slope of the density varies by an unknown amount along the transect. Let $\rho'(x)$ denote the derivative of the density function. Given the measurements, the following info-gap model represents the spatial uncertainty in the true density function:

$$\mathcal{U}(h) = \left\{\rho(x) : \ \rho(x_i) = 0, \ i = 1, \ldots, N, \ |\rho'(x)| \le h\right\}, \quad h \ge 0 \tag{112}$$

(a) Suppose you perform two measurements, one at each end of the interval. Formulate and evaluate the robustness to spatial uncertainty.

(b) Suppose you perform $N + 1$ evenly spaced measurements, including one at each end of the interval. Formulate and evaluate the robustness to spatial uncertainty.

34. **Spatial monitoring.** (p.197) The density $\rho(x)$ of some material of interest (e.g., rare plants, valuable minerals, chemical impurity, seismic faults, etc.) varies along a transect from $x = 0$ to $x = 1$. You will perform N measurements, obtaining the results $m_i = \rho(x_i)$, $i = 1, \ldots, N$. Your estimate of the mean density is $\overline{m} = (1/N)\sum_{i=1}^N m_i$. The true value of the average density is $\mu = \int_0^1 \rho(x)\,\mathrm{d}x$. The density tends to be constant, but the actual slope of the density varies by an unknown amount along the transect. Let $\rho'(x)$ denote the derivative of the density function. Given the measurements, the following info-gap model represents the spatial uncertainty in the true density function:

$$\mathcal{U}(h) = \left\{\rho(x) : \ \rho(x_i) = m_i, \ i = 1, \ldots, N, \ |\rho'(x)| \le h\right\}, \quad h \ge 0 \tag{113}$$

You require that the absolute difference between the estimate, $\overline{m}$, and the true value, μ, be no greater than ε.

(a) Suppose you perform a single measurement at the midpoint, $x_1 = 1/2$. Formulate and evaluate the robustness to spatial uncertainty.

(b) Suppose you perform two measurements, one at each end of the interval. Formulate and evaluate the robustness to spatial uncertainty.

35. **Investment for bio-diversity.** (p.198) You will invest a quantity q of resources in order to increase the bio-diversity of a nature reserve. The number of new species which will thrive in the reserve after the investment is:

$$N(q, u) = u_1 q + u_2 q^2 \tag{114}$$

where the coefficients u_i are uncertain. The project is a failure if the number of new species is less than N_{c}.

(a) The best estimates of the coefficients u_i are $\widetilde{u}_i$, where $\widetilde{u}_1 > 0$ and $\widetilde{u}_2 < 0$. These estimates are highly uncertain and we have no further information other than that $u_1 \ge 0$ and $u_2 \le 0$. Use the fractional-error info-gap model:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u : \ \begin{array}{rcl} \max[0, \ (1-h)\widetilde{u}_1] \le & u_1 & \le (1+h)\widetilde{u}_1 \\ (1+h)\widetilde{u}_2 \le & u_2 & \le \min[0, \ (1-h)\widetilde{u}_2] \end{array} \right\}, \quad h \ge 0 \tag{115}$$

Evaluate the robustness of investment q with requirement $N_{\rm c}$. Discuss the significance of the possible crossing of the robustness curves.

(b) Now consider additional information. We have an estimate, σ_i, of the error of the estimated value $\widetilde{u}_i$. Now use the following modification of the info-gap model in eq.(115):

$$\mathcal{U}(h,\widetilde{u}) = \left\{ u : \begin{array}{rcl} \max[0,\ \widetilde{u}_1 - h\sigma_1] \leq & u_1 & \leq \widetilde{u}_1 + h\sigma_1 \\ \widetilde{u}_2 - h\sigma_2 \leq & u_2 & \leq \min[0,\ \widetilde{u}_2 + h\sigma_2] \end{array} \right\}, \quad h \geq 0 \tag{116}$$

Evaluate the robustness of investment q with requirement $N_{\rm c}$ and discuss the significance of the possible crossing of the robustness curves.

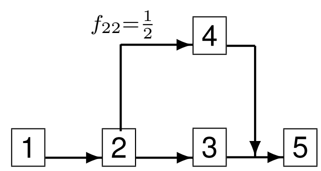

Figure 6: A 5-task project schedule for problem 17.

36. **Project management.** Consider the 5-task project shown in fig. 6. This project has two task-paths:

Path 1: 1 $\longrightarrow$ 2 $\longrightarrow$ 3 $\longrightarrow$ 5
Path 2: 1 $\longrightarrow$ 2 $\longrightarrow$ 4 $\longrightarrow$ 5

In path 2, task 4 is initiated when task 2 is half finished, as indicated by $f_{22} = 0.5$.

The nominal task durations are:

$$\widetilde{t}_1 = 1,\ \widetilde{t}_2 = 1.3,\ \widetilde{t}_5 = 1.5, \quad \widetilde{t}_3 = q, \quad \widetilde{t}_4 = 1 - q \tag{117}$$

where q is a parameter which the project manager is free to choose in the interval $[0,\ 1]$. q represents a valuable resource which must be allocated between tasks 3 and 4.

The uncertainty in the actual task durations is represented by an interval info-gap model:

$$\mathcal{U}(h,\widetilde{t}) = \left\{ t : \ \frac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \leq h,\ n = 1, \ldots, 5 \right\} \quad h \geq 0 \tag{118}$$

The project must be completed within about 4 time units. Construct the robustness and opportuneness functions and demonstrate their use in choosing q.

37. **Project management.** Consider the 5-task project shown in fig. 4 on p.18. The modification from problem 17 is that we now consider both *allocation* between two tasks, and *total budget change.* This project has two task-paths:

Path 1: 1 $\longrightarrow$ 2 $\longrightarrow$ 3 $\longrightarrow$ 5
Path 2: 1 $\longrightarrow$ 2 $\longrightarrow$ 4 $\longrightarrow$ 5

In path 2, task 4 is initiated when task 2 is half finished, as indicated by $f_{22} = 0.5$.

The nominal task durations are:

$$\widetilde{t}_1 = \widetilde{t}_2 = \widetilde{t}_5 = 1, \quad \widetilde{t}_3 = q, \quad \widetilde{t}_4 = Q - q \tag{119}$$

where both Q (the total budget) and q (the allocation to one task) are parameters which the project manager is free to choose in the interval $[0,\, Q]$.

The uncertainty in the actual task durations is represented by an interval info-gap model:

$$\mathcal{U}(h, \widetilde{t}) = \left\{ t : \ \frac{|t_n - \widetilde{t}_n|}{\widetilde{t}_n} \le h, \ n = 1, \ldots, 5 \right\} \quad h \ge 0 \tag{120}$$

The project must be completed within about 4 time units. Construct the robustness and opportuneness functions and demonstrate their use in choosing q.

38. **Estimate spring stiffness with model uncertainty.** (p.200) We will use measurements to estimate the stiffness, k, of a spring with a linear model, where force f is related to displacement x by $f = kx$.

(a) Derive the least-squares estimate of k, given n measurements (x_i, f_i), $i = 1, \ldots, n$.

(b) Now consider model uncertainty. The spring we will measure is different from the actual spring for which we wish to use the model. We suspect that there might be a cubic term in the actual spring model: $f = kx + k_3 x^3$. However, we will not include this term in the estimated model, because the measured spring has no cubic term at all. We wish to choose the linear coefficient, k, so that the mean squared error (MSE) is small even in the presence of a cubic term. Specifically, we seek a value of k for which the MSE is no greater than S_{c}, and for which the robustness to the magnitude of the uncertain cubic term is large. Consider this info-gap model for uncertainty in the cubic term:

$$\mathcal{U}(h) = \{k_3 : \ |k_3| \le h\}, \quad h \ge 0 \tag{121}$$

Formulate and derive the robustness function. (Suggestion: derive the inverse of the robustness function.)

39. **Managing mean and variance.** (p.201) The performance of a system is evaluated as $\psi^T x$ where ψ is a known vector and x is uncertain. We require that $\psi^T x$ be small, no larger than a critical value $r_{\rm c}$. The mean of x is $y\mu$ where y is a non-negative parameter to be chosen and μ is a known vector. Assume that $\psi^T \mu > 0$. The covariance matrix of x is $\frac{1}{y^2}W$ where W is a known positive definite symmetric real matrix. The pdf of x is unknown.

(a) Formulate an info-gap model for uncertainty in x, and derive the robustness function for y.

(b) Find the robust-satisficing choice of y.

(c) Find the choice of y which optimizes the average performance: minimizes $\psi^T(y\mu)$. Compare this with the result in (b): when are the results the same?

(d) Consider the following variance-weighted performance function: $\psi^T(y\mu) + \frac{\epsilon}{y^2}\mathbf{1}^T W \mathbf{1}$, where $\epsilon > 0$. Find the choice of y which minimizes this performance function and compare it to the result in (b): when are the results the same? What are the implications of this?

40. **Choosing between two nature reserves.** (p.202) You must choose between two nature reserves, one carefully studied and well understood, and the other not studied at all. The utility (e.g., duration until biodiversity will be threatened) is confidently known to be u_1 for the first reserve. For the second reserve the utility is either u_0 with probability p_0, or u_2 with probability p_2, where $u_0 < u_1 < u_2$.

The utilities u_0 and u_2 are poorly known. Their best estimates, and rough errors of these estimates, are $\widetilde{u}_i$ and σ_i, for $i = 0$ and 2. A fractional-error info-gap model describes the uncertainties in these estimated utilities:

$$\mathcal{U}(h) = \left\{ u : \ \left| \frac{u_i - \widetilde{u}_i}{\sigma_i} \right| \le h, \ i = 0, \ 2 \right\}, \quad h \ge 0 \tag{122}$$

We allow utility to be negative. In some applications we may wish to require non-negative utility, which would require modification of the info-gap model in eq.(122).

(a) Suppose that you require the expected utility for option 2 to be no less than $E_{\rm c}$. Derive an expression for the robustness function.

(b) The estimated expected utility from the second option is $\mathrm{EU}(\widetilde{u}) = \widetilde{u}_0 p_0 + \widetilde{u}_2 p_2$. Use the robustness function to assist the choice between the two options if:

- $\mathrm{EU}(\widetilde{u}) \approx u_1$.
- $\mathrm{EU}(\widetilde{u}) \gg u_1$.
- $\mathrm{EU}(\widetilde{u}) \ll u_1$.

(c) Unfortunately, due to delay in making a decision, the first option—with known utility u_1—has been sold and converted into a parking lot. But don't worry, a third option has opened up for which the best estimate of the expected utility equals u_1. The estimates of the low and high utilities for this third option are $\widetilde{v}_0$ and $\widetilde{v}_2$ with rough error estimates s_0 and s_2. The known probabilities of low and high utility are p_0 and p_2 which are the same as before. The estimate of the expected utility is $\mathrm{EU}(\widetilde{v}) = u_1$. However, the actual low and high utilities are uncertain, with info-gap model, analogous to eq.(122):

$$\mathcal{V}(h) = \left\{ v : \ \left| \frac{v_i - \widetilde{v}_i}{s_i} \right| \le h, \ i = 0, \ 2 \right\}, \quad h \ge 0 \tag{123}$$

Use the result from part (a) to express the robustness of the third option. Now discuss the choice between the 2nd and 3rd options in the following two cases:

• $\mathrm{EU}(\widetilde{u}) > \mathrm{EU}(\widetilde{v})$ (so option 2 is expected to be better than option 3), but $\mathrm{EU}(\sigma) > \mathrm{EU}(s)$ (so option 2 is more uncertain than option 3).

• $\mathrm{EU}(\widetilde{u}) > \mathrm{EU}(\widetilde{v})$ (so option 2 is expected to be better than option 3), but $\mathrm{EU}(\sigma) < \mathrm{EU}(s)$ (so option 2 is less uncertain than option 3).

(d) Use matlab program natres01.m to explore the implications of different values of estimated utility, $\widetilde{u}$, and errors, σ. Begin by comparing three nature reserves, where $\widetilde{u}_i$ is the column vector of estimated utilities for the ith reserve, and σ_i is the column vector of estimated errors. The vector of probabilities of low and high utility is $p = (p_0,\ p_2)$. The available information is:

$$[\widetilde{u}_1\ \widetilde{u}_2\ \widetilde{u}_3] = \begin{bmatrix} 20 & 22 & 18 \\ 25 & 27 & 21 \end{bmatrix}, \quad [\sigma_1\ \sigma_2\ \sigma_3] = \begin{bmatrix} 5 & 7 & 4 \\ 6 & 9 & 6 \end{bmatrix}, \quad p = (0.3,\ 0.7) \tag{124}$$

Discuss the significance of crossing robustness curves, and consider the importance of anticipated expected utility as well as sensitivity to uncertainty. Explore nature reserves with different $\widetilde{u}$'s and σ's.

(e) Now consider the fact that outcomes can be better than expected: uncertainty can be propitious. Derive the opportuneness function for utility as large as E_{w}. Show that if the robustness curves for reserves i and j *do* cross, then their opportuneness curves *do not* cross. Explain the significance of this for choosing a nature reserve.

Matlab program: \lectures\WShop+ShrtCrs\Brisbane2008\natres01.m

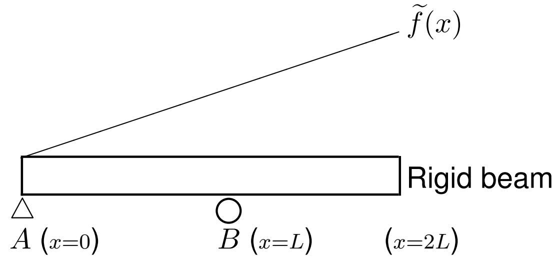

Figure 7: Rigid beam for problem 41.

41. **Trigger mechanism.** (p.205) Consider a completely rigid beam of length $2L$ as shown in fig. 7, with simple supports A and B at points $x = 0$ and $x = L$. The distributed load acts perpendicularly to the beam, with positive force directed downward. The estimated load is:

$$\widetilde{f}(x) = \mu x / L \tag{125}$$

where $\mu > 0$.

The uncertainty in the load is represented by:

$$\mathcal{U}(h) = \left\{ f(x) : \ \left| \frac{f(x) - \widetilde{f}(x)}{\mu} \right| \le h \right\}, \quad h \ge 0 \tag{126}$$

We require that the reaction force at support B be no less than the critical value $R_{\rm c}$.

Derive an explicit expression for the robustness function.

42. **Cantilever–1.** (p.206) Consider a uniform cantilever beam of length L. The axis of the beam is along the x axis and the beam is subject to a continuous load applied in the x-y plane. Let $\imath$ denote the unit vector in the x direction, and let $\jmath$ denote the unit vector in the y direction. The force, in units of N/m, at point x along the beam is:

$$f(x) = \imath F \cos\theta(x) + \jmath F \sin\theta(x) \tag{127}$$

where F is a positive constant. The function $\theta(x)$ varies along the beam.

The function $\theta(x)$ is uncertain. The best estimate is $\widetilde{\theta}(x)$ and the uncertainty is represented by an info-gap model:

$$\mathcal{U}(h) = \left\{ \theta(x) : \ 0 \le \theta(x) \le \pi, \ |\theta(x) - \widetilde{\theta}(x)| \le h \ \text{for all } x \right\}, \quad h \ge 0 \tag{128}$$

Let $M(\theta)$ denote the bending moment at $x = 0$, which depends on the function $\theta(x)$. We require:

$$|M(\theta)| \le M_{\rm c} \tag{129}$$

(a) Given that $\widetilde{\theta}(x) = \pi/2$, derive an explicit expression for the robustness function.

(b) Given that $\widetilde{\theta}(x) = \pi$, derive an explicit expression for the robustness function.

(c) Now consider a different info-gap model:

$$\mathcal{U}(h) = \left\{ \theta(x) : \ 0 \le \theta(x) \le \pi, \ |\theta(x) - \widetilde{\theta}(x)| \le h|\widetilde{\theta}(x)| \ \text{for all } x \right\}, \quad h \ge 0 \tag{130}$$

Given that $\widetilde{\theta}(x) = \pi/2$, derive an explicit expression for the robustness function. Use the performance requirement in eq.(129).

(d) Given that $\widetilde{\theta}(x) = \pi$, derive an explicit expression for the robustness function. Use the info-gap model in eq.(130) and the performance requirement in eq.(129).

(e) Compare the expressions for the robustness which were obtained in steps (b) and (d). Explain the relation between them by considering the info-gap models which were used.

(f) Let $\widehat{h}$ denote the robustness for the generic formulation in eqs.(127)–(129). Now add the following information: the horizon of uncertainty, h, is a random variable with exponential probability density function:

$$p(h) = \lambda {\rm e}^{-\lambda h}, \ \ h \ge 0 \tag{131}$$

where λ is a known positive constant. Derive an expression for a lower bound, greater than zero, on the probability of satisfying eq.(129). Hint: use $\widehat{h}$.

(g) We now completely change the formulation. Suppose that the load is present as n uncertain point forces $f_1, \ldots, f_n$ at known locations $x = (x_1, \ldots, x_n)^T$. Let $\widetilde{f}$ denote the estimated load vector, while f is the uncertain true load vector. The uncertainty in the load is represented by:

$$\mathcal{U}(h) = \left\{ f : \ \frac{1}{F^2} \sum_{i=1}^{n} (f_i - \widetilde{f}_i)^2 \le h^2 \right\}, \quad h \ge 0 \tag{132}$$

where F is a known positive constant. Derive an explicit expression for the robustness, given the performance requirement in eq.(129). Assume that the nominal bending moment, $x^T \widetilde{f}$, is positive.

43. **Tichonov estimate with model uncertainty.** (p.209). We wish to choose the slope, s, of a linear scalar model:

$$y = sx \tag{133}$$

We have a prior estimate of the slope, $\widetilde{s}$, and we have data, (x_i, y_i), $i = 1, \ldots, M$. The Tichonov estimate of s minimizes:

$$T = \lambda(\widetilde{s} - s)^2 + (1 - \lambda)\frac{1}{M}\sum_{i=1}^{M}(y_i - sx_i)^2 \tag{134}$$

where $0 \le \lambda \le 1$. We will assume that x and y are dimensionless quantities.

(a) Derive an expression for the estimate of s which minimizes T.

(b) Now consider model uncertainty, with two different info-gap models:

$$\mathcal{U}(h) = \left\{y = sx + u : \ |u| \le h\right\}, \quad h \ge 0 \tag{135}$$

$$\mathcal{U}(h) = \left\{y = sx + ux^2 : \ |u| \le h\right\}, \quad h \ge 0 \tag{136}$$

For each info-gap model, derive an expression for the robustness of an estimate of the slope. How does the robust-satisficing estimate differ between the two models? How do they differ from the Tichonov estimate? Note that, because x and y are dimensionless, the horizons of uncertainty in these two info-gap models are also dimensionless. This makes the robustnesses which are evaluated with these two info-gap models comparable.

If x and y have units, and even if they have the same units, then the units of T in eq.(134) are undefined. This means that the relative weights of the two terms in T are controlled by the units, not by the value of λ.

If x and y have units then it is be necessary to calibrate the two robustnesses, which requires judgment and cannot be done uniquely. However, if x and y have units then we also face a different problem, noted in footnote .

44. **One-wheel vehicle on rough terrain,** (p.211). Consider a vertically stabilized 1-wheel vehicle traversing rough terrain. The suspension system is modelled as a 1-dimensional spring. The vehicle moves at constant horizontal velocity. The vertical displacement of the center of mass is represented by:

$$m\ddot{x}(t) + kx(t) = ky(t) \tag{137}$$

where $y(t)$ is the vertical displacement of the terrain.

(a) The nominal profile of the terrain is $\widetilde{y}(t)$, and the uncertainty in the terrain is represented as:

$$\mathcal{U}(h) = \{y(t):\ |y(t) - \widetilde{y}(t)| \le sh\}, \quad h \ge 0 \tag{138}$$

where s is a known positive number. We require that the vertical displacement of the vehicle not exceed x_{c}. Derive an expression for the robustness.

(b) Consider a different info-gap model. An uncertain spectral representation of the terrain is:

$$y(t) = \widetilde{y}(t) + \sum_{i=1}^{N} c_i \cos\omega_i t = \widetilde{y}(t) + c^T\gamma(t) \tag{139}$$

An info-gap model for uncertain spectral coefficients is:

$$\mathcal{U}(h) = \left\{y(t) = \widetilde{y}(t) + c^T\gamma(t):\ c^TWc \le h^2\right\}, \quad h \ge 0 \tag{140}$$

where W is a known, positive definite, symmetric real matrix. Derive the robustness for the requirement $x \le x_{\mathrm{c}}$.

(c) Now consider the acceleration, which must not exceed a critical value, a_{c}. Derive an expression for the robustness using the info-gap model in eq.(138). (Hint: use eq.(137), and its solution for $x(t)$, to express the acceleration.)

(d) Derive the robustness with the info-gap model in eq.(138) and the requirement that $x(t) \ge x_{\mathrm{l}}$. Consider the special case than $t = n\pi/\omega$ and compare this to the result of part (a) and explain the relation between these two robustness functions. Specifically, if a change in design causes one robustness function to increase, will the other also increase? That is, are they sympathetic or antagonistic?

See Yakov Ben-Haim and Isaac Elishakoff, 1990, *Convex Models of Uncertainty in Applied Mechanics,* Elsevier, section 3.2.

45. **Braking system–2.** (p.213). Consider a linear braking system for which the stopping distance is described by:

$$s(g,f) = \int_0^t g(\tau) f(t-\tau)\,\mathrm{d}\tau \tag{141}$$

where $f(\cdot)$ is the force and $g(\cdot)$ is the system response function. The uncertainty in the force is described by:

$$\mathcal{U}(h) = \left\{ f(t) : \int_0^t (f(\tau) - \widetilde{f}(\tau))^2\,\mathrm{d}\tau \le h^2 \right\}, \quad h \ge 0 \tag{142}$$

The nominal forcing function, $\widetilde{f}(t)$, is known. It is required that the stopping distance not exceed the critical value s_{c}.

(a) Derive an explicit expression for the robustness function.

(b) Now consider a special case and choose between two proposed designs.

The nominal forcing function is:

$$\widetilde{f}(t-\tau) = \tau \tag{143}$$

The two designs are characterized by the following response functions:

$$g_1(\tau) = t - \tau \tag{144}$$

$$g_2(\tau) = \phi\tau \tag{145}$$

where ϕ is a known positive constant.

For what values of s_{c} and ϕ should one prefer g_1 over g_2 based on robustness to uncertainty in the force as described by the info-gap model of eq.(142)?

See also problem 28 on p.27.

46. **Robustness and opportuneness of failure probability,** (p.214). The response of a system to input x is:

$$f(x) = \frac{a}{x} \tag{146}$$

where $a > 0$ and x is a random variable with an exponential distribution:

$$p(x) = \lambda \mathrm{e}^{-\lambda x}, \quad x \geq 0 \tag{147}$$

The failure criterion is probabilistic. The system fails if f exceeds f_{c}. The system requirement is that the probability of failure not exceed the critical value, P_{fc}.

(a) Derive an expression for the probability of failure, assuming that λ and a are known precisely.

(b) The coefficient a is estimated to equal $\widetilde{a}$ with error approximately s, and a is known to be positive. However, a may vary due to uncontrolled factors. Derive an expression for the robustness to uncertainty in a. What is the sign of the slope of the robustness curve? What does this sign indicate? At what value of critical failure probability does the robustness become zero?

(c) Continuing part 46b, consider the choice between two systems with parameters:

$$\lambda_1 < \lambda_2 \quad \text{and} \quad s_1 > s_2 \quad \text{and} \quad \lambda_1 s_1 < \lambda_2 s_2 \tag{148}$$

For what values of P_{fc} do you prefer option 1? Why? What do these three inequalities mean?

(d) Let P_{fw} be a lower probability than P_{fc}. Windfall occurs if the probability is no greater than P_{fw} that f exceeds f_{c}. Derive an expression for the opportuneness and discuss its relation to the robustness derived earlier. Specifically, at what value of $P_{\mathrm{fc}} = P_{\mathrm{fw}}$ do these curves cross one another, and what is the significance of this?

47. **Extrapolating an S–N curve,** (p.216). An S–N curve is the functional relation between the load amplitude S and the number of cycles to fatigue failure, N. A (very simple and inaccurate) model is:

$$S(N) = \frac{a}{b+N} \tag{149}$$

(a) Given observed cycles to failure with corresponding load amplitudes, (N_i, S_i), $i = 1, \ldots, K$, derive a least-squares estimate of the coefficient a, assuming that b is known.

(b) Let $N_{\max}$ denote the greatest lifetime which has been observed. We want to predict the load which will yield a lifetime $N_0 > N_{\max}$ for some specified value of N_0.

Let us suppose that eq.(149) accurately describes the S–N curve. Let us furthermore suppose that the value of a is known precisely. However, we are not sure that the value of b used for lower lifetimes is still valid when we extrapolate. Let us write eq.(149) as $S(N, b)$.

Suppose that b is estimated at $\widetilde{b}$ with approximate error w_b, but the true value of b is unknown. An info-gap model for uncertainty in b is:

$$\mathcal{U}(h) = \left\{ b : \left| \frac{b - \widetilde{b}}{w_b} \right| \le h \right\}, \quad h \ge 0 \tag{150}$$

The true failure load at lifetime N_0 is $S(N_0, b)$, described by eq.(149) with known a and uncertain b. We will extrapolate (predict) the failure load at lifetime N_0 by using eq.(149) with a value for b of our choice, call it $b_{\rm c}$. We require that the true load at lifetime N_0 not exceed the predicted load by more than ε:

$$S(N_0, b) \le S(N_0, b_{\rm c}) + \varepsilon \tag{151}$$

Derive an expression for the robustness of the choice $b_{\rm c}$.

48. **Signal transmission in an uncertain absorbing medium** (p.217). A transmitter is located a distance D from a receiver. The absorptive property of the medium between transmitter and receiver is represented by a function $\rho(x)$. The strength at the receiver, of a signal of strength t at the transmitter, is:

$$r = t \exp\left(-\int_0^D \rho(x)\,\mathrm{d}x\right) \tag{152}$$

We require that the signal strength at the receiver be no less than a fraction f of the transmitted signal, where $0 < f < 1$ is specified by the design requirement.

(a) The absorption function is estimated to be constant at the value $\widetilde{\rho}$. The error of this estimate is approximated as σ, but the actual function can vary substantially and we have no reliable or meaningful worst-case estimate. An info-gap model for uncertainty in the absorption function is:

$$\mathcal{U}(h) = \left\{\rho(x) : \ \left|\frac{\rho(x) - \widetilde{\rho}}{\sigma}\right| \le h\right\}, \quad h \ge 0 \tag{153}$$

Derive an explicit expression for the robustness of the transmission. At what value of f does the robustness become zero? Explain the significance of this value. Explain the significance of the sign of the slope of the robustness curve for values of f at which the robustness is positive.

(b) The absorption function is estimated to be a sine function:

$$\widetilde{\rho}(x) = \sin\frac{2\pi x}{D_0} \tag{154}$$

where D_0 is a known constant. Negative absorption means amplification.

The error of this estimate is approximated as σ, but the actual function can vary substantially and we have no reliable or meaningful worst-case estimate. An info-gap model for uncertainty in the absorption function is:

$$\mathcal{U}(h) = \left\{\rho(x) : \ \left|\frac{\rho(x) - \widetilde{\rho}(x)}{\sigma}\right| \le h\right\}, \quad h \ge 0 \tag{155}$$

For any transmission distance D (which might differ from D_0), derive an explicit expression for the robustness of the transmission.

49. **Let's play golf** (p.217). You will putt a golf ball towards a hole D meters away on a flat grassy horizontal surface. The velocity on the axis between you and the hole is $v_1(t)$, and the velocity on the perpendicular axis is $v_2(t)$, given by:

$$v_1(t) = v_1(0) - gt \tag{156}$$

$$v_2(t) = \varepsilon v_1(t) \tag{157}$$

g is a constant deceleration due to the grass, and ε is a dimensionless axis-coupling constant. Both g and ε are uncertain, with estimated values $\widetilde{g}$ and $\widetilde{\varepsilon}$. A fractional-error info-gap model is:

$$\mathcal{U}(h) = \left\{g, \varepsilon : \ \left|\frac{g - \widetilde{g}}{\widetilde{g}}\right| \le h, \ \left|\frac{\varepsilon - \widetilde{\varepsilon}}{\widetilde{\varepsilon}}\right| \le h\right\}, \quad h \ge 0 \tag{158}$$

We want the ball to go into the hole. This implies two requirements. First, the ball must not stop before it reaches the hole. Second, when its axial coordinate (the '1' axis) equals D, its perpendicular coordinate must not exceed the radius of the hole, R. Derive an expression for the robustness to uncertainty. (Hint: Let $x_1^\star$ denote the axial distance that ball would travel up to the time at which $v_1(t) = 0$, if there were no hole. The first requirement is $x_1^\star \ge D$.)

50. **Search and evasion, 1.** (p.219) A hunter is trying to catch an evasive target. The target can move either left or right (but not both). The probability that the target moves left is p. The hunter can move either left or right (but not both) and must decide which way to move. The hunter catches the target if and only if they move in the same direction. The hunter's decisions are denoted $m = 1$ for moving left and $m = 0$ for moving right. The hunter's utility is u if the target is caught, and zero otherwise. Thus the hunter's expected utility is:

$$V = mup + (1-m)u(1-p) \tag{159}$$

(a) The hunter's utility is estimated to be $\widetilde{u}$, with error s_u. The info-gap model for uncertain utility is:

$$\mathcal{U}(h) = \left\{ u : \left| \frac{u - \widetilde{u}}{s_u} \right| \leq h \right\}, \quad h \geq 0 \tag{160}$$

The hunter requires utility no less than V_c. Derive an expression for the hunter's robustness. Which of the hunter's strategies is preferred, in terms of robustness to uncertainty?

(b) Now suppose that both the hunter's utility and the target's move-probability are uncertain. The probability is estimated as $\widetilde{p}$ with error s_p. The info-gap model is:

$$\mathcal{U}(h) = \left\{ u, p : \left| \frac{u - \widetilde{u}}{s_u} \right| \leq h,\ p \in [0,\, 1],\ \left| \frac{p - \widetilde{p}}{s_p} \right| \leq h \right\}, \quad h \geq 0 \tag{161}$$

The hunter requires utility no less than V_c. Which of the hunter's strategies is preferred, in terms of robustness to uncertainty? Suggestion: derive the inverse of the robustness function.

51. **Search and evasion, 2.** (p.221) A hunter is trying to catch an evasive target. The target can move either left or right (but not both) and the consequences of the moves are different (unlike in problem 50). That is, from the hunter's point of view, one move might be "bad" and the other "really bad". The probability that the target moves left is p. The hunter can move either left or right (but not both) and must decide which way to move. The hunter catches the target if and only if they move in the same direction. The hunter's decisions are denoted $m = 1$ for moving left and $m = 0$ for moving right. The hunter's utility is u_0 if the target is caught moving to the right, u_1 if the target is caught moving to the left, and zero if the target is not caught. Thus the hunter's expected utility from taking action m is:

$$V(m) = m u_m p + (1 - m) u_m (1 - p) \tag{162}$$

(a) The hunter's utilities are estimated to be $\widetilde{u}_0$ and $\widetilde{u}_1$, with errors s_{u_0} and s_{u_1} respectively. The info-gap model for uncertain utility is:

$$\mathcal{U}(h) = \left\{ u : \left| \frac{u_m - \widetilde{u}_m}{s_{u_m}} \right| \le h, \ m = 0, 1 \right\}, \quad h \ge 0 \tag{163}$$

The hunter requires utility no less than $V_{\rm c}$. Derive an expression for the hunter's robustness. Which of the hunter's strategies is preferred, in terms of robustness to uncertainty?

(b) Now suppose that both the hunter's utilities and the target's move-probability are uncertain. The probability is estimated as $\widetilde{p}$ with error s_p. The info-gap model is:

$$\mathcal{U}(h) = \left\{ u, p : \left| \frac{u_m - \widetilde{u}_m}{s_{u_m}} \right| \le h, \ m = 0, 1, \ p \in [0, 1], \ \left| \frac{p - \widetilde{p}}{s_p} \right| \le h \right\}, \quad h \ge 0 \tag{164}$$

The hunter requires utility no less than $V_{\rm c}$. Which of the hunter's strategies is preferred, in terms of robustness to uncertainty? Suggestion: derive the inverse of the robustness function.

(c) Now consider the target's decisions. Suppose the target (T) has studied info-gap theory and believes that the hunter (H) has also. Furthermore, T believes that H is uncertain only about the utilities, so T believes that H has solved the problem in part (a) (or part (b) if you prefer), and that this will underlie H's decision. T has estimates of H's estimates of utility, so T can approximately reproduce H's robustness calcuations. T's only concern is not to get caught. What should T do? Move left or right?

(d) Let us quantitatively extend part (c). Let q denote T's degree of belief that H uses a robust satisficing strategy. H's alternative strategy is to choose the action whose estimated outcome is predicted to be optimal. Let $m_{\rm rs}$, which equals either 0 or 1, denote T's guess of H's decision if H is a robust satisficer. Similarly, let $m_{\rm op}$ denote T's guess of H's decision if H is an optimizer. Let n denote T's decision, where $n = 0$ or $n = 1$ means that T moves left or right respectively. Let w_n be a negative number denoting the utility to T of getting caught moving left ($n = 0$) or right ($n = 1$). T's utility is zero if T is not caught. Let $\mathcal{I}(x)$ be an indicator function which equals 1 if $x = 0$ and zero otherwise. T's expected utility of making decision n is:

$$W(n) = q \mathcal{I}(n - m_{\rm rs}) w_{m_{\rm rs}} + (1 - q) \mathcal{I}(n - m_{\rm op}) w_{m_{\rm op}} \tag{165}$$

T is uncertain about the utilities of getting caught, as reflected in this info-gap model:

$$\mathcal{U}(h) = \left\{ w : \left| \frac{w_m - \widetilde{w}_m}{s_{w_m}} \right| \le h, \ m = 0, 1 \right\}, \quad h \ge 0 \tag{166}$$

T requires utility no less than $W_{\rm c}$. Derive an expression for T's robustness. Which of T's strategies is preferred, in terms of robustness to uncertainty?

52. **Uncertain truss.** (p.224)

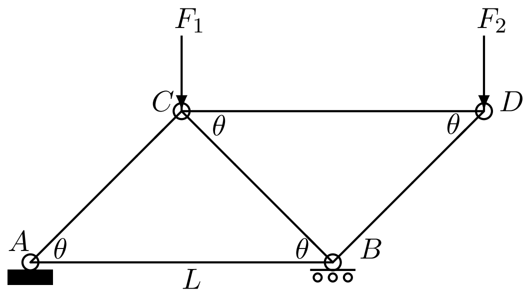

Figure 8: Uncertain truss, problem 52.

Consider the truss in fig. 8, fixed at A and simply supported at B, with vertical loads at C and D. The members are all straight bars and the joints are all frictionless.

(a) The loads are uncertain as described by:

$$\mathcal{U}(h) = \left\{ F : \left| \frac{F_i - \widetilde{F}_i}{s_i} \right| \le h, \ i = 1, 2 \right\}, \quad h \ge 0 \tag{167}$$

where s_i is a positive error estimate. We require that the force in bar AC not exceed the critical value, $F_{\rm c}$. Determine the robustness to uncertainty in the loads.

(b) In continuation of part (a), consider two designs with different values of θ, where $0 < \theta_1 < \theta_2 < \pi/2$. Which design is robust-preferred, as a function of the critical force $F_{\rm c}$?

(c) In continuation of part (a), consider two designs with different values of θ and of the error estimates s_i. Denote these designs $q = (\theta, s_1, s_2)$ and $q' = (\theta', s_1', s_2')$, where:

$$0 < \theta < \theta' < \pi/2 \quad \text{and} \quad s_1 + s_2 < s_1' + s_2' \tag{168}$$

Which design is robust-preferred, as a function of the critical force $F_{\rm c}$?

(d) Now consider a more uncertain truss, in which the loads, lengths, and angles are uncertain, though the members are all straight. The four angles θ in fig. 8 are the same, though the value of θ is uncertain. The uncertainty is described by:

$$\mathcal{U}(h) = \left\{ F : \left| \frac{F_i - \widetilde{F}_i}{s_i} \right| \le h, \ \ i = 1, 2, \ \left| \frac{\theta - \widetilde{\theta}}{s_\theta} \right| \le h, \ \theta > 0, \ \left| \frac{L - \widetilde{L}}{s_L} \right| \le h, \right\}, \quad h \ge 0 \tag{169}$$

where s_i, s_θ and s_L are positive error estimates. We require that the force in bar AC not exceed the critical value, $F_{\rm c}$. Derive an expression for the inverse of the robustness to uncertainty in the loads.

(e) In continuation of part (d), consider the following special case:

$$\widetilde{F}_1 = 1, \ \widetilde{F}_2 = 2, \ \widetilde{\theta} = 45^\circ, \ s_1 = 0.5, \ s_2 = 1, \ s_\theta = 5^\circ \tag{170}$$

(i) What is the largest critical load at which the robustness is zero? *(ii)* What is the robustness when the critical load is zero? *(iii)* What is the critical load when the robustness equals 2?

(f) Once again consider uncertainty only in the loads, and suppose that the difference, $F_1 - F_2$, is a random variable, denoted Δ. The estimated pdf of Δ is $\widetilde{p}(\Delta)$, and uncertainty in this pdf is given by:

$$\mathcal{U}(h) = \left\{ p(\Delta) : \ p(\Delta) \geq 0, \ \int_{-\infty}^{\infty} p(\Delta) \, \mathrm{d}\Delta = 1, \ |p(\Delta) - \widetilde{p}(\Delta)| \leq \widetilde{p}(\Delta) h \right\}, \quad h \geq 0 \qquad (171)$$

The truss fails if the force in bar AC exceeds the critical value F_c. We require that the probability of failure not exceed ε. Derive an expression for the robustness. Assume that the estimated probability of failure, based on $\widetilde{p}(\Delta)$, is much less than 1.

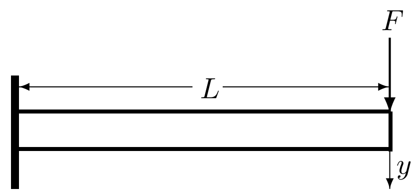

Figure 9: Cantilever for problem 53.

53. **Cantilever–2.** (p.227) Consider the cantilever in fig. 9. The force F is applied perpendicular to the elastic beam of length L which is rigidly constrained at the base. The bending stiffness of the beam is EI and the end deflection is $y = FL^3/(3EI)$.

(a) The anticipated force is $\widetilde{F}$, which is positive. The uncertainty in the true force, F, is represented by the info-gap model:

$$\mathcal{U}(h) = \left\{ F : \ \left| \frac{F - \widetilde{F}}{\sigma} \right| \leq h \right\}, \quad h \geq 0 \tag{172}$$

where σ is known and positive. The performance requirement is that the end deflection be no less than the critical value y_{c}. Derive an explicit expression for the robustness to uncertainty.

(b) Continue part (a) and compare two designs with different bending stiffnesses and load uncertainties:

$$(EI)_1 > (EI)_2 \quad \text{and} \quad \sigma_1 < \sigma_2 \tag{173}$$

For what values of critical deflection, y_{c}, is design $[(EI)_1, \sigma_1]$ preferred over design $[(EI)_2, \sigma_2]$?

(c) Now consider a different performance requirement: the bending moment at the base of the beam must not exceed the critical value M_{c}. Use the info-gap model of eq.(172) to derive an explicit expression for the robustness to uncertainty.

(d) Derive an expression, based on parts (a) and (c), for the robustness to uncertainty when both of the performance requirements must be satisfied.

(e) Let F be a non-negative random variable with probability density function (pdf) $p(F)$ whose estimated form is exponential: $\widetilde{p}(F) = \lambda \mathrm{e}^{-\lambda F}$. The uncertainty in the pdf is represented by:

$$\mathcal{U}(h) = \left\{ p(F) : \ p(F) \geq 0, \ \int_0^\infty p(F)\, \mathrm{d}F = 1, \ |p(F) - \widetilde{p}(F)| \leq h\widetilde{p}(F) \right\}, \quad h \geq 0 \tag{174}$$

The mechanical system fails if the deflection, y, is less than y_{c}. The performance requirement is that the probability of failure must not exceed P_{c}. Derive an explicit expression for the robustness of this performance function, for P_{c} much less than 1.

(f) Now suppose that N forces, $f = (f_1, \ldots, f_N)$, are applied perpendicularly to the beam, where f_i is applied at a distance ℓ_i from the base. As in part (c), the performance requirement is that the bending moment at the base of the beam must not exceed the critical value M_{c}. The nominal force vector is $\widetilde{f}$, and uncertainty is represented as:

$$\mathcal{U}(h) = \left\{ f : \ (f - \widetilde{f})^T W (f - \widetilde{f}) \leq h^2 \right\}, \quad h \geq 0 \tag{175}$$

where W is a known, positive definite, symmetric matrix. Derive an explicit expression for the robustness.

(g) Return to part (a) and denote the robustness $\widehat{h}_y$. Suppose that the horizon of uncertainty, h, is a random variable with exponential distribution: $p(h) = \lambda \mathrm{e}^{-\lambda h}$. The system fails if the end deflection is less than y_{c}. Derive an upper bound for the probability of failure, as a function of $\widehat{h}_y$. This upper bound is less than one.

(h) Consider the end-loaded beam in fig. 9, where $L = 1$m and $F = 1000$N. The end deflection was measured 5 times with normal noise, and the observed deflections are 0.016, 0.010, 0.013, 0.011 and 0.012m. Use a statistical test to decide between the following two hypotheses:

$$H_0: \qquad EI = 2 \times 10^4 \text{Nm}^2 \tag{176}$$

$$H_1: \qquad EI > 2 \times 10^4 \text{Nm}^2 \tag{177}$$

Do you reject H_0 at 0.05 level of significance?

(i) The beam in fig. 9 is loaded repeatedly and the deflection is measured and categorized as "low", "medium" or "high". Under normal conditions the probabilities of these categories are:

$$p_{\mathrm{low}} = 0.35, \ p_{\mathrm{med}} = 0.55, \ p_{\mathrm{high}} = 0.10 \tag{178}$$

In the last batch of loadings the observations are:

$$n_{\mathrm{low}} = 55, \ n_{\mathrm{med}} = 75, \ n_{\mathrm{high}} = 20 \tag{179}$$

The null hypothesis is that the conditions are normal. Do you reject the null hypothesis at 0.05 level of significance?

54. **Exposure limit from dose-response data.** (p.230) Consider a new and dangerous material. We will use dose-response data to set a maximum allowed exposure dose in order to keep the response below a critical value.

(a) Given N observed dose-response pairs, $(d_i, \widetilde{r}_i)$, determine the least-squares coefficient assuming a linear homogeneous relation $\widetilde{r}_i = cd_i$. Let d and $\widetilde{r}$ denote the vectors of doses and corresponding observed responses. Denote the least squares estimated coefficient $\widehat{c}(\widetilde{r})$.

(b) The observations are from test animals, while our exposure limit will be applied to humans. We are unsure that the measured data realistically represent human dose-response relations. In fact, it is suspected that humans are more sensitive than the test animals, but to an unknown extent. That is, the human response to dose d_i, denoted r_i, is expected to exceed (by an unknown amount) the response $\widetilde{r}_i$ observed in the test animals. A rough estimate of the human-animal disparity for dose d_i is s_i, but this is not a worst case or upper limit of error. The doses d_i are certain. We represent the uncertain disparity between animal and human response by the following asymmetric fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ (d_i, r_i) :\ 0 \le \frac{r_i - \widetilde{r}_i}{s_i} \le h,\ i = 1 \ldots N \right\}, \quad h \ge 0 \tag{180}$$

Let us assume that the dose-response relationship is truly linear and homogeneous, and that if we had human response data r the least-squares estimated relation $r_i = \widehat{c}(r) d_i$ would be reliable. If we choose the exposure limit d_{c}, then the maximal response in humans would be $\widehat{c}(r) d_{\mathrm{c}}$. The largest tolerable response is r_{c}, so we must choose d_{c} so that $\widehat{c}(r) d_{\mathrm{c}} \le r_{\mathrm{c}}$.

The problem is that we don't have human response data, r, but rather animal response data, $\widetilde{r}$, where r and $\widetilde{r}$ differ by an unknown amount as expressed by eq.(180). For any choice of the exposure limit, d_{c}, what is the robustness to the unknown disparity between human and animal response behavior? Use this to choose the exposure threshold.

(c) Suppose that you could reduce the uncertainty about the human-animal disparity by investing in research. Two different scenarios—the first before research and the second after—have different uncertainty estimates s_i. Specifically, scenario 2 is less uncertain than scenario 1, as expressed by:

$$(s^T d)_1 > (s^T d)_2 \tag{181}$$

Assume that the doses, d, and the observed animal responses, $\widetilde{r}$, are the same in both scenarios. The lower uncertainty in scenario 2 may allow you to adopt a larger allowed exposure dose:

$$d_{\mathrm{c}1} < d_{\mathrm{c}2} \tag{182}$$

Use the robustness function to evaluate these two scenarios.

55. **Adaptive force balancing.** (p.231) A downward distributed load is applied on a straight unit interval. Denote the load $L(x)$ for $0 \le x \le 1$. Uncertainty in the load is described by:

$$\mathcal{U}(h) = \left\{ L(x) : \ \left| \frac{L(x) - \widetilde{L}}{\widetilde{L}} \right| \le h \right\}, \quad h \ge 0 \tag{183}$$

where $\widetilde{L}$ is known and positive. The designer must choose a distributed restoring force directed upward along the same unit interval. Denote the restoring force $R(x)$ for $0 \le x \le 1$. We require that the net moment of force around $x = 0$ not exceed the critical value M_{c}. Construct the robustness function for each of the following designs, and discuss your preferences among the designs:

(a) Designer 1 suggests choosing $R(x) = \widetilde{L}$.

(b) Designer 2 suggests an adaptive procedure whereby the restoring force is constant along the interval, and equal to the average of the actually realized force: $R(x) = \int_0^1 L(y)\,\mathrm{d}y$.

(c) Designer 3 suggests an adaptive procedure whereby the restoring force is constant along the interval, and equal to the average of the actually realized force: $R(x) = \int_0^1 L(y)\,\mathrm{d}y$. However, the adaptive procedure introduces additional uncertainty to the load, so eq.(183) is replaced by:

$$\mathcal{U}(h) = \left\{ L(x) : \ \left| \frac{L(x) - \widetilde{L}}{w\widetilde{L}} \right| \le h \right\}, \quad h \ge 0 \tag{184}$$

where $w > 1$ and known.

(d) Designer 4 suggests an adaptive procedure whereby the restoring force is linearly increasing along the interval, and equal at the midpoint to the average of the actually realized force: $R(x) = 2x \int_0^1 L(y)\,\mathrm{d}y$.

(e) Designer 5 suggests an adaptive procedure whereby the restoring force is linearly decreasing along the interval, and equal at the midpoint to the average of the actually realized force: $R(x) = 2(1-x) \int_0^1 L(y)\,\mathrm{d}y$.

56. **Mesh size extrapolation.** (p.234) Numerical computation of the response of a complex system uses a spatial discretization with mesh size x. We are able to calculate only at positive mesh size, and we estimate the value at zero mesh size by extrapolation.

Given a calculated value, $c(x)$, at mesh size x, the extrapolation to zero mesh size is given by the relation:

$$y(x, A, p) = c(x) + Ax^p \tag{185}$$

where A and p are constants, $p > 0$, and one or both are uncertain. $\widetilde{A}$ and $\widetilde{p}$ are known estimates of A and p. We require that the absolute error of the extrapolation be no greater than ε:

$$\left| y(x, A, p) - y(x, \widetilde{A}, \widetilde{p}) \right| \le \varepsilon \tag{186}$$

(a) Assume that $\widetilde{p}$ is known to be accurate and consider a fractional error info-gap model for uncertainty in A:

$$\mathcal{U}(h) = \left\{ A : \left| \frac{A - \widetilde{A}}{\widetilde{A}} \right| \le h \right\}, \quad h \ge 0 \tag{187}$$

Derive an explicit expression for the robustness function.

(b) Now consider the situation in which $y(x, A, p)$ in eq.(185) is an approximation to the correction extrapolation function $y(x)$, where the functional form of $y(x)$ is unknown. Consider fractional uncertainty in the form of the extrapolation function:

$$\mathcal{U}(h) = \left\{ y(x) : \left| \frac{y(x) - y(x, \widetilde{A}, \widetilde{p})}{y(x, \widetilde{A}, \widetilde{p})} \right| \le h \right\}, \quad h \ge 0 \tag{188}$$

where $y(x, \widetilde{A}, \widetilde{p})$ is given in eq.(185) where $\widetilde{A}$ and $\widetilde{p}$ are known and $\widetilde{p}$ is positive. Derive an explicit expression for the robustness function.

(c) Now return to the case that eq.(185) is known to be the correct functional form, and suppose that $\widetilde{A}$ is known and that only p is uncertain. Use the fractional error info-gap model:

$$\mathcal{U}(h) = \left\{ p : \left| \frac{p - \widetilde{p}}{\widetilde{p}} \right| \le h, \ \ p > 0 \right\}, \ \ h \ge 0 \tag{189}$$

Suppose that $\widetilde{A} < 0$. Derive an expression for the robustness assuming that $0 < x < 1$.

(d) Extend the info-gap model of eq.(187) and consider fractional uncertainty in both parameters, A and p:

$$\mathcal{U}(h) = \left\{ (A, p) : \left| \frac{A - \widetilde{A}}{\widetilde{A}} \right| \le h, \ \left| \frac{p - \widetilde{p}}{\widetilde{p}} \right| \le h, \ p > 0 \right\}, \quad h \ge 0 \tag{190}$$

Derive an explicit expression for the inverse of the robustness function where $\widetilde{A}$ and $\widetilde{p}$ are known and $\widetilde{p}$ is positive. Assume $0 < x < 1$.

57. **Waste water system design.** (p.236) The volume of liquid sewage in a processing plant, s, grows at the constant in-flow rate r [m^3/s] and decreases at the constant processing rate ρ [m^3/s]. Thus the volume of sewage at the end of the processing period t is $s = (r - \rho)t$. The in-flow rate varies randomly from period to period with an exponential distribution:

$$p(r) = \lambda e^{-\lambda r}, \quad r \geq 0 \tag{191}$$

The exponent, λ, is non-negative and typically equals $\widetilde{\lambda}$, with a typical range of variation ε though λ may vary even more. Use a fractional-error info-gap model to represent uncertainty in λ:

$$\mathcal{U}(h) = \left\{ \lambda : \ \lambda \geq 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{\varepsilon} \right| \leq h \right\}, \quad h \geq 0 \tag{192}$$

We want the volume of sewage in the plant not to exceed the critical value s_c. However, the volume of sewage is a random variable. We require that the sewage volume exceed the critical value s_c with probability no greater than P_c:

$$\mathrm{Prob}(s \geq s_c) \leq P_c \tag{193}$$

(a) Evaluate the robustness to uncertainty in λ.

(b) Discuss the preference between two processing plants with different processing rates ρ.

58. **Pipeline replacement.** (p.238) We will replace one of two alternative pipelines. The first pipeline is in poor condition and has higher probability of failure than the second pipeline. However, the consequence of failure of the second pipeline is greater. The estimated probabilities of failure are $\widetilde{p}_1$ and $\widetilde{p}_2$, and the consequences of failure are positive values, c_1 and c_2, where:

$$\widetilde{p}_1 > \widetilde{p}_2, \quad c_1 < c_2 \tag{194}$$

The true failure probabilities, p_1 and p_2, are uncertain. Use a fractional error info-gap model:

$$\mathcal{U}(h) = \left\{ (p_1, p_2) : \ 0 \le p_i \le 1, \ \left| \frac{p_i - \widetilde{p}_i}{\widetilde{p}_i} \right| \le h, \ i = 1, \, 2 \right\}, \quad h \ge 0 \tag{195}$$

Assume the failure probability of a replaced pipeline is zero.

Let $f_i = 1$ if the ith pipeline is fixed, and let $f_i = 0$ if the ith pipeline is not fixed. We must choose between two replacement plans $f = (f_1, f_2)$: either $f = (0, 1)$ or $f = (1, 0)$.

The expected consequence of replacement plan f is:

$$\mathrm{E}(f|p) = (1 - f_1)p_1c_1 + (1 - f_2)p_2c_2 \tag{196}$$

We require the expected consequence to be no greater than the critical expected consequence, E_c. Evaluate the robustness of each plan and discuss the choice between them. In particular, consider each of the following special cases:

$$\widetilde{p}_1 c_1 \ > \ \widetilde{p}_2 c_2 \tag{197}$$

$$\widetilde{p}_1 c_1 \ < \ \widetilde{p}_2 c_2 \tag{198}$$

59. **Heat flux.** (p.240) The heat flux q (W) is linearly related the temperature gradient ΔT (K) and area A (m^2) by the heat transfer coefficient h ($\text{W/m}^2\text{K}$):

$$q = hA\Delta T \tag{199}$$

We consider implications of uncertainty in the heat transfer coefficient.
(a) Let $\widetilde{h}$ be an estimate of the heat transfer coefficient, with estimation error s but suppose no probability distribution is known. Use a fractional-error info-gap model to represent uncertainty in h (let α represent the horizon of uncertainty):

$$\mathcal{U}(\alpha) = \left\{ h:\ h \geq 0,\ \left| \frac{h - \widetilde{h}}{s} \right| \leq \alpha \right\}, \quad \alpha \geq 0 \tag{200}$$

We require that the heat flux be no less than the positive critical value q_{c}. Derive an explicit algebraic expression for the robustness.
(b) It would be wonderful if the heat flux is as large as q_{w} which is greater than q_{c}. Derive an expression for the opportuneness function using the info-gap model of eq.(200).
(c) Return to part (a) and compare the following two designs:

$$\widetilde{h}_1 < \widetilde{h}_2, \quad \frac{\widetilde{h}_1}{s_1} > \frac{\widetilde{h}_2}{s_2} \tag{201}$$

For what values of q_{c} is design 1 preferred over design 2, based on the robustness function?
(d) Now consider the heat transfer coefficient, h, to be a random variable with an exponential distribution, $p(h) = \lambda \exp(-\lambda h)$. The system fails if the heat flux is less than q_{c}. Derive an explicit algebraic expression for the probability of failure.
(e) Continuing part (d), let $\widetilde{\lambda}$ be a known estimate of λ, with no other information about the true value of λ other than that it is positive. Use a fractional-error info-gap model to represent uncertainty in λ:

$$\mathcal{U}(\alpha) = \left\{ \lambda:\ \lambda \geq 0,\ \left| \frac{\lambda - \widetilde{\lambda}}{\widetilde{\lambda}} \right| \leq \alpha \right\}, \quad \alpha \geq 0 \tag{202}$$

We require that the probability of failure be no greater than the positive critical value P_{c}. Derive an explicit algebraic expression for the robustness.
(f) Now consider heat flux at n points on a surface, with heat transfer coefficients $h_1, \ldots, h_n$ at these points. The heat flux q_j at point j is described by eq.(199) with coefficient h_j. Assume that $A\Delta T$ is the same at each point. Let h denote the vector of heat transfer coefficients, with estimated vector $\widetilde{h}$. Let W represent a matrix of covariances of the coefficients at different points. Use the following ellipsoidal-bound info-gap model to represent uncertainty in h:

$$\mathcal{U}(\alpha) = \left\{ h:\ (h - \widetilde{h})^T W^{-1} (h - \widetilde{h}) \leq \alpha^2 \right\}, \quad \alpha \geq 0 \tag{203}$$

The total heat flux is the sum over the n points: $Q = \sum_{j=1}^{n} q_j$. We require that the total heat flux be no less than the positive critical value q_{c}. Derive an explicit algebraic expression for the robustness.
(g) Suppose that the heat transfer coefficient, h, is a fixed value and the heat flux is a random variable described by:

$$q = hA\Delta T + \varepsilon \tag{204}$$

where ε is a zero-mean normal random variable with variance σ^2, denoted $\varepsilon \sim \mathcal{N}(0, \sigma^2)$. Let $\overline{q}$ denote the mean of N statistically independent measurements of q. What is the probability distribution of $\overline{q}$?

(h) Continuing part (g), suppose that the heat transfer coefficient can take one of two values, either $h = h_0$ or $h = h_1$, where $h_1 > h_0$. Consider the following two hypotheses regarding the value of h:

$$H_0 : \qquad h = h_0 \tag{205}$$

$$H_1 : \qquad h = h_1 \tag{206}$$

The mean of a random sample of size N of heat flux values is observed to equal $\overline{q}_{\rm obs}$. Derive an explicit algebraic expression for the probability that the sample mean exceeds $\overline{q}_{\rm obs}$, if H_0 holds. That is, derive an expression for the probability that you err if you reject H_0. What is the probability of error if $\overline{q}_{\rm obs} = 2.3$, $h_0 A\Delta T = 1$, $\sigma = 1.5$ and $N = 7$?

60. **Risk-adjusted average.** (p.243) Consider a random variable x in the interval $[0,\, 1]$ whose probability density function is $p(x) = 2\delta x + 1 - \delta$, where δ is a parameter in the interval $[-1,\, 1]$. Note that $p(x)$ goes through the point $(\frac{1}{2}, 1)$ with slope determined by δ. The mean and variance of x are:

$$\mathrm{E}(x) = \frac{1}{2} + \frac{\delta}{6}, \quad \sigma^2(x) = \frac{1}{12} - \frac{\delta^2}{36} \tag{207}$$

A large value of x is better than a small value, and the performance is assessed with the risk-adjusted average as:

$$R(\delta) = \mathrm{E}(x) - \alpha\sigma(x) \tag{208}$$

where α is a non-negative parameter expressing the reduction in mean value resulting from risk as expressed by the standard deviation. Note that $\mathrm{E}(x)$ increases, and $\sigma(x)$ decreases, as δ increases. Thus the performance measure $R(\delta)$ improves (increases) as δ increases.

We must choose between 2 systems, where system 2 is estimated to be better but more uncertain than system 1. The estimated δ values are $0 < \widetilde{\delta}_1 < \widetilde{\delta}_2$. $\widetilde{\delta}_i$ is known to be an upper estimate of the true value, δ_i. However, the fractional error of the estimate, $\frac{\widetilde{\delta}_i - \delta_i}{\widetilde{\delta}_i}$, though positive, is thought to be not too large but its value is unknown. In any case, δ_i is no worse (no less) than $-\widetilde{\delta}_i$. We represent this information with the following info-gap model:

$$\mathcal{U}(h, \widetilde{\delta}_i) = \left\{ \delta_i : \ -\widetilde{\delta}_i \le \delta_i, \ 0 \le \frac{\widetilde{\delta}_i - \delta_i}{\widetilde{\delta}_i} \le h \right\}, \quad h \ge 0 \tag{209}$$

We require that the performance, $R(\delta_i)$, be no less than a critical value R_{c}. Derive an expression for the robustness function. Which system do you prefer, as a function of R_{c}?

61. **National energy policy.** (p.244) You must allocate a fixed budget for developing various technological alternatives which together will satisfy the future national energy requirement. The technological alternatives are solar energy, wind energy, oil, coal, natural gas, nuclear, distribution infrastructure, etc. There are N alternatives that can be funded. The investment in alternative i is \$$q_i$, for $i = 1, \ldots, N$, which are non-negative values to be chosen. The anticipated energy return per dollar invested in option i is $\widetilde{u}_i$ MW/\$. The actual rate of return from alternative i, u_i, is uncertain. The total energy production, in MW, of allocation q is $E = q^T u$. We require that E be no less than E_{c}.

(a) Uncertainty in the energy return on investment is represented by the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ u : \left| \frac{u_i - \widetilde{u}_i}{\widetilde{u}_i} \right| \le h, \ i = 1, \ldots, N \right\}, \quad h \ge 0 \tag{210}$$

Note that we allow u_i to be negative, implying that the technology may in fact consume more energy than it produces. Derive an explicit algebraic expression for the robustness of budget allocation q.

(b) Uncertainty in the energy return on investment is represented by the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ u : \ u_i \ge 0, \ \left| \frac{u_i - \widetilde{u}_i}{\widetilde{u}_i} \right| \le h, \ i = 1, \ldots, N \right\}, \quad h \ge 0 \tag{211}$$

Note that we do *not* allow u_i to be negative, implying that development of a technology would be terminated if it would consume more energy than it produces. Derive an explicit algebraic expression for the robustness of budget allocation q.

(c) Uncertainty in the energy return on investment is represented by the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ u : \left| \frac{u_i - \widetilde{u}_i}{s_i} \right| \le h, \ i = 1, \ldots, N \right\}, \quad h \ge 0 \tag{212}$$

where s_i is a known positive uncertainty weight for alternative i. Derive an explicit algebraic expression for the robustness of budget allocation q.

(d) Mr A and Mr B advocate different technological alternatives. The anticipated vector of returns for Mr A's package is $\widetilde{u}^{(1)}$, and for Mr B's package is $\widetilde{u}^{(2)}$. They both use the info-gap model of eq.(212), and their vectors of uncertainty weights are $s^{(1)}$ and $s^{(2)}$ respectively. The following relations hold for a given budget allocation q:

$$q^T \widetilde{u}^{(1)} > q^T \widetilde{u}^{(2)} \tag{213}$$

$$q^T s^{(1)} > q^T s^{(2)} \tag{214}$$

Derive an explicit algebraic expression for the values of the critical energy requirement, E_{c}, for which you robust-prefer Mr A's package.

(e) We now consider uncertain interactions between the energy returns, using the following ellipsoid-bound info-gap model:

$$\mathcal{U}(h) = \left\{ u : \ (u - \widetilde{u})^T W (u - \widetilde{u}) \le h^2 \right\}, \quad h \ge 0 \tag{215}$$

where W is a known, real, symmetric, positive definite matrix. Derive an explicit algebraic expression for the robustness of budget allocation q.

(f) For a specific budget allocation, q, we have an estimated probability density function for the total energy production, $\widetilde{p}(E) = \widetilde{\lambda}\mathrm{e}^{-\widetilde{\lambda}E}$ for $E \geq 0$. We are uncertain about the exponential coefficient, as represented by the following info-gap model:

$$\mathcal{U}(h) = \left\{ p(E) = \lambda \mathrm{e}^{-\lambda E} : \ \lambda \geq 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{\widetilde{\lambda}} \right| \leq h \right\}, \quad h \geq 0 \tag{216}$$

We require that the probability that the total energy production is less than E_{c}, be less that ε. That is, we require:

$$\mathrm{Prob}(E \leq E_{\mathrm{c}}) \leq \varepsilon \tag{217}$$

Derive an explicit algebraic expression for the robustness.

62. **Snake robots.** (p.246) Consider a snake robot with N segments, as illustrated in fig. 10 for $N = 4$. The angles θ_i are positive or negative according to the following rule. θ_i is positive if the rotation from the thin to the thick line is counter clockwise; θ_i is negative otherwise. Let $\theta = (\theta_1, \ldots, \theta_N)$ denote the vector of angles. Throughout this problem treat angles θ_i as real numbers between $-\infty$ and $+\infty$. Thus, for instance $\theta_i = 2\pi$ represents a joint that has wrapped entirely around in a full circle, and thus is a larger angle than $\theta_i = 0$.

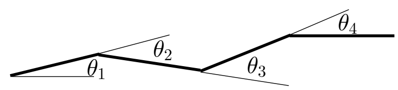

Figure 10: Snake robot for problem 62.

(a) We require that the snake have a net orientation from left to right. Specifically, we require:

$$\left| \sum_{i=1}^{N} \theta_i \right| \leq \varepsilon \tag{218}$$

where ε is a specified performance requirement. However, the angles are uncertain as represented by the following info-gap model:

$$\mathcal{U}(h) = \{\theta : \ |\theta_i| \leq h, \ i = 1, \ldots, N\}, \quad h \geq 0 \tag{219}$$

Derive an explicit algebraic expression for the robustness function.

(b) We require that the snake **not** have a net orientation from left to right. Specifically, we require:

$$\left| \sum_{i=1}^{N} \theta_i \right| \geq \varepsilon \tag{220}$$

where ε is a specified performance requirement. However, the angles are uncertain as represented by the info-gap model of eq.(219). Derive an explicit algebraic expression for the robustness function.

(c) We require that the snake **not** have a net orientation from left to right. Specifically, we require:

$$\sum_{i=1}^{N} \theta_i \geq \varepsilon \tag{221}$$

where ε is a specified performance requirement. However, the angles are uncertain as represented by the info-gap model of eq.(219). Derive an explicit algebraic expression for the robustness function.

(d) Let $\widetilde{\theta}$ denote a vector of nominal angles where we require that the net error in orientation be less than ε:

$$\left| \sum_{i=1}^{N} (\theta_i - \widetilde{\theta}_i) \right| \leq \varepsilon \tag{222}$$

However, the angles θ are uncertain as represented by the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ \theta : \ \left| \frac{\theta_i - \widetilde{\theta}_i}{\widetilde{\theta}_i} \right| \leq h, \ i = 1, \ldots, N \right\}, \quad h \geq 0 \tag{223}$$

where $\widetilde{\theta}_i \neq 0$ for all i. Derive an explicit algebraic expression for the robustness function.

(e) Let φ denote a vector of angles where we require that the net error in orientation with respect to φ be less than ε:

$$\left| \sum_{i=1}^{N} (\theta_i - \varphi_i) \right| \le \varepsilon \tag{224}$$

However, the angles θ are uncertain as represented by the info-gap model of eq.(223). Derive an explicit algebraic expression for the robustness function.

(f) Let φ denote a vector of angles where we require that the absolute deviation in orientation with respect to φ be no less than ε:

$$\left| \sum_{i=1}^{N} (\theta_i - \varphi_i) \right| \ge \varepsilon \tag{225}$$

However, the angles θ are uncertain as represented by the info-gap model of eq.(223). Derive an explicit algebraic expression for the value of the angles θ_i that minimize the sum in eq.(225) at horizon of uncertainty h. Derive an explicit algebraic expression for the **inverse** of the robustness function.

(g) Continue from part 62e, and consider two different nominal configurations, $\widetilde{\theta}^{(a)}$ and $\widetilde{\theta}^{(b)}$, where:

$$\begin{aligned} \left| \sum_{i=1}^{N} (\widetilde{\theta}_i^{(b)} - \varphi_i) \right| &\le \left| \sum_{i=1}^{N} (\widetilde{\theta}_i^{(a)} - \varphi_i) \right| & (226) \\ \sum_{i=1}^{N} \left| \widetilde{\theta}_i^{(b)} \right| &\ge \sum_{i=1}^{N} \left| \widetilde{\theta}_i^{(a)} \right| & (227) \end{aligned}$$

Derive an explicit algebraic expression for the range of ε values for which nominal configuration $\widetilde{\theta}^{(a)}$ is preferred over $\widetilde{\theta}^{(b)}$ based on robustness.

(h) For any given configuration, θ, the effort needed to straighten the snake is proportional to the sum of the squares of the angles. We require that this effort not exceed E:

$$\sum_{i=1}^{N} \theta_i^2 \le E \tag{228}$$

However, the angles are uncertain according to the following ellipsoidal-bound info-gap model:

$$\mathcal{U}(h) = \left\{ \theta : \ \theta^T W \theta \le h^2 \right\}, \quad h \ge 0 \tag{229}$$

where W is a known, real, symmetric, positive definite matrix. Derive an explicit algebraic expression for the robustness function.

(i) Let x denote the sum of the angles, $x = \sum_{i=1}^{N} \theta_i$. Assume that x is a random variable with a uniform probability density on the interval $[-b, \ b]$:

$$p(x|b) = \frac{1}{2b}, \ x \in [-b, \ b] \tag{230}$$

The probability density depends on the parameter b, which is uncertain:

$$\mathcal{U}(h) = \left\{ b : \ b > 0, \ |b - \widetilde{b}| \le h \right\}, \quad h \ge 0 \tag{231}$$

$\widetilde{b}$ is known and positive. Let x_c be a positive critical value, $x_c > 0$. We require that the probability that x is less than x_c, must exceed ε:

$$\mathrm{Prob}(x \le x_c) \ge \varepsilon \tag{232}$$

Because $x_{\rm c} > 0$ we only consider $\varepsilon > 1/2$. Derive an explicit algebraic expression for the robustness function.

(j) Let x denote the sum of the angles, $x = \sum_{i=1}^{N} \theta_i$. Assume that x is a random variable with a uniform probability density on the interval $[0,\ b]$:

$$p(x|b) = \frac{1}{b},\ x \in [0,\ b] \tag{233}$$

The probability density depends on the parameter b, which is uncertain:

$$\mathcal{U}(h) = \left\{ b:\ b > 0,\ \frac{|b - \widetilde{b}|}{\widetilde{b}} \le h \right\}, \quad h \ge 0 \tag{234}$$

where $\widetilde{b}$ is known and positive. Let $x_{\rm c}$ be a positive critical value in the interval $[0,\ \widetilde{b}]$. We require that the probability that x is less than $x_{\rm c}$, must not exceed ε:

$$\mathrm{Prob}(x \le x_{\rm c}) \le \varepsilon \tag{235}$$

Derive an explicit algebraic expression for the robustness function.

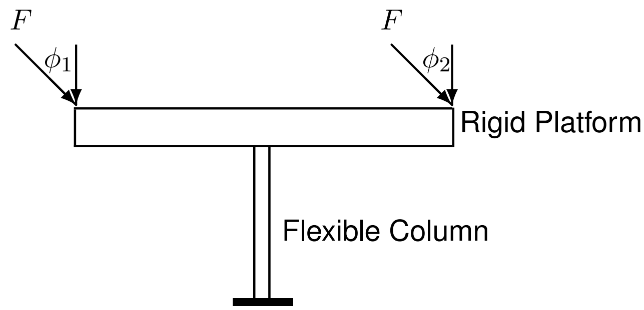

Figure 11: Platform for problem 63.

63. **Stability of a platform.** (p.250) A thin rigid beam-like platform is supported from below at its midpoint by a flexible column that is at elastic equilibrium when the platform is horizontal, as shown in fig. 11. The flexural stiffness of the elastic column is k [Nm/radian] and it applies a restoring moment of force $M = k\theta$ at the midpoint of the platform when the platform is tilted by θ radians. The width of the platform is $2L$ [m]. Ignore the thickness of the beam. The platform is loaded, in the plane, at its two ends by a static force F, at different angles from the vertical, ϕ_1 and ϕ_2.

    (a) The force, F, is known. The angles are nominally zero but deviate from zero by an unknown amount. That is, ϕ_1 and ϕ_2 belong to the following info-gap model of uncertainty:

$$\mathcal{U}(h) = \{\phi_1, \phi_2 : \ |\phi_i| \leq h, \ i = 1, \ 2\}, \quad 0 \leq h \leq \pi \tag{236}$$

    The platform is satisfactorily level if the absolute angle of tilt at static equilibrium is never greater than the critical value $\theta_{\rm c}$:

$$|\theta| \leq \theta_{\rm c} \tag{237}$$

    Derive the robustness function of the platform. The decision vector is $q = (k, L)^T$. Explain how the robustness changes as these design variables are changed.

    (b) Now consider a continuously distributed load in the plane, along the beam from $x = -L$ at the left to $x = +L$ at the right. The magnitude of the load density at each point is f [N/m], which is known and constant. The angle of the load with respect to the vertical, at position x, is $\phi(x)$. The angle of the load at each point is nominally zero but may deviate from zero by an unknown amount. That is, $\phi(x)$ belongs to the following info-gap model of uncertainty:

$$\mathcal{U}(h) = \{\phi(x) : \ |\phi(x)| \leq h, \ -L \leq x \leq L\}, \quad 0 \leq h \leq \pi \tag{238}$$

    Derive the robustness function for satisfying the performance requirement in eq.(237). Explain how the robustness changes as the design variables, k and L, are changed.

64. **Reliability of a milling process–2.** (p.252) An automated cutting tool moves at constant horizontal velocity across a work piece. The height $y(t)$ of the tool varies in transit. The desired height profile is $\widetilde{y}(t)$. The actual height profile differs from $\widetilde{y}(t)$ in an uncertain manner:

$$y(t) = \widetilde{y}(t) + \sum_{n=n_1}^{n_2} b_n \cos \frac{n\pi t}{T} = \widetilde{y}(t) + b^T \gamma(t), \quad 0 \le t \le T \tag{239}$$

where b is the vector of uncertain Fourier coefficients and $\gamma(t)$ is the vector of corresponding cosine functions. Uncertainty in the actual height profile is represented by the following info-gap model:

$$\mathcal{U}(h) = \left\{ y(t) = \widetilde{y}(t) + b^T \gamma(t) : \ b^T W b \le h^2 \right\}, \quad h \ge 0 \tag{240}$$

where W is a known, real, symmetric, positive definite matrix.

The milling process fails if the cutting tool is too far above the planned height at the end of the run, $t = T$. That is, failure is defined as:

$$y(T) - \widetilde{y}(T) > D_{\mathrm{c}} \tag{241}$$

(a) Derive a generic algebraic expression for the robustness.

(b) Consider the specific case that W is the identity matrix. How does the robustness vary with the size and location of the frequency window, $n_1, \ldots, n_2$?

(c) Now suppose that W is a diagonal matrix whose diagonal elements are $\frac{1}{n_1}, \ldots, \frac{1}{n_2}$. What does this imply about the relative uncertainty of the different modes? Use the result from part 64a to numerically evaluate the robustness vs. n_2 for a range of n_2 values, with $n_1 = 1$. What does this indicate about the impact, on the robustness, of the bandwidth of uncertain modes? Let $D_{\mathrm{c}} = 1$.

65. **Uncertain loads on a linear elastic system.** (p.254) Consider the deflection of a point in a linear elastic system subject to uncertain loads. The magnitude of deflection, y, is related to the vector of N loads on the system, f, according to:

$$y = k^T f \tag{242}$$

where k is a known vector of flexibility coefficients. The system fails if the deflection exceeds a critical value, y_{c}. Consider the following different situations.

(a) Each of the N elements of the load vector may be either positive or negative. It is known that the length of the load vector is bounded but the value of the bound is unknown. Represent the uncertain load vector with the following info-gap model:

$$\mathcal{U}(h) = \left\{ f : \ f^T f \le h^2 \right\}, \quad h \ge 0 \tag{243}$$

Derive an explicit expression for the robustness function.

(b) Each of the N elements of the load vector may be either positive or negative. It is known that the length of the load vector is bounded but the value of the bound is unknown. Furthermore, the greatest magnitude of the ith element of the load vector may exceed the greatest magnitude the $(i-1)$th element by as much as a factor 2. Represent the uncertain load vector with the following info-gap model:

$$\mathcal{U}(h) = \left\{ f : \ f^T W f \le h^2 \right\}, \quad h \ge 0 \tag{244}$$

where W is a diagonal $N \times N$ matrix whose non-zero terms are $W_{nn} = 4^{-(n-1)}$, $n = 1, \ldots, N$. Explain why this info-gap model reflects the available information. Derive an explicit expression for the robustness function.

(c) Now consider a generic ellipsoidal uncertainty where the load vector belongs to the info-gap model:

$$\mathcal{U}(h) = \left\{ f : \ (f - \widetilde{f})^T W (f - \widetilde{f}) \le h^2 \right\}, \quad h \ge 0 \tag{245}$$

where the nominal load, $\widetilde{f}$, and the shape matrix, W, are known and W is real, symmetric, and positive definite. Derive an explicit expression for the robustness function.

(d) Consider the choice between two designs with flexibility vectors k_1 and k_2 for which:

$$k_1^T \widetilde{f} \ > \ k_2^T \widetilde{f} \tag{246}$$

$$k_1^T W^{-1} k_1 \ < \ k_2^T W^{-1} k_2 \tag{247}$$

Using the robustness function from part 65c, for what values of y_{c} do you prefer design k_1, and when do you prefer k_2?

66. **Braking system–3.** (p.257). Consider a linear braking system for which the stopping distance is described by:

$$s(g, f) = \int_0^t f(\tau) g(t - \tau)\, \mathrm{d}\tau \tag{248}$$

where $f(t)$ is the uncertain forcing function and $g(t)$ is the known impulse response function. We require that the braking distance not exceed the critical value s_{c} at a specified time T.

(a) Derive an explicit algebraic expression for the robustness for each of the following four info-gap models of uncertainty in the force.

$$\mathcal{U}_1(h) = \left\{ f(t) : \frac{1}{T} \int_0^T f(t)^2\, \mathrm{d}t \le h^2 \right\}, \quad h \ge 0 \tag{249}$$

$$\mathcal{U}_2(h) = \{ f(t) : |f(t)| \le h, \forall\, t \ge 0 \}, \quad h \ge 0 \tag{250}$$

$$\mathcal{U}_3(h) = \left\{ f(t) : \frac{1}{T} \int_0^T (f(t) - \widetilde{f}(t))^2\, \mathrm{d}t \le h^2 \right\}, \quad h \ge 0 \tag{251}$$

$$\mathcal{U}_4(h) = \left\{ f(t) : |f(t) - \widetilde{f}(t)| \le h, \forall\, t \ge 0 \right\}, \quad h \ge 0 \tag{252}$$

The nominal forcing function, $\widetilde{f}(t)$, is known and positive. Note that the horizon of uncertainty, h, has the same units for all 4 info-gap models, so the corresponding robustnesses are commensurable.

(b) Employ the Cauchy-Schwarz inequality to show that the following relation holds:

$$\int_0^T |g(t)|\, \mathrm{d}t \le \sqrt{T \int_0^T g^2(t)\, \mathrm{d}t} \tag{253}$$

Hint: $|g| = 1 \times |g|$.

(c) The info-gap models of eqs.(249) and (250) represent two different states of knowledge (or ignorance). Based on the robustness functions, which state of knowledge do you prefer?

(d) The info-gap models of eqs.(249) and (252) represent two different states of knowledge (or ignorance). Based on the robustness functions, which state of knowledge do you prefer?

(e) The info-gap models of eqs.(250) and (251) represent two different states of knowledge (or ignorance). Based on the robustness functions, which state of knowledge do you prefer?

67. **Ballistics–2.** (p.260) A missile is designed to follow the trajectory:

$$y(x) = ax(b-x) \tag{254}$$

where x is the horizontal position along the ground from the launch site to the landing site, and $y(x)$ is the height of the missile along the trajectory. a is positive. We require that the missile land at a distance no less than x_c from the launch site.

(a) The coefficient b is uncertain, which introduces uncertainty in the trajectory:

$$\mathcal{U}(h) = \left\{ b : \ \left| \frac{b - \widetilde{b}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{255}$$

$\widetilde{b}$ and s are known and positive. Derive an explicit algebraic expression for the robustness.

(b) Use the robustness function from part 67a to choose between two designs. System 1 has better flight control while system 2 has longer range. Specifically:

$$s_1 < s_2 \tag{256}$$
$$\widetilde{b}_1 < \widetilde{b}_2 \tag{257}$$

where s_i and $\widetilde{b}_i$ are the parameters of the info-gap model, eq.(255), for system i.

(c) Now consider a different info-gap model for the uncertain flight path:

$$\mathcal{U}(h) = \left\{ y(x) : \ y(x) \ge 0, \ \left| \frac{y(x) - \widetilde{y}(x)}{y_0} \right| \le h \right\}, \quad h \ge 0 \tag{258}$$

where $\widetilde{y}(x)$ is known and specified in eq.(254), and y_0 is positive and known. Derive an explicit algebraic expression for the robustness. Evaluate and plot the robustness curves for $a = 1$ and the following two sets of values: $(y_0, b) = (0.4, 1)$ and $(0.2, 0.8)$. What is the intuitive meaning of these values for b and y_0? For what values of x_c do you robust-prefer each option?

68. **Safety factor in a rotating beam,** (p.262). Consider a rigid beam supported at one end with a rotational spring of stiffness k. The moment of inertia of the beam, for rotation around its support, is J. The natural frequency of rotation is ω. The rotation is frictionless, and a moment $u(t)$ is applied to the free end of the beam. The angle of rotation from zero initial conditions, as a function of time, is:

$$\theta_u(t) = \frac{1}{J\omega}\int_0^t u(\tau)\sin\omega(t-\tau)\,\mathrm{d}\tau \tag{259}$$

We require that the absolute angle of rotation, at a specified time T, not exceed the critical value θ_{c}:

$$|\theta_u(T)| \le \theta_{\mathrm{c}} \tag{260}$$

The estimated applied moment is:

$$\widetilde{u}(t) = u_0 \sin\omega(T-t) \tag{261}$$

where u_0 is a known positive constant. The uncertainty in the applied moment is represented by the following info-gap model:

$$\mathcal{U}(h) = \left\{ u(t) : \ \left| \frac{u(t)-\widetilde{u}(t)}{u_0} \right| \le h \right\}, \quad h \ge 0 \tag{262}$$

(a) Derive an explicit algebraic expression for the robustness function. Let $T = n\pi/\omega$ for a known positive integer n.

(b) For a specified value of θ_{c}, let J_0 be the moment of inertia for which the estimated absolute angle of deflection at time T exactly equals θ_{c}. What is the robustness to load uncertainty at this value of θ_{c}? Hint: It is not necessary to calculate $\theta_{\widetilde{u}}(T)$ directly.

(c) We want to increase the moment of inertia from J_0 to $(1+\varepsilon)J_0$ in order to enhance the safety of the system. The horizon of uncertainty, h, in the info-gap model of eq.(262) is unknown. However, we are fairly confident that load amplitude will not exceed the nominal value, u_0, by more than a known factor f. Use the robustness function from part 68a to derive an explicit algebraic expression for the safety factor ε.

(d) Now suppose that we know that $|\theta(T)|$ is a random variable with an exponential distribution:

$$p(|\theta(T)|) = \mu \mathrm{e}^{-\mu|\theta(T)|}, \quad |\theta(T)| \ge 0 \tag{263}$$

We know that μ cannot be negative, and a known estimate of μ is $\widetilde{\mu}$, but we don't know the fractional error of $\widetilde{\mu}$. Hence we adopt the following info-gap model:

$$\mathcal{U}(h) = \left\{ \mu : \ \mu > 0, \ \left| \frac{\mu - \widetilde{\mu}}{\widetilde{\mu}} \right| \le h \right\}, \quad h \ge 0 \tag{264}$$

We require small probability of violating the requirement in eq.(260). Specifically, we require:

$$\mathrm{Prob}\left(|\theta_u(T)| > \theta_{\mathrm{c}}\right) \le P_{\mathrm{c}} \tag{265}$$

Derive an explicit expression for the robustness function.

(e) Return to part 68a and suppose that we know that h is a random variable with an exponential distribution:

$$p(h) = \lambda \mathrm{e}^{-\lambda h}, \quad h \ge 0 \tag{266}$$

with known exponential coefficient λ. Using the robustness of part 68a, derive an explicit algebraic expression for a meaningful lower bound of the probability of satisfying eq.(260). Hint: Consider the relation between "$h \le \widehat{h}$" and "$|\theta_u(T)| \le \theta_{\mathrm{c}}$".

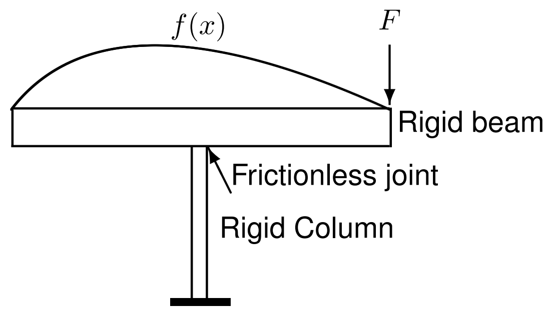

Figure 12: Beam for problem 69.

69. **Balancing a beam,** (p.264). Consider a rigid beam supported at the midpoint on a frictionless joint, as shown in fig. 12. A positive force F acts downward at the right end. A distributed force $f(x)$ acts along the beam, where positive force acts downward. The length of the beam is $2L$. The net moment of force on the beam is:

$$M(f) = FL + \int_{-L}^{L} x f(x)\, \mathrm{d}x \tag{267}$$

We require that the absolute moment not exceed the critical value M_{c}:

$$|M(f)| \le M_{\mathrm{c}} \tag{268}$$

(a) Derive an explicit algebraic expression for the robustness function with this info-gap model:

$$\mathcal{U}(h) = \left\{ f(x) : \left| \frac{f(x) - f_0}{f_0} \right| \le h \right\}, \quad h \ge 0 \tag{269}$$

where f_0 is a known positive constant.

(b) Derive an explicit algebraic expression for the robustness function with this info-gap model:

$$\mathcal{U}(h) = \left\{ f(x) : \left| \frac{f(x) - f_0}{s} \right| \le h \right\}, \quad h \ge 0 \tag{270}$$

where f_0 and s are known positive constants.

(c) Derive an explicit algebraic expression for the robustness function with this info-gap model:

$$\mathcal{U}(h) = \left\{ f(x) : \left| \frac{f(x) - f_0 \cos \frac{\pi x}{L}}{f_0} \right| \le h \right\}, \quad h \ge 0 \tag{271}$$

where f_0 is a known positive constant.

(d) Derive an explicit algebraic expression for the robustness function with this info-gap model:

$$\mathcal{U}(h) = \left\{ f(x) = \sum_{n=1}^{N} a_n \sin \frac{n\pi x}{L} = a^T \sigma(x) : \ a^T W a \le h^2 \right\}, \quad h \ge 0 \tag{272}$$

where W is a known positive definite real symmetric matrix.

(e) Derive an explicit algebraic expression for the robustness function with this info-gap model:

$$\mathcal{U}(h) = \left\{ f(x) : \ \int_{-L}^{L} (f(x) - f_0)^2 \, \mathrm{d}x \le h^2 \right\}, \quad h \ge 0 \tag{273}$$

where f_0 is a known positive constant.

70. **Random events and failure,** (p.266). Consider a system in which adverse events occur randomly, independently, and at constant average rate λ/sec. Failure of the system is defined to occur if n or more events occur within a specified duration T, for a specified value of n.
   (a) If failure occurs after the 1st event, that is, $n = 1$, what is the probability of failure of the system in time T?
   (b) If failure occurs after the 2nd event, that is, $n = 2$, what is the probability of failure of the system in time T?
   (c) Suppose that failure occurs after the first event, that is, $n = 1$, but λ is uncertain according to the info-gap model:
$$\mathcal{U}(h) = \left\{ \lambda : \ \lambda > 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{s} \right| \le h \right\}, \quad h \ge 0 \qquad (274)$$
   We require that the probability of failure be no greater than $P_{\rm c}$. Derive an explicit algebraic expression for the robustness.
   (d) Continuing part 70c, consider a budget allocation problem. The cost of the system depends independently on both $\widetilde{\lambda}$ and s. A small average event rate, $\widetilde{\lambda}$, is better than a large average event rate. Similarly, a small uncertainty weight, s, is better than a large uncertainty weight. $\widetilde{\lambda}$ can be reduced by investing resources. Specifically, $\widetilde{\lambda}$ decreases linearly with expenditure: ${\rm d}\widetilde{\lambda}/{\rm d}\$ = -a$ where a is a known positive constant. Likewise, s decreases linearly with expenditure: ${\rm d}s/{\rm d}\$ = -b$ where b is a known positive constant. You have a tiny budget, c, say \$1. Should you allocate it to decreasing $\widetilde{\lambda}$, or to decreasing s, or should you divide it between the two items and if so, how?
   (e) Now suppose that failure occurs after the second event, that is, $n = 2$, but λ is uncertain according to the info-gap model in eq.(274). Derive an explicit algebraic expression for the inverse of the robustness function.

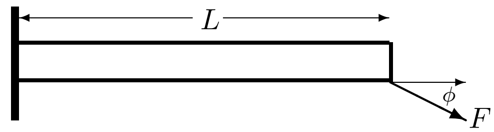

Figure 13: Cantilever for problem 71.

71. **Random loads on a beam,** (p.268). Consider the cantilever in fig. 13 with load F at the free end and rotational stiffness k at the base. The angle of deflection of the beam around its base is:

$$\theta = \frac{FL\sin\phi}{k} \tag{275}$$

where $0 \le \phi \le \pi$. We require that the absolute deflection not exceed the known critical value $\theta_{\rm c}$:

$$|\theta| \le \theta_{\rm c} \tag{276}$$

(a) The load, F, is uncertain according to the info-gap model:

$$\mathcal{U}(h) = \{F: \ |F| \le h\}, \quad h \ge 0 \tag{277}$$

Derive an explicit algebraic expression for the robustness.

(b) The load, F, and the stiffness, k, are uncertain according to the info-gap model:

$$\mathcal{U}(h) = \left\{F, k: \ \left|\frac{F-\widetilde{F}}{\widetilde{F}}\right| \le h, \ k > 0, \ \left|\frac{k-\widetilde{k}}{\widetilde{k}}\right| \le h\right\}, \quad h \ge 0 \tag{278}$$

Derive an explicit algebraic expression for the robustness.

(c) Now consider k to be known and the load, F, to be a random variable with an exponential distribution:

$$p(F) = \lambda \mathrm{e}^{-\lambda F}, \quad F \ge 0 \tag{279}$$

The system fails if eq.(276) is not satisfied. Let $P_{\rm f}$ denote the probability that the system fails. Derive an explicit algebraic expression for the probability of failure.

(d) Continuing part 71c, suppose that the exponential coefficient λ is uncertain as described by the following info-gap model:

$$\mathcal{U}(h) = \left\{\lambda: \ \lambda > 0, \ \left|\frac{\lambda-\widetilde{\lambda}}{\widetilde{\lambda}}\right| \le h\right\}, \quad h \ge 0 \tag{280}$$

We require that the probability of failure not exceed the critical value $P_{\rm c}$. Derive an explicit algebraic expression for the robustness.

(e) One person claims that the true angle of deflection of the beam is 0.26, but another person claims the angle is greater:

$$H_0: \qquad \theta = 0.26 \tag{281}$$

$$H_1: \qquad \theta > 0.26 \tag{282}$$

We have performed 5 statistically independent and normally distributed measurements of the angle θ: 0.26, 0.31, 0.29, 0.22, 0.32. Given this random sample, do you accept or reject H_0 at the 0.01 level of significance?

(f) The beam is subject to successive independent loadings. Each loading is either "normal" or "excessive". One person claims that, given a large set of loadings, 10% of the loadings are "excessive" and the rest are "normal". Another person claims that this is false:

$$H_0 : \qquad p_1 = 0.1, \; p_2 = 0.9 \tag{283}$$

$$H_1 : \qquad \neg H_0 \tag{284}$$

The loading has been observed 35 times, and exactly 5 cases were "excessive". Do you accept or reject H_0 at level of significance 0.05?

(g) A large batch of many millions of beams has been produced, where the probability that any one beam is defective is 0.005. In a sample of 1000 beams, what is the probability that more than 1 beam is defective?

(h) Beams are produced in sequence, where the average number of defective beams is 5 per 1000 beams. Defects in any one beam are independent of defects in other beams. What is the probability of more than 1 defective beam in the first 1000 beams?

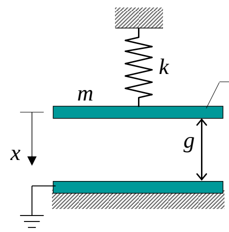

Figure 14: Gap-closing electrostatic actuator for problem 72. (Fig. thanks to Prof. David Elata, head, Mechanical Engineering Micro Systems (MEMS) lab, Technion.)

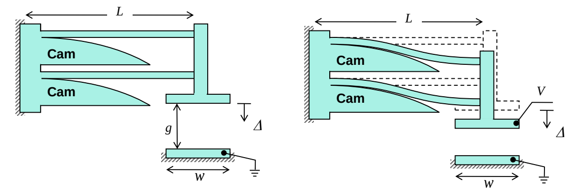

Figure 15: Mechanically linearized Gap-closing electrostatic actuator for problem 72. (Fig. thanks to Prof. David Elata)

72. **Gap-closing electrostatic actuators,** (p.270). The non-linear force-displacement relation for the gap-closing electrostatic actuator in fig. 14 is:

$$F = kx - \frac{\varepsilon A V^2}{2(g-x)^2} \tag{285}$$

where ε is the dielectric constant, A is the area of the plates, V is the electric potential on the device, k is the spring stiffness and g is the initial gap size.

Fig. 15 shows a mechanically linearized modification of the device in fig. 14 for which the force-displacement relation is, nominally, linear:

$$F = Kx \tag{286}$$

The degree of linearity depends on the shapes of the cams and on the degree of mechanical and structural uniformity of the pair of beams. We will explore the robustness to various forms of uncertainty in the linearity of the beam. We will also explore probabilistic models and statistical decisions.

(a) We require that application of a known force F results in a displacement no less than $x_{\rm c}$. Uncertainty in the linear stiffness coefficient K is represented by the info-gap model:

$$\mathcal{U}(h) = \left\{ K:\ K>0,\ \left| \frac{K-\widetilde{K}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{287}$$

where $\widetilde{K}$ is the known nominal linear stiffness and s is a known positive error coefficient. Derive an explicit algebraic expression for the robustness.

(b) Repeat part 72a where we aspire to displacement as large as $x_{\rm w}$. Derive an explicit algebraic expression for the opportuneness.

(c) We require that application of a known force F results in a displacement no less than $x_{\rm c}$. However, the nominal linear force-displacement relation in eq.(286) is replaced by:

$$x = \frac{F}{K} + \sum_{n=1}^{N} a_n F^n = \frac{F}{K} + a^T \phi \tag{288}$$

where ϕ is the vector of powers of F and a is the vector of coefficients whose uncertainty is represented by the info-gap model:

$$\mathcal{U}(h) = \left\{ a:\ a^T W a \le h^2 \right\}, \quad h \ge 0 \tag{289}$$

where W is a known, real, symmetric, positive definite matrix. Derive an explicit algebraic expression for the robustness.

(d) Now consider K in eq.(286) to be a random variable with a uniform probability density:

$$p(K) = \frac{1}{K_{\rm max}}, \quad 0 \le K \le K_{\rm max} \tag{290}$$

Failure occurs if

$$x < x_{\rm c} \tag{291}$$

Derive an explicit algebraic expression for the probability of failure. Assume $F \le x_{\rm c} K_{\rm max}$.

(e) Continuing part 72d, suppose that F and $K_{\rm max}$ are both info-gap-uncertain as described by the following info-gap model:

$$\mathcal{U}(h) = \left\{ F, K_{\rm max}: \ F > 0, \ \left| \frac{F - \widetilde{F}}{\widetilde{F}} \right| \le h, \ K_{\rm max} > 0, \ \left| \frac{K_{\rm max} - \widetilde{K}_{\rm max}}{\widetilde{K}_{\rm max}} \right| \le h, \right\}, \quad h \ge 0 \tag{292}$$

We require that the probability of failure not exceed the critical value $P_{\rm c}$. Derive an explicit algebraic expression for the robustness. Assume $\widetilde{F} \le x_{\rm c} \widetilde{K}_{\rm max}$.

(f) Let K be a random variable whose estimated pdf, $\widetilde{p}(K)$, is normal with mean μ and variance σ^2. We are confident that this estimate is accurate for K within an interval around μ of known size $\pm\delta_s$. However, outside of this interval of K values the fractional error of the pdf is unknown. Our info-gap model is:

$$\mathcal{U}(h) = \left\{ p(K): \quad \begin{array}{l} \displaystyle\int_{-\infty}^{\infty} p(K)\,{\rm d}K = 1, \ p(K) \ge 0, \ \forall K, \\ p(K) = \widetilde{p}(K), \ |K - \mu| \le \delta_s \\ \displaystyle\left| \frac{p(K) - \widetilde{p}(K)}{\widetilde{p}(K)} \right| \le h, \ |K - \mu| > \delta_s \end{array} \right\}, \ h \ge 0 \tag{293}$$

The system fails if $x < x_{\rm c}$ where $x = F/K$ as stated in eq.(286), where F is a known positive constant. x is now a random variable so the performance requirement is that the probability of failure not exceed a critical value $P_{\rm c}$. Derive an explicit algebraic expression for the robustness function. Assume that $F/x_{\rm c} \ge \mu + \delta_s$.

(g) We are testing a MEMS system but we don't know if it is "raw" like fig. 14 or "linearized" like fig. 15. The loads are random and, if the beam is linearized, they produce small, medium and large deflections with frequencies 0.5, 0.3 and 0.2, respectively. If the beam is not linearized then the frequencies are different. We observe 41 small, 32 medium, and 27 large displacements. For the following hypotheses, do you accept or reject H_0 at the 0.05 level of significance?

$$H_0: \quad p_{\rm sml} = 0.5, \ \ p_{\rm med} = 0.3, \ \ p_{\rm lrg} = 0.2 \tag{294}$$

$$H_1: \quad \neg H_0 \tag{295}$$

(h) The lifetime of the device is distributed according to a Weibull distribution whose probability distribution function is:

$$P(t) = 1 - {\rm e}^{-(\lambda t)^{\alpha}}, \quad t \ge 0 \tag{296}$$

where λ and α are positive constants. A specific unit has been observed to be operational at time t_0. Derive an explicit algebraic expression for the probability that this unit will be operational at time t_1.

(i) We have an endless supply of devices where a fraction p are "raw" like fig. 14 and the rest are "linearized" like fig. 15. We select N devices randomly and independently. Derive an explicit algebraic expression for the probability that J or more devices are "raw".

(j) Apply a known force, F, measure the resulting displacement x, and let y denote the difference between the measurement and the predicted displacement based on eq.(286). Assume the measurement is corrupted by zero-mean normally distributed noise whose variance is unknown. The values of y in a random sample of size $N = 6$ are 1, 3, 3, 4, 1, 2. The advocate of the linear model in eq.(286) claims that the true value of y is zero, while the critic claims that this is false:

$$H_0 : \qquad y = 0 \tag{297}$$

$$H_1 : \qquad y \neq 0 \tag{298}$$

Do you accept or reject H_0 at the 0.01 level of significance?

73. **Supply network,** (p.276).

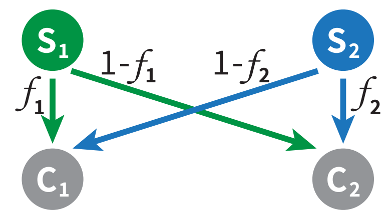

Figure 16: Topology for two sources and two consumers, problem 73.

(a) Consider a network that supplies a good (e.g. water, electricity or cookies) to consumers. The network has two suppliers and two consumers. Source i produces quantity s_i and supplies a fraction f_i to consumer i, and a fraction $1-f_i$ to consumer j, as in fig. 16. Source j acts similarly. Each consumer consumes whatever is supplied. Thus the consumption by consumer i is:

$$c_i = f_i s_i + (1 - f_j) s_j \tag{299}$$

where $i = 1,\, 2$ and $j = 3 - i$.

There is fractional-error uncertainty in the source properties:

$$\mathcal{U}(h) = \left\{ f_i, s_i : \ \left| \frac{f_i - \widetilde{f}_i}{\widetilde{f}_i} \right| \le h,\ f_i \in [0,\, 1],\ \left| \frac{s_i - \widetilde{s}_i}{\widetilde{s}_i} \right| \le h,\ s_i \ge 0,\ i = 1,\, 2 \right\}, \quad h \ge 0 \tag{300}$$

We require that each consumer be within δ of a specified value, $\overline{c}$:

$$|c_i - \overline{c}| \le \delta \tag{301}$$

Derive an explicit expression for the inverse of the robustness function for consumer i. Evaluate and compare the robustnesses for $\overline{c} = 1$, $\widetilde{f} = 1/2$ and $\widetilde{s} = 0.9$ or 1.0. Which option is preferred? Why, and what does this mean?

(b) Modify part 73a as follows. The nominal consumption by each consumer is $\widetilde{c}$ but the actual consumption is:

$$c = \widetilde{c} + \varepsilon \tag{302}$$

where ε is an exponentially distributed random variable whose pdf is $p(\varepsilon) = \lambda \mathrm{e}^{-\lambda\varepsilon}$, $\varepsilon \ge 0$. Derive an explicit algebraic expression for the probability that $c \le \overline{c}$ where $\overline{c}$ is a known positive value greater than $\widetilde{c}$.

(c) Continuing part 73b, consider uncertainty in the exponential coefficient, λ, represented by the info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda : \ \lambda > 0,\ \left| \frac{\lambda - \widetilde{\lambda}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{303}$$

where $\widetilde{\lambda}$ and w are known positive values. We require that the probability that $c \le \overline{c}$ be less than δ where $0 < \delta < 1$. Derive an explicit algebraic expression for the robustness.

(d) Continuing part 73c, consider two different designs with values $\widetilde{\lambda}_i$ and w_i, for $i = 1$ and 2. Based on the robustness function, derive an explicit algebraic expression for the values of δ for which you prefer system 1.

(e) A particular consumer (you, perhaps) is supplied by N sources resulting in consumption equal to:

$$c = \sum_{i=1}^{N} f_i s_i \tag{304}$$

The fractions f_i and source terms s_i are uncertain:

$$\mathcal{U}(h) = \left\{ f, s:\ f_i \ge 0,\ \left| \frac{f_i - \widetilde{f}_i}{\widetilde{f}_i} \right| \le h,\ s_i \ge 0,\ \left| \frac{s_i - \widetilde{s}_i}{\widetilde{s}_i} \right| \le h, i = 1, \ldots, N \right\}, \quad h \ge 0 \tag{305}$$

We require that the consumption be no less than the critical value $\overline{c}$:

$$c \ge \overline{c} \tag{306}$$

Derive an explicit algebraic expression for the robustness function.

(f) Repeat part 73e with the modification that the fractions, f_i, are positive and known for sure and the source terms are uncertain according to:

$$\mathcal{U}(h) = \left\{ s:\ (s - \widetilde{s})^T W^{-1} (s - \widetilde{s}) \le h^2 \right\}, \quad h \ge 0 \tag{307}$$

where $\widetilde{s}$ is a known vector and W is a known, positive definite, real, symmetric matrix. The performance requirement is eq.(306). Derive an explicit algebraic expression for the robustness function.

(g) Consider the following modification of the 2-source and 2-consumer network, in which we introduce a mutual commitment. Under ordinary conditions, each source supplies a single consumer at each discrete time step:

$$c_1(t) = s_1, \quad c_2(t) = s_2, \qquad t = 0, 1, 2, \ldots \tag{308}$$

Each consumer has its own private supply, and each consumer requires a positive consumption $\overline{c}_i$, $i = 1, 2$.

However, the consumers have mutual commitments. If consumer i loses its supply at some time step, then in the next time step consumer j is committed to supply consumer i with a fraction γ of i's requirement, $\overline{c}_i$, though j cannot supply more than s_j provides.

Suppose that at some time step, call it $t = 0$, consumer 1 loses its supply, so the consumption in this step is:

$$c_1(0) = 0, \quad c_2(0) = s_2 \tag{309}$$

In the next time step, consumer 2 must transfer to consumer 1 a part of 2's supply so the consumption is:

$$c_1(1) = \min(s_2, \gamma \overline{c}_1), \quad c_2(1) = (s_2 - \gamma \overline{c}_1)^+ \tag{310}$$

where $x^+ = x$ if $x \ge 0$ and equals zero otherwise.

The sources are uncertain according to a fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ s:\ s_i \ge 0,\ \left| \frac{s_i - \widetilde{s}_i}{w_i} \right| \le h, i = 1, 2 \right\}, \quad h \ge 0 \tag{311}$$

where $\widetilde{s}_i$ and w_i are known positive constants.

Derive an explicit algebraic expression for the robustness of consumer 2 at step 1.

(h) Using the robustness function from part 73g, consider the following two commitment situations, (γ, w_2) and (γ', w_2'), where:

$$\gamma' < \gamma, \quad w_2' > w_2 \tag{312}$$

The 'prime' configuration entails lower commitment by consumer 2, but greater uncertainty in consumer 2's source. Use the robustness function to discuss the values of consumer 2's required consumption, $\overline{c}_2$, for which 2 prefers the 'prime' configuration.

74. **Serial and parallel networks,** (p.279). We will consider the networks shown in fig. 17. Let P_i denote the probability that sub-unit i is functional, and assume that sub-unit failures are statistically independent.

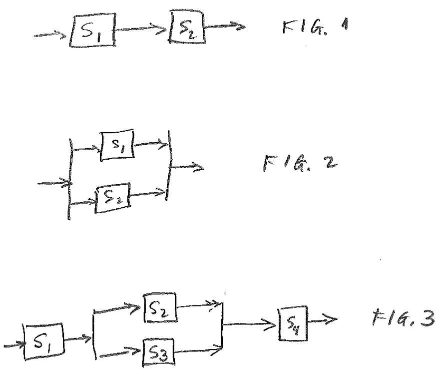

Figure 17: Three networks for problem 74. Fig. 1: serial. Fig. 2: parallel. Fig. 3: hybrid.

(a) Consider the 2-element serial network in part 1 of fig. 17. Derive an explicit expression for the probabilistic reliability of the network: the probability that the network is functional.

(b) Consider the 2-element parallel network in part 2 of fig. 17. Derive an explicit expression for the probabilistic reliability of the network.

(c) Consider the 4-element hybrid network in part 3 of fig. 17. Derive an explicit expression for the probabilistic reliability of the network.

(d) Continue part 74c but assume that the true probabilities, P_i, are uncertain. Let $\widetilde{P}_i$ be an estimate of P_i with error s_i, where the fractional deviation of P_i from $\widetilde{P}_i$, in units of s_i, is unknown. Derive an explicit expression for the inverse of the robustness function if we require that the reliability be no less than $R_{\rm c}$.

75. **Uncertain truss,** (p.281). Consider a truss subject to loads F_1 and F_2 resulting in tension T in one of the bars:

$$T = F_1 \sin\theta - F_2 \tag{313}$$

where θ is known and $0 < \theta < \pi/2$.

(a) We require that the tension not exceed a know critical value, $T_{\rm c}$:

$$T \le T_{\rm c} \tag{314}$$

The loading forces are uncertain as specified by a fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ F : \left| \frac{F_i - \widetilde{F}_i}{s_i} \right| \le h, \ i = 1, 2 \right\}, \quad h \ge 0 \tag{315}$$

where $\widetilde{F}_i$ and s_i are known and positive. Derive an explicit algebraic expression for the robustness.

(b) Continuing part 75a, consider two alternative configurations, where the angles of the alternatives satisfy:

$$0 < \theta^{(1)} < \theta^{(2)} < \frac{\pi}{2} \tag{316}$$

All other parameters are the same for both alternatives. For what values of $T_{\rm c}$ is the first design robust-preferred?

(c) Return to the initial definition of the problem, prior to part 75a, and let F_1 be a random variable with a uniform probability density function on the interval $[-b, \ b]$. The tension is specified by eq.(313). Derive an explicit algebraic expression for the probability of violating eq.(314).

(d) Continuing part 75c, suppose that the value of b is uncertain, as specified by the info-gap model:

$$\mathcal{U}(h) = \left\{ b : \ b > 0, \ \left| \frac{b - \widetilde{b}}{\widetilde{b}} \right| \le h \right\}, \quad h \ge 0 \tag{317}$$

In part 75c we evaluated the probability of violating eq.(314). We require that this probability not exceed the critical value $P_{\rm c}$. Derive an explicit algebraic expression for the robustness. Assume that $0 \le (T_{\rm c} + F_2)/\sin\theta \le \widetilde{b}$.

(e) Now let's generalize the problem and consider a vector, F, of N forces resulting in tension T in a particular bar given by:

$$T = g^T F \tag{318}$$

where g is a known vector. The uncertainty in the loading vector is specified by the following ellipsoid-bound info-gap model:

$$\mathcal{U}(h) = \left\{ F : \ (F - \widetilde{F})^T W (F - \widetilde{F}) \le h^2 \right\}, \quad h \ge 0 \tag{319}$$

where $\widetilde{F}$ and W are known and W is a real, symmetric, positive definite matrix. We require that eq.(314) hold. Derive an explicit algebraic expression for the value of F that maximizes the tension, T, at horizon of uncertainty h.

76. **Uncertain loading** (based on exam, 2014), (p.283). Consider a cantilever beam of length L. A distributed force of constant positive magnitude, F_0 [N/m], but variable angle of application, $\theta(x)$, is applied along the beam in the vertical plane. The bending moment at the base of the beam is:

$$M(\theta) = \int_0^L xF_0 \sin\theta(x)\,\mathrm{d}x \tag{320}$$

We require that the bending moment not exceed a critical value, M_{c}:

$$M \le M_{\mathrm{c}} \tag{321}$$

(a) The angle of application, $\theta(x)$, is uncertain as expressed by the following info-gap model:

$$\mathcal{U}(h) = \left\{ \theta(x) : \left| \frac{\theta(x) - \theta_0}{\theta_0} \right| \le h \right\}, \quad h \ge 0 \tag{322}$$

where θ_0 is known, positive and less than $\pi/2$. Derive an explicit algebraic expression for the inverse of the robustness function of the beam.

(b) Now suppose that the system is altered so that the system model, instead of eq.(320), is:

$$M(F) = \int_0^L xF(x)\,\mathrm{d}x \tag{323}$$

The loading, $F(x)$, is uncertain as expressed by the info-gap model:

$$\mathcal{U}(h) = \left\{ F(x) : \left| \frac{F(x) - \widetilde{F}(x)}{s} \right| \le h \right\}, \quad h \ge 0 \tag{324}$$

where s is known and positive and $\widetilde{F}(x) = ax$ and a is known and positive. The performance requirement is eq.(321). Derive an explicit algebraic expression for the robustness function.

(c) Continuing part 76b, consider two alternative designs specified by (a_1, s_1) and (a_2, s_2) where:

$$0 < a_1 < a_2 \quad \text{and} \quad 0 < s_2 < s_1 \tag{325}$$

Explain the dilemma facing the designer who must choose between these two designs. Which design is preferred, based on the robustness criterion, when the performance requirement is in the interval:

$$\frac{a_1 L^3}{3} \le M_{\mathrm{c}} \le \frac{a_2 L^3}{3} \tag{326}$$

(d) Return to the system model of eq.(320) and suppose that $\theta(x) = \theta_0$ which is known, constant, positive and no greater than $\pi/2$. Also suppose that F_0 is a random variable with the following probability density function:

$$p(F_0) = \frac{2F_0}{F_{\max}^2}, \quad 0 \le F_0 \le F_{\max} \tag{327}$$

and zero otherwise. Derive an explicit algebraic expression for the probability of violating the condition in eq.(321).

(e) The probability of violating the condition in eq.(321) depends on the value of M_{c}. Suppose, unlike problem 76d, that the probability of violating the condition in eq.(321) is:

$$P_{\mathrm{f}} = \mathrm{e}^{-\lambda M_{\mathrm{c}}} \tag{328}$$

where $M_{\rm c}$ can only take non-negative values. We can specify a critical value, $\widetilde{M}_{\rm c}$. However, the actual critical value that we should use, $M_{\rm c}$, is uncertain, as expressed by this info-gap model:

$$\mathcal{U}(h) = \left\{ M_{\rm c} : \ M_{\rm c} \ge 0, \ \left| \frac{M_{\rm c} - \widetilde{M}_{\rm c}}{\widetilde{M}_{\rm c}} \right| \le h \right\}, \quad h \ge 0 \tag{329}$$

We require that $P_{\rm f}$ not exceed the value, $P_{\rm fc}$:

$$P_{\rm f} \le P_{\rm fc} \tag{330}$$

Derive an explicit algebraic expression for the robustness function.

(f) Return to the system model of eq.(320) and suppose that $\theta(x) = \theta_0$ which is known, constant, positive and no greater than $\pi/2$. Also suppose that F_0 is a random variable with a normal distribution with mean μ and unknown variance. The value of F_0 has been measured in a random sample of size N. The mean and variance of this sample are $\overline{x}_{\rm obs}$ and $s_{\rm obs}^2$. There is dispute about the value of μ. One person claims that it equals the known value μ_0, while another person claims that it is greater. The corresponding hypotheses are:

$$H_0 : \qquad \mu = \mu_0 \tag{331}$$

$$H_1 : \qquad \mu > \mu_0 \tag{332}$$

Suppose that:

$$\frac{\overline{x}_{\rm obs} - \mu_0}{s_{\rm obs}} = 1 \tag{333}$$

What is the smallest sample size, N, at which you would reject H_0 at level of confidence equal to 0.005?

77. **Uncertain elastic system** (based on exam, 2014), (p.285). Displacement x is related to force f as:

$$x = cf \tag{334}$$

where c is a positive constant.

(a) We require that the displacement, x, be no less than the critical value x_{c}:

$$x \geq x_{\mathrm{c}} \tag{335}$$

However, the force, f, is uncertain according to the info-gap model:

$$\mathcal{U}(h) = \left\{ f: \ f \geq 0, \ \left| \frac{f - \widetilde{f}}{s} \right| \leq h \right\}, \quad h \geq 0 \tag{336}$$

where $\widetilde{f}$ and s are known positive constants. Derive an explicit algebraic expression for the robustness function.

(b) Now suppose, unlike part 77a, that f is a random variable with an exponential distribution whose probability density function is $p(f) = \lambda \mathrm{e}^{-\lambda f}$ for $f \geq 0$. Derive an explicit algebraic expression for the probability of violating the condition in eq.(335).

(c) Continuing part 77b, suppose that c is uncertain as described by this info-gap model:

$$\mathcal{U}(h) = \left\{ c: \ c \geq 0, \ \left| \frac{c - \widetilde{c}}{w} \right| \leq h \right\}, \quad h \geq 0 \tag{337}$$

where $\widetilde{c}$ and w are known positive constants. We require that the probability of violating the condition in eq.(335) be no greater than the value P_{c}. Suppose that $x_{\mathrm{c}} \geq 0$. Derive an explicit algebraic expression for the robustness function.

(d) We now modify eq.(334) to introduce a non-linearity:

$$x = cf^2 \tag{338}$$

Both the coefficient c and the force f are uncertain:

$$\mathcal{U}(h) = \left\{ f, u: \ f \geq 0, \ \left| \frac{f - \widetilde{f}}{s} \right| \leq h, \ -h \leq c \leq 0 \right\}, \quad h \geq 0 \tag{339}$$

where $\widetilde{f}$ and s are known positive constants. Derive an explicit algebraic expression for the smallest (most negative) value of x at horizon of uncertainty h.

(e) We now generalize part 77a as follows. The displacement, x, is a function of a vector of forces, f_i, with a vector of real (positive or negative) coefficients, c_i:

$$x = \sum_{i=1}^{N} c_i f_i \tag{340}$$

The force vector is uncertain according to the info-gap model:

$$\mathcal{U}(h) = \left\{ f: \ \left| \frac{f_i - \widetilde{f}_i}{s_i} \right| \leq h, \ i = 1, \ldots, N \right\}, \quad h \geq 0 \tag{341}$$

where $\widetilde{f}_i$ and s_i are known positive constants. We require that eq.(335) hold. Derive an explicit algebraic expression for the robustness.

(f) Eq.(334) describes the force-displacement relationship. Measurements of the displacement are corrupted by noise which is statistically independent but identically distributed from one measurement to the next. The displacement was measured many times. However, each displacement was recorded only as "small", "medium" or "large". Actual values of displacement were not recorded. The number of "small", "medium" or "large" measurements are n_1, n_2 and n_3 respectively. The total number of measurements is N. Under ordinary conditions, the probability is p_i to obtain a "small", "medium" or "large" displacement, for $i = 1, 2, 3$. These probabilities change if failure occurs. Define the statistic:

$$\chi^2 = \sum_{i=1}^{3} \frac{(n_i - Np_i)^2}{Np_i} \tag{342}$$

i. Under "ordinary" conditions and with many measurements, what is the probability that χ^2 will exceed the value 7.38?
ii. The null hypothesis, H_0, is that the system is "ordinary", that is, no failure has occurred. The alternative hypothesis, H_1, is that failure has occurred. Many observations have been made and the χ^2 statistic has been evaluated. Do you reject H_0 at level of significance 0.05 if $\chi^2 = 9.21$?
iii. Continuing part 77(f)ii, what is the smallest level of significance at which you would reject H_0?

78. **Uncertain dynamical system** (036057, based on exam, 30.12.2014), (p.287). Consider a dynamical system whose output, $y(t)$, is:

$$y(t) = \int_0^t u(s) f(t-s)\,\mathrm{d}s \tag{343}$$

where the input function, $u(s)$, is uncertain and the system properties are expressed by the function $f(s) = \mathrm{e}^{-\lambda s}$ and λ is positive and known.

(a) Uncertainty in the input function is expressed by a fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ u(s) : \ \left| \frac{u(s) - \widetilde{u}(s)}{\widetilde{u}(s)} \right| \le h \right\}, \quad h \ge 0 \tag{344}$$

where the nominal input is $\widetilde{u}(s) = \mathrm{e}^{-\mu s}$, and μ is a known positive constant different from μ. We require that $y(t) \ge y_{\mathrm{c}}$ at a specified time t. Derive an explicit expression for the robustness function.

(b) Revise problem 78a by changing the info-gap model as follows. The input is a truncated Fourier series: $u(s) = \sum_{j=1}^{N} c_j \cos \frac{j\pi s}{t} = c^T \gamma(s)$ and the vector of Fourier coefficients, c, is uncertain:

$$\mathcal{U}(h) = \left\{ u(s) = c^T\gamma(s) : \ (c - \widetilde{c})^T W (c - \widetilde{c}) \le h^2 \right\}, \quad h \ge 0 \tag{345}$$

where W is a known, real, symmetric, positive definite matrix and $\widetilde{c}$ is a known real vector. Derive the robustness function.

(c) Revise the system dynamics from eq.(343) to:

$$y(t) = \lambda \widetilde{u}(t) \tag{346}$$

where λ is a known positive constant and the nominal input function, $\widetilde{u}(s) = \mathrm{e}^{-\mu s}$, drives the system. The output is measured with a noisy sensor at N different times, $t_1, \ldots, t_N$, and the measured outputs are $y_1, \ldots, y_N$. The actual system response at time t_j is $y(t_j, \widetilde{u}) = \lambda \widetilde{u}(t_j)$ and the squared error of the measurements is:

$$S(\lambda, \widetilde{u}) = \sum_{j=1}^{N} [y_j - y(t_j, \widetilde{u})]^2 \tag{347}$$

Derive an explicit algebraic expression for the least squared error estimate of λ.

(d) Revise the system dynamics from eq.(346) to:

$$y(t) = \lambda u(t) \tag{348}$$

where λ is a known positive constant and $u(t)$ is uncertain according to the info-gap model of eq.(344) with known nominal input function, $\widetilde{u}(s) = \mathrm{e}^{-\mu s}$ where $\mu > 0$. The output is measured with a noisy sensor at N different times, $t_1, \ldots, t_N$, and the measured outputs are $y_1, \ldots, y_N$. The actual system response at time t_j is $y(t_j, u) = \lambda u(t_j)$. The same function, $u(s)$, drives the system at each measurement time, but the form of this function is uncertain. The measurement times are all different so the values of $u(t_1), \ldots, u(t_N)$ may be different. We require that the squared error, $S(\lambda, u)$, not exceed the critical value S_{c}. Derive an explicit algebraic expression for the inverse of the robustness function.

79. **Tube obstructions** (036057, based on exam, 20.1.2015), (p.289). A hidden tube (e.g. blood vessel or subterranean pipeline) has obstructions at various unknown points. We can investigate the tube in an interval $[y-r,\ y+r]$. If we find the obstruction, then the resulting value is:

$$V(b) = \int_{y-r}^{y+r} b(x)\,\mathrm{d}x \tag{349}$$

where $b(x)$ is a real-valued function: $b(x) > 0$ means benefit, while $b(x) < 0$ means loss. The function $b(x)$ is uncertain, with estimate $\widetilde{b}(x)$. The sign of $\widetilde{b}(x)$ varies along its length, representing estimated benefits and losses. Uncertainty is represented by this info-gap model:

$$\mathcal{U}(h) = \left\{ b(x) :\ \left| \frac{b(x) - \widetilde{b}(x)}{\widetilde{b}(x)} \right| \le h \right\}, \quad h \ge 0 \tag{350}$$

(a) We require that the value of the benefit be no less than V_{c}. Derive an explicit algebraic expression for the robustness function.

(b) Continue part (79a) with a different info-gap model. The benefit function is a polynomial in x:

$$b(x) = \sum_{i=1}^{N} a_i x^i = a^T \xi \tag{351}$$

where a is a vector of coefficients and $\xi = (x^1,\, x^2,\, \ldots, x^N)^T$. The coefficients are uncertain and the info-gap model is:

$$\mathcal{U}(h) = \left\{ b(x) = a^T \xi :\ (a-\widetilde{a})^T W (a-\widetilde{a}) \le h^2 \right\}, \quad h \ge 0 \tag{352}$$

where $\widetilde{a}$ and W are known and W is a real, symmetric, positive definite matrix. Derive an explicit algebraic expression for the robustness function.

(c) We now modify the problem. Investigation of the tube in the interval $[y-r,\ y+r]$ has probability p for finding an obstruction. The benefit from finding an obstruction has value b. However, if we don't find an obstruction, the damage done causes loss with value $-d$ (where d is a positive number). What is the average value of an investigation?

(d) Continue part (79c) but now let b, d and p be uncertain according to the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ b, d, p :\ \left| \frac{b-\widetilde{b}}{\widetilde{b}} \right| \le h,\ b \ge 0,\ \left| \frac{d-\widetilde{d}}{\widetilde{d}} \right| \le h,\ \ d \ge 0,\ \left| \frac{p-\widetilde{p}}{s_p} \right| \le h,\ p \in [0,1] \right\}, \quad h \ge 0 \tag{353}$$

where $\widetilde{b}$, $\widetilde{d}$, $\widetilde{p}$ and s_p are known positive numbers. We require that the average value of the investigation be no less than the value V_{c}. Derive an explicit algebraic expression for the inverse of the robustness function.

(e) Return to part (79a) but with a different info-gap model:

$$\mathcal{U}(h) = \left\{ b(x) :\ \left| b(x) - \widetilde{b}_0 \right| \le h w(x) \right\}, \quad h \ge 0 \tag{354}$$

where $\widetilde{b}_0$ is a known positive constant and:

$$w(x) = \begin{cases} w_0 & \text{if } y-r \le x \le x_0 \\ 0 & \text{if } x_0 < x \le y+r \end{cases} \tag{355}$$

where w_0 and x_0 are known positive constants and $y-r \le x_0 \le y+r$. Derive an explicit algebraic expression for the robustness, and explain its dependence on the value of x_0.

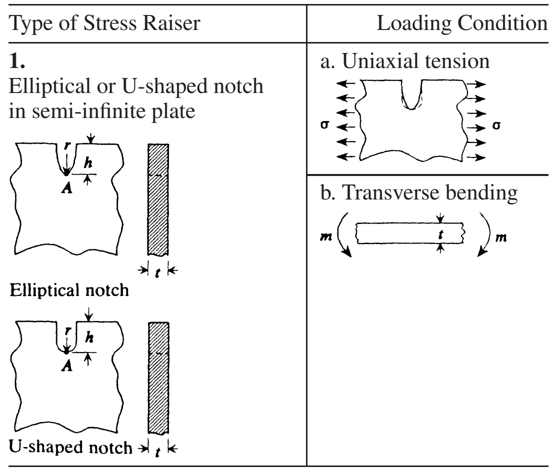

Figure 18: Stress concentration geometry for problem 80.

80. **Stress concentration factor** (035018, based on exam, 31.5.2015), (p.291). Consider a small notch in the surface of a large solid under uniaxial tension σ, as in fig. 18. The depth of the notch is d (denoted h in the figure) and the radius of curvature of the tip of the notch is r. The stress concentration factor (SCF), K, is the ratio of the maximal stress at the tip of the notch to the stress, σ, far from the notch. A theoretically based empirical relation is:

$$K = a + b\sqrt{\frac{d}{r}} \tag{356}$$

where a and b are positive empirical coefficients.

(a) The radius of curvature is estimated to be $\widetilde{r}$ with error s_r. The fractional error is unknown and uncertainty is represented with this info-gap model:

$$\mathcal{U}(h) = \left\{ r :\ r \geq 0,\ \left| \frac{r - \widetilde{r}}{s_r} \right| \leq h \right\}, \quad h \geq 0 \tag{357}$$

We require that the SCF be no greater than the critical value $K_{\rm c}$:

$$K \leq K_{\rm c} \tag{358}$$

Derive an explicit algebraic expression for the robustness.

(b) Define $c = (a, b)^T$ as the vector of coefficients in eq.(356) and define the vector $g = (1, \sqrt{d/r})^T$. Thus $K = c^T g$. The coefficients are uncertain as represented by an ellipsoid-bound info-gap model:

$$\mathcal{U}(h) = \left\{ c :\ (c - \widetilde{c})^T W (c - \widetilde{c}) \leq h^2 \right\}, \quad h \geq 0 \tag{359}$$

where $\widetilde{c}$ is a known vector and W is a known, real, positive definite, symmetric matrix. The performance requirement is eq.(358). Derive an explicit algebraic expression for the robustness.

(c) Assume that the radius, r, in eq.(356) is a random variable with a normal distribution with mean μ and variance s^2. The probability of failure, which we denote $P_{\rm f}$, is the probability of violating eq.(358). Derive an explicit algebraic expression for $P_{\rm f}$.

This figure is from http://www.ewp.rpi.edu/hartford/˜ ernesto/Su2012/EP/MaterialsforStudents/Aiello/Roark-Ch06.pdf

(d) Continuing part 80c, let the mean, μ, and standard deviation, s, be uncertain according to this fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ \mu, s: \ \left| \frac{\mu - \widetilde{\mu}}{\widetilde{\mu}} \right| \le h, \ s \ge 0, \ \left| \frac{s - \widetilde{s}}{\widetilde{s}} \right| \le h \right\}, \quad h \ge 0 \tag{360}$$

We require that the probability of failure be no greater than the critical value $P_{\rm c}$:

$$P_{\rm f} \le P_{\rm c} \tag{361}$$

Combining eqs.(356) and (358), let us define a "critical radius" in terms of the critical SCF: $r_{\rm c} = d\left(\frac{b}{K_{\rm c}-a}\right)^2$. Assume that the estimated mean, $\widetilde{\mu}$, exceeds $r_{\rm c}$. Derive an explicit algebraic expression for the inverse of the robustness function.

(e) Consider the SCF at small radii, for which eq.(356) can be approximated as:

$$K = b\sqrt{\frac{d}{r}} \tag{362}$$

The estimated values of the coefficient b for two different materials are $\widetilde{b}_1$ and $\widetilde{b}_2$ where $\widetilde{b}_1 > \widetilde{b}_2$. However, the fractional errors of the true values are uncertain, as represented by this info-gap model:

$$\mathcal{U}(h) = \left\{ b_1, b_2: \ b_i \ge 0, \ \left| \frac{b_i - \widetilde{b}_i}{\widetilde{b}_i} \right| \le h, \ i = 1, 2 \right\}, \quad h \ge 0 \tag{363}$$

For each material, derive an explicit algebraic expression for the robustness of satisfying the performance requirement in eq.(358). For what range of $K_{\rm c}$ values do you robustly prefer option 2?

(f) Consider the SCF at all radii, for which we must use eq.(356). The estimated values of the coefficients a and b for two different materials are $(\widetilde{a}_1, \widetilde{b}_1)$ and $(\widetilde{a}_2, \widetilde{b}_2)$ where $\widetilde{a}_1 < \widetilde{a}_2$ and $\widetilde{b}_1 > \widetilde{b}_2$. Let $K_i(r)$ denote the SCF for material i as a function of the radius r. From eq.(356) we see that $K_1(r) > K_2(r)$ at small radii, and $K_1(r) < K_2(r)$ at large radii. Let $r_\times$ denote the radius at which the SCF curves of the two materials cross.

The fractional errors of the true values of a_i and b_i are uncertain, as represented by this info-gap model:

$$\mathcal{U}(h) = \left\{ (\widetilde{a}_i, \widetilde{b}_i): \ a_i \ge 0, \ \left| \frac{a_i - \widetilde{a}_i}{\widetilde{a}_i} \right| \le h, \ b_i \ge 0, \ \left| \frac{b_i - \widetilde{b}_i}{\widetilde{b}_i} \right| \le h, \ i = 1, 2 \right\}, \quad h \ge 0 \tag{364}$$

For each material, derive an explicit algebraic expression for the robustness of satisfying the performance requirement in eq.(358). For what range of $K_{\rm c}$ values do you robustly prefer material 1?

Figure 19: Two sensors for locating target in problem 81.

81. **Triangulation** (035018, based on exam, 8.7.2015), (p.293). Consider two sensors on the x axis that triangulate the position, (x_t, y_t), of a target in the x-y plane, as in fig. 19. The distance between the two sensors is D and their angular measurements are θ_1 and θ_2. The triangulated coordinates are:

$$x_t(D) = D\frac{\tan\theta_2}{\tan\theta_1 + \tan\theta_2} = \rho D \tag{365}$$

$$y_t(D) = D\frac{\tan\theta_1 \tan\theta_2}{\tan\theta_1 + \tan\theta_2} \tag{366}$$

where ρ is defined in eq.(365). Assume the angles θ_1 and θ_2 are between 0 and $\pi/2$ radians.

(a) The estimated distance between the sensors is $\widetilde{D}$, but the true value, D, is uncertain as represented by this info-gap model:

$$\mathcal{U}(h) = \left\{ D : D \ge 0,\ \left| \frac{D - \widetilde{D}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{367}$$

where s is a known positive constant. We require that the estimated x coordinate, $x_t(\widetilde{D})$, differ from the correct value, $x_t(D)$, by no more than ε:

$$\left| x_t(\widetilde{D}) - x_t(D) \right| \le \varepsilon \tag{368}$$

Derive an explicit algebraic expression for the robustness function.

(b) Let $D_{\rm e}$ be an alternative positive estimate of the distance between the sensors, where $D_{\rm e} \ge \widetilde{D}$. We will use the observed angles and the value of $D_{\rm e}$ to estimate x_t as $x_t(D_{\rm e})$ with eq.(365). The true value of x_t could be evaluated as $x_t(D)$ from eq.(365), but D is uncertain as expressed by eq.(367). We require an estimation error no greater than ε:

$$|x_t(D_{\rm e}) - x_t(D)| \le \varepsilon \tag{369}$$

Derive an explicit algebraic expression for the robustness function. For what values of ε do we robust-prefer to estimate with $D_{\rm e} > \widetilde{D}$, and for what values of ε do we prefer $D_{\rm e} = \widetilde{D}$?

(c) The measurement team has spotted a target and reported triangulation angles θ_1 and θ_2. They think these measurements were made when the distance between the sensors was $\widetilde{D}$, a known value. They thus recommend evaluating x_t as $x_t(\widetilde{D})$ with eq.(365). Actually, however, D was a random variable with an exponential distribution:

$$p(D) = \lambda {\rm e}^{-\lambda D}, \quad D \ge 0 \tag{370}$$

where $\lambda = 1/\widetilde{D}$. Derive an explicit algebraic expression for the probability that the true distance, $x_t(D)$, exceeds the estimate, $x_t(\widetilde{D})$, by at least ε. Denote this probably $P_{\rm f}$.

(d) Continue from part 81c and suppose that the exponential coefficient, λ, is uncertain according to this info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda : \ \lambda > 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{371}$$

where $\widetilde{\lambda} = 1/\widetilde{D}$ is known. We require that $P_{\rm f}$ be no greater than the critical value $P_{\rm c}$. Derive an explicit algebraic expression for the robustness.

(e) We now have n pairs of sensors on the x axis, each pair like those in part 81a, all triangulating on the same target. The estimated distance between the sensors of the jth pair is the known value $\widetilde{D}_j$, while the true distance at the time of measurement, D_j, is uncertain according to this info-gap model:

$$\mathcal{U}(h) = \left\{ D = (D_1, \ldots, D_n) : D_j \ge 0, \ \left| \frac{D_j - \widetilde{D}_j}{s_j} \right| \le h, \ j = 1, \ldots, n \right\}, \quad h \ge 0 \tag{372}$$

where each s_j is a known positive constant. The measurement team for each pair of sensors has reported triangulation angles on the target, and has reported its estimate of x_t as $x_t(\widetilde{D}_j)$ using eq.(365). The true value would have been obtained by each team as $x_t(D_j)$, using eq.(365), but the true value D_j is unknown. We calculate the average of the n reported values as:

$$\overline{x}(\widetilde{D}) = \frac{1}{n} \sum_{j=1}^{n} x_t(\widetilde{D}_j) \tag{373}$$

where $\widetilde{D}$ is the vector of estimated distances. We require that the estimate deviate from the true value, x_t, by no more than ε:

$$\left| \overline{x}(\widetilde{D}) - x_t \right| \le \varepsilon \tag{374}$$

Derive an explicit algebraic expression for the robustness function.

(f) We have made 5 statistically independent measurements of x_t where each measurement is corrupted by identically distributed normal noise. The measurements are 7.4, 7.0, 7.2, 6.9 and 7.6. One team claims that the true value of x_t is $X = 6.9$, and another team claims that this is false. Stating these claims as statistical hypotheses:

$$H_0 : \quad x_t = 6.9 \tag{375}$$

$$H_1 : \quad x_t \ne 6.9 \tag{376}$$

Do you accept or reject H_0 at the 0.01 level of significance? Explain.

(g) We have a single target located at (x_t, y_t), and n pairs of sensors on the x axis used to measure x_t. These measurements are statistically independent and normally distributed around the true value of x_t, which we denote μ. Thus the measurement from the jth pair of sensors is:

$$x_{t,j} \sim \mathcal{N}(\mu, \sigma^2) \tag{377}$$

where σ^2 is the known variance of each estimate. Let $\overline{x}$ denote the mean of the n measurements. For any given value of a constant, δ, derive an explicit algebraic expression for the probability that $\overline{x}$ exceeds $\mu + \delta$. Denote this probability $P_{\rm f}$.

(h) Continue from part 81g and let $\delta = \sigma$. Find the smallest integer value of n (smallest sample size) so that $P_{\rm f} \le 0.05$.

82. **New zoonotic disease** (p.296). A new and possibly zoonotic disease is moving up an estuary from the sea. The distance of the disease front from the coast seems to be progressing as a function of time according to the relation:

$$x(t) = a\sqrt{t} \tag{378}$$

The coefficient a has been roughly estimated as $\widetilde{a}$, but the true value of a could deviate from $\widetilde{a}$ by a fraction f or more, though a must be non-negative. We represent the uncertainty in the value of a with this fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ a: \ a \ge 0, \ \left| \frac{a - \widetilde{a}}{f\widetilde{a}} \right| \le h \right\}, \quad h \ge 0 \tag{379}$$

(a) The disease has not reached human settlement, and the town closest to the sea is a distance x_{c} from the coast. The public health department wants to know how much time, T, they have until the disease reaches the town. Derive the robustness function and use it to discuss a confident choice of T.

(b) We repeat part (82a) with a different info-gap model. Eq.(378), with the estimated coefficient $\widetilde{a}$, is our best estimate of the rate of progress of the disease front. We denote that function as $\widetilde{x}(t)$. However, this functional form may be wrong. We do believe that the front does not regress, but the shape of the true function, $x(t)$, may deviate fractionally from $\widetilde{x}(t)$ by an unknown amount. We represent this with the following info-gap model:

$$\mathcal{U}(h) = \left\{ x(t): \ \frac{\mathrm{d}x(t)}{\mathrm{d}t} \ge 0, \ \left| \frac{x(t) - \widetilde{x}(t)}{\widetilde{x}(t)} \right| \le h \right\}, \quad h \ge 0 \tag{380}$$

Derive the robustness function and discuss the choice of a value of T.

(c) We modify the problem as follows. Eq.(378) is the correct functional form, but the value of a is uncertain and represented as an exponential random variable whose probability density function is:

$$p(a) = \lambda \mathrm{e}^{-\lambda a}, \quad a \ge 0 \tag{381}$$

Derive an expression for the probability that the disease front will reach the town no later than time T. Call this probability $P_{\mathrm{f}}(T)$, the probability of failure.

(d) Continuing part 82c, we consider the exponential coefficient λ as being info-gap-uncertain. Its estimated value is $\widetilde{\lambda}$, which could err by as much as s or more. We use the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda: \ \lambda \ge 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{382}$$

We require that the probability of failure be no greater than the critical value P_{c}. Derive the robustness function and discuss the choice of a value of T.

83. **Surveillance and remediation** (p.299). You have a fixed budget, B, that you can divide between N tasks: surveillance, remediation, etc. The allocations, $b_1, \ldots, b_N$, are non-negative and use up the entire budget.

(a) The benefit resulting from allocating budget b_i to task i is:

$$r_i(b_i) = \lambda_i b_i, \quad i = 1, \ldots, N \tag{383}$$

where λ_i is uncertain according to a fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda : \ \lambda_i \geq 0, \ \left| \frac{\lambda_i - \widetilde{\lambda}_i}{\widetilde{\lambda}_i} \right| \leq h, \ i = 1, \ldots, N \right\}, \quad h \geq 0 \tag{384}$$

The $\widetilde{\lambda}_i$ are known and positive. We require that the total benefit be no less than the positive value r_c. Derive the robustness function and find the robustness-maximizing allocation.

(b) Now consider a different fractional-error info-gap model, representing different uncertainties for the various tasks. Instead of eq.(384) we have:

$$\mathcal{U}(h) = \left\{ \lambda : \ \lambda_i \geq 0, \ \left| \frac{\lambda_i - \widetilde{\lambda}_i}{s_i} \right| \leq h, \ i = 1, \ldots, N \right\}, \quad h \geq 0 \tag{385}$$

where each $\widetilde{\lambda}_i$ and s_i is known and positive. Derive the robustness function.

(c) Continue part 83b. Let $s^{(1)}$ denote the vector of uncertainty weights, s_i in eq.(385). An expert consultant proposes a different vector of uncertainty weights, $s^{(2)}$. Some of the elements of $s^{(2)}$ are larger, and some smaller, than the corresponding elements of $s^{(1)}$. The consultant explains that some of our error-estimates were overly optimistic, and some overly pessimistic (which is which?). The consultant claims that $s^{(2)}$ reflects better knowledge. Does this better knowledge entail greater robustness to the residual uncertainty?

(d) Now consider a different info-gap model, in which the uncertain coefficients λ_i are correlated. Specifically, consider the ellipsoidal-bound info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda : \ (\lambda - \widetilde{\lambda})^T S (\lambda - \widetilde{\lambda}) \leq h^2 \right\}, \quad h \geq 0 \tag{386}$$

where S is a known, real, positive definite, symmetric matrix. Derive the robustness function and find the robustness-maximizing budget allocation.

84. **Managerial attention,** (p.301). A manager must allocate time between two tasks. The time allocated to the ith task is t_i, and the total time available is T. The reward from task i resulting from this allocation is:

$$r_i(t_i) = \lambda_i t_i \tag{387}$$

The total reward, $r(t)$, is the sum of the two single-task rewards. We require that the total reward exceed $r_{\rm c}$.

(a) The coefficients λ_i are uncertain, as expressed by the following info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda_i : \ \lambda_i > 0, \ \left| \frac{\lambda_i - \widetilde{\lambda}_i}{\widetilde{\lambda}_i} \right| \le h, \ i = 1, \, 2 \right\}, \quad h \ge 0 \tag{388}$$

where $\widetilde{\lambda}_i$ is known and positive and $\widetilde{\lambda}_1 < \widetilde{\lambda}_2$. Derive an expression for the robustness and find the putative-optimal and robust-optimal time allocations.

(b) The coefficients λ_i are uncertain, as expressed by the following info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda_i : \ \lambda_i > 0, \ \left| \frac{\lambda_i - \widetilde{\lambda}_i}{s_i} \right| \le h, \ i = 1, \, 2 \right\}, \quad h \ge 0 \tag{389}$$

where s_i and $\widetilde{\lambda}_i$ are known and positive and where:

$$\widetilde{\lambda}_1 < \widetilde{\lambda}_2 \quad \text{and} \quad \frac{\widetilde{\lambda}_1}{s_1} > \frac{\widetilde{\lambda}_2}{s_2} \tag{390}$$

Derive an expression for the robustness and find the putative-optimal and robust-optimal time allocations.

See related problem: #18 in \lectures\p_mgt\p-mgt-hw02.tex.

85. **Search and destroy,** (p. 302). Consider allocation of resources to search for and destroy an agent (invasive species, enemy alien, terrorist, etc.). The region of responsibility is divided into n sites where, for site i:

$p_i =$ probability that the agent is present.

$x_i =$ surveillance effort, in units of cost.

$\lambda_i =$ surveillance efficiency.

$\mathrm{e}^{-\lambda_i x_i} =$ probability of failing to detect the agent if it is present.

$C_i^{\mathrm{D}} =$ expected cost of incursion management if the agent is detected.

$C_i^{\mathrm{U}} =$ expected cost if the agent is present but undetected (and hence does damage). We assume

that $C_i^{\mathrm{U}} > C_i^{\mathrm{D}}$.

$T_i(x_i) =$ expected combined surveillance and incursion management cost, where:

$$T_i(x_i) = x_i + \underbrace{\left[\left(1 - \mathrm{e}^{-\lambda_i x_i}\right) C_i^{\mathrm{D}} + \mathrm{e}^{-\lambda_i x_i} C_i^{\mathrm{U}}\right]}_{\gamma_i(x_i)} \; p_i \tag{391}$$

This defines γ_i which is positive because $C_i^{\mathrm{U}} > C_i^{\mathrm{D}}$.

$B =$ total budget.

$n =$ number of sites.

The probability of presence, p_i, is estimated to be $\widetilde{p}_i$, which is small, typically between 0 and 0.09. The estimated error of $\widetilde{p}_i$ is s_i, but the true probability, p_i, may differ from $\widetilde{p}_i$ by more. Use a fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ p: \; 0 \le p_i \le 1, \; \left| \frac{p_i - \widetilde{p}_i}{s_i} \right| \le h, \; i = 1, \ldots, n \right\}, \quad h \ge 0 \tag{392}$$

(a) First consider only a single site, i. We must choose the surveillance effort, x_i, and we want the combined cost, $T_i(x_i)$, to be no greater than T_{c}. Derive the robustness function. Will the putative-optimal surveillance effort be the most robust effort for all values of T_{c}?

(b) Now consider all n sites, and the budget constraint:

$$B = \sum_{i=1}^{n} x_i \tag{393}$$

The total combined expected cost considering all n sites is:

$$T(x) = \sum_{i=1}^{n} T_i(x_i) = \sum_{i=1}^{n} [x_i + p_i \gamma_i(x) i)] \tag{394}$$

Now we require that the total cost not exceed the critical value, T_{c}. Derive an expression for the inverse of the robustness function for a given allocation, x.

Based on models developed in: Cindy E. Hauser and Michael A. McCarthy, 2009, Streamlining 'search and destroy': Cost-effective surveillance for invasive species management, *Ecology Letters,* 12: 683–692.

86. **Regression with asymmetric uncertainty,** (035018 exam, 21.10.2015) (p.305) Two measurements, $\overline{y}_1$ and $\overline{y}_2$, have been made at successive time steps. It is suspected that $\overline{y}_1$ is an over estimate of the true value, y_1, by as much as s_1 or more. Furthermore, it is suspected that $\overline{y}_2$ deviates from the true value, y_2, either above or below, by s_2 or more. We will represent this uncertainty with this info-gap model:

$$\mathcal{U}(h) = \left\{ y_1, y_2 :\ 0 \le \frac{\overline{y}_1 - y_1}{s_1} \le h,\ \left| \frac{y_2 - \overline{y}_2}{s_2} \right| \le h \right\}, \quad h \ge 0 \tag{395}$$

We wish to choose a linear regression of the form:

$$y_i^{\mathrm{r}} = ci + b,\ i = 1,\, 2,\, \ldots \tag{396}$$

The squared error, with respect to measurements $y = (y_1, y_2)$, of a regression with coefficients $q = (c, b)$ is:

$$S(q, y) = \sum_{i=1}^{2} (y_i - y_i^{\mathrm{r}})^2 \tag{397}$$

If we knew that the measurements $\overline{y} = (\overline{y}_1, \overline{y}_2)$ were reliable we might choose the coefficients q to minimize $S(q, \overline{y})$. However, given the uncertainty in the measurements, and especially the asymmetric uncertainty in y_1, we wish to choose the coefficients q so that the regression reflects the measurements, $\overline{y}$, as well as the information about the uncertainty in these measurements. That is, we want the squared error to be small for a wide range of realizations of y as represented by the info-gap model. Consequently our performance requirement is:

$$S(q, y) \le S_{\mathrm{c}} \tag{398}$$

(a) Let $\widetilde{q} = (\widetilde{c}, \widetilde{b})$ denote the regression coefficients for which the regression **intersects** both measurements, $\overline{y}$. Derive the robustness function for these coefficients.

(b) Let $q = (c, b)$ denote the regression coefficients for a regression that **intersects** measurement $\overline{y}_2$ and that falls **below** the value of $\overline{y}_1$. Derive the inverse of the robustness function for these coefficients.

(c) Let $q = (c, b)$ denote the regression coefficients for a regression that **intersects** measurement $\overline{y}_2$ and that falls **above** the value of $\overline{y}_1$. Derive the inverse of the robustness function for these coefficients.

(d) If y_i^{r} is the correct regression then a prediction, x, that is based on the regression, is normal with mean 0 and standard deviation 1.5. We have observed 6 statistically independent values of x, all evaluated with the same y_i^{r} and for data from the same situation (so if this y_i^{r} is correct for one then it is correct for all). These observed x values fall in 4 different ranges:

$n_1 = 1$	value in the range	$[-\infty,\ -2)$
$n_2 = 3$	values in the range	$[-2,\ 0)$
$n_3 = 2$	values in the range	$[0,\ 2)$
$n_4 = 0$	values in the range	$[2,\ \infty)$

Consider the following two hypotheses:

$$H_0 : \quad y_i^{\mathrm{r}} \text{ is the correct regression} \tag{399}$$

$$H_1 : \quad y_i^{\mathrm{r}} \text{ is not the correct regression} \tag{400}$$

Given the observations, do you accept H_0 at 0.05 level of significance?

(e) We now know that y is distributed uniformly in the interval $[x,\ \overline{y}]$. Find the value, denoted $y_\alpha(x)$, such that:

$$\mathrm{Prob}[y \le y_\alpha(x)] = \alpha \tag{401}$$

(f) Continue part 86e but consider uncertainty in the value of x. We think the correct value is $\widetilde{x}$, but this could err by as much as w or more. Represent this uncertainty with this info-gap model:

$$\mathcal{U}(h) = \left\{ x : \ \left| \frac{x - \widetilde{x}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{402}$$

We require that $y_\alpha(\widetilde{x})$ over-estimate the true value, $y_\alpha(x)$, by no more than ε:

$$y_\alpha(\widetilde{x}) - y_\alpha(x) \le \varepsilon \tag{403}$$

Derive an explicit algebraic expression for the robustness function.

87. **Quantiles with asymmetric uncertainty,** (p.308) x is a non-negative random variable with probability density function (pdf) $p(x)$. The system we are designing will fail if x is too large. We want to know the largest value of x for which the probability of not exceeding this value is $1-\alpha$. This value is called the $(1-\alpha)$ quantile of x, denoted q_α, and defined in the relation:

$$1-\alpha = \int_0^{q_\alpha} p(x)\,\mathrm{d}x \tag{404}$$

(a) Derive an explicit algebraic expression for the $(1-\alpha)$ quantile of x using the exponential distribution:

$$\widetilde{p}(x) = \widetilde{\lambda}\mathrm{e}^{-\widetilde{\lambda}x} \tag{405}$$

(b) Now suppose that the true pdf of x, denoted $p(x)$, is exponential but the coefficient of the distribution, λ, is uncertain. The best available estimate is $\widetilde{\lambda}$ (which is positive) but we suspect that this is an under estimate. We represent the uncertainty in the pdf of x with this info-gap model:

$$\mathcal{U}(h) = \left\{ p(x) = \lambda\mathrm{e}^{-\lambda x} :\ 0 \le \frac{\lambda-\widetilde{\lambda}}{s} \le h \right\}, \quad h \ge 0 \tag{406}$$

where s is a known positive constant. We will estimate the $(1-\alpha)$ quantile using $\widetilde{p}(x)$ in eq.(405), but this will be an over estimate (explain why):

$$0 \le q_\alpha(p) \le q_\alpha(\widetilde{p}) \tag{407}$$

We require that this over estimate not err by more than ε:

$$q_\alpha(\widetilde{p}) - q_\alpha(p) \le \varepsilon \tag{408}$$

Derive an explicit algebraic expression for the robustness if we estimate the quantile as $q_\alpha(\widetilde{p})$.

(c) We continue with the info-gap model of eq.(406) but we estimate the quantile with an exponential distribution whose coefficient, λ_e, is greater than $\widetilde{\lambda}$. For convenience we will denote quantiles according to the exponential coefficient, so our estimate of the quantile is $q_\alpha(\lambda_\mathrm{e})$ and we require that the absolute error of this estimate not exceed ε:

$$|q_\alpha(\lambda_\mathrm{e}) - q_\alpha(\lambda)| \le \varepsilon \tag{409}$$

Derive an algebraic expression for the inverse of the robustness function. Explore the crossing of these robustness curves with the robustness curve of part 87b.

88. **Evaluating a complex system with sub-systems of uncertain importance** (p.309).

We consider the design of a complex system with sub-systems and sub-sub-systems. We evaluate the overall system with a quadratic function expressing the importance of the sub- and sub-sub-systems. This evaluation is uncertain, so the design is uncertain. We evaluate the robustness to this uncertainty, as the basis for design decisions.

(a) Consider N different sub-systems, where each sub-system has J sub-sub-systems. Let q_{nj} denote the quantity of resources devoted to sub-sub-system j in sub-system n. Q is the $N \times J$ matrix of quantities q_{nj}. The overall effectiveness of the system is evaluated as:

$$E = \sum_{n=1}^{N} v_n \sum_{j=1}^{J} w_j q_{nj} = v^T Q w \tag{410}$$

where $v \in \Re^N$ is the vector of "values" of the sub-systems, and $w \in \Re^J$ is the vector of "worths" of the sub-sub-systems. We would like to choose the quantities, Q, so that the effectiveness if large. However, the values are uncertain according to this info-gap model:

$$\mathcal{U}(h) = \left\{ v : \ (v - \widetilde{v})^T A (v - \widetilde{v}) \le h^2 \right\}, \quad h \ge 0 \tag{411}$$

where A is a known real, symmetric, positive definite matrix and $\widetilde{v}$ is a known vector. We require that the effectiveness be no less than the critical value, $E_{\rm c}$. Derive an explicit algebraic expression for the robustness.

(b) We continue part 88a where we now consider uncertainty in both the values and the worths:

$$\mathcal{U}(h) = \left\{ v : \ (v - \widetilde{v})^T A (v - \widetilde{v}) \le h^2, \ (w - \widetilde{w})^T B (w - \widetilde{w}) \le h^2 \right\}, \quad h \ge 0 \tag{412}$$

where we assume that the matrices A and B are defined so that both quadratic forms are dimensionless. Derive an explicit algebraic expression for the robustness.

(c) We now modify part 88a so that the sub-sub-system worths take different values in each sub-system. Thus, w_{jn} is the worth of sub-sub-system j in sub-system n. W is the $J \times N$ matrix of worths. The total effectiveness of the system is:

$$E = \sum_{n=1}^{N} v_n \sum_{j=1}^{J} q_{nj} w_{nj} \tag{413}$$

The values v and worths W are uncertain according to this info-gap model:

$$\mathcal{U}(h) = \left\{ v, W : \ v_n \ge 0, \ \left| \frac{v_n - \widetilde{v}_n}{s_n} \right| \le h, \ \forall\, n. \ w_{nj} \ge 0, \ \left| \frac{w_{nj} - \widetilde{w}_{nj}}{t_{nj}} \right| \le h, \ \forall\, j, n \right\}, \quad h \ge 0 \tag{414}$$

where the s_n's and t_{jn}'s are known and positive. Derive an explicit algebraic expression for the inverse of the robustness.

(d) We now repeat part 88c with the following modified info-gap model:

$$\mathcal{U}(h) = \left\{ v, W, Q : \ \ v_n \ge 0, \ \left| \frac{v_n - \widetilde{v}_n}{s_n} \right| \le h, \ \forall\, n. \ \ w_{nj} \ge 0, \ \left| \frac{w_{nj} - \widetilde{w}_{nj}}{t_{nj}} \right| \le h, \ \forall\, j, n. \right.$$
$$\left. q_{nj} \ge 0, \ \left| \frac{q_{nj} - \widetilde{q}_{nj}}{u_{nj}} \right| \le h, \ \forall\, j, n \ \right\}, \quad h \ge 0 \tag{415}$$

This problem is explored in detail in section 16 of the Lecture Notes on Robustness and Opportuneness.

where the s_n's, t_{jn}'s and u_{nj}'s are known and positive. Uncertainty in v and W reflects uncertainty in assessing the importance of various sub-systems. Uncertainty in Q reflects uncertainty in the actual quantities that would be produced. This production uncertainty is particularly relevant for new technologies whose production may entail unknown development challenges. Derive an explicit algebraic expression for the inverse of the robustness.

(e) We now repeat part 88c with the following modified info-gap model:

$$\mathcal{U}(h) = \left\{ v, W : \left| \frac{v_n - \widetilde{v}_n}{s_n} \right| \le h, \ \forall \, n. \ \left| \frac{w_{nj} - \widetilde{w}_{nj}}{t_{nj}} \right| \le h, \ \forall \, j, n \right\}, \quad h \ge 0 \tag{416}$$

Derive an explicit algebraic expression for the inverse of the robustness.

89. **Uncertain linear elasticity** (based on exam 035018, 22.5.2016) (p.316). Consider a linear elastic system whose stress-strain relation is described by:

$$\varepsilon = \frac{\varepsilon_1}{\sigma_1}\sigma \quad \text{for} \quad 0 \le \sigma \tag{417}$$

The values of ε_1 and σ_1 define the endpoint of the linear-elastic domain in an idealized elasto-plastic model, though we will employ this model for all positive values of stress, σ.

(a) The values of ε_1 and σ_1 are uncertain, as expressed by this info-gap model:

$$\mathcal{U}(h) = \left\{ (\varepsilon_1, \sigma_1) : \ \varepsilon_1 \ge 0, \ \left| \frac{\varepsilon_1 - \widetilde{\varepsilon}_1}{\widetilde{\varepsilon}_1} \right| \le h, \ \sigma_1 \ge 0, \ \left| \frac{\sigma_1 - \widetilde{\sigma}_1}{\widetilde{\sigma}_1} \right| \le h \right\}, \quad h \ge 0 \tag{418}$$

where $\widetilde{\varepsilon}_1$ and $\widetilde{\sigma}_1$ are known and positive. A known positive stress, σ_0, will be applied, and we require that the strain not exceed the value ε_0. Derive an explicit algebraic expression for the robustness function.

(b) Return to the linear elastic model in eq.(417) and suppose the ε_1 is known but σ_1 is a random variable with an exponential distribution:

$$p(\sigma_1) = \lambda \mathrm{e}^{-\lambda \sigma_1}, \quad \sigma_1 \ge 0 \tag{419}$$

As before, we require that the strain not exceed the value ε_0. Derive an explicit algebraic expression for the probability of failure, namely, the probability that the strain exceeds ε_0.

(c) Continuing from part 89b, suppose that the threshold for mechanical failure, ε_0, is uncertain as represented by this info-gap model:

$$\mathcal{U}(h) = \left\{ \varepsilon_0 : \ \varepsilon_0 \ge 0, \ \left| \frac{\varepsilon_0 - \widetilde{\varepsilon}_0}{s} \right| \le h \right\}, \quad h \ge 0 \tag{420}$$

We require that the probability of failure not exceed the critical value P_{c}, which is strictly less than 1. Derive an explicit algebraic expression for the robustness function for this probabilistic requirement.

(d) Return to part 89a and derive an explicit algebraic expression for the opportuneness function, if we aspire to achieve a strain that is at least as small as ε_w, which is strictly greater than zero.

(e) Let us continue with eq.(417) but assume that ε_1 and σ_1 are known. However, the stress σ is the result of a vector of forces, f, acting on the body:

$$\sigma = \psi^T f \tag{421}$$

where ψ is a known vector. The force vector, f, is uncertain, as described by this ellipsoid-bound info-gap model:

$$\mathcal{U}(h) = \left\{ f : \ \left(f - \widetilde{f}\right)^T W^{-1} \left(f - \widetilde{f}\right) \le h^2 \right\}, \quad h \ge 0 \tag{422}$$

where $\widetilde{f}$ and W are known and W is a real, symmetric, positive definite matrix. We require that the strain not exceed the critical value ε_0. Derive an explicit algebraic expression for the robustness function.

(f) ‡ We now modify the stress-strain relation in eq.(417) by delimiting the range of validity of the linear relation:

$$\varepsilon = \frac{\varepsilon_1}{\sigma_1}\sigma \quad \text{for} \quad 0 \le \sigma \le \sigma_1 \tag{423}$$

The values of ε_1 and σ_1 define the endpoint of the linear-elastic domain in an idealized elasto-plastic model. However, the values of ε_1 and σ_1 are uncertain as specified in the info-gap model of eq.(418) which we now denote $\mathcal{U}_1(h)$. Hence, we do not know the upper limit, σ_1, of the domain of applicability of the linear relation. Let us suppose additional information about the stress-strain relation. Specifically, for $\sigma > \sigma_1$, the fractional error of the true strain function, $\varepsilon(\sigma)$, with respect to the linear model in eq.(417), is unknown:

$$\left| \varepsilon(\sigma) - \frac{\varepsilon_1}{\sigma_1}\sigma \right| \le \frac{\varepsilon_1}{\sigma_1}\sigma h, \quad \text{for } \sigma > \sigma_1 \tag{424}$$

We now formulate the overall info-gap model:

$$\mathcal{U}(h) = \left\{ \varepsilon(\sigma) : \ \text{for } 0 \le \sigma \le \sigma_1 : \ \varepsilon(\sigma) = \frac{\varepsilon_1}{\sigma_1}\sigma, \ \ (\varepsilon_1, \sigma_1) \in \mathcal{U}_1(h) \right.$$
$$\left. \text{for } \sigma_1 < \sigma : \ \left| \varepsilon(\sigma) - \frac{\varepsilon_1}{\sigma_1}\sigma \right| \le \frac{\varepsilon_1}{\sigma_1}\sigma h \right\} \tag{425}$$

Derive the inverse of the robustness function for the requirement that the strain not exceed the critical value ε_0 with known positive applied stress σ_0.

90. **Time to recovery,** (p.318). An important property of many technological systems is the time to recovery after a disruptive event. A building after an earthquake, an airplane after loss of an engine, a micro-sensor after excessive shock loading, etc., all need to recover critical functionalities within specified times.

We will compare two generic design concepts, one denoted 'State of the Art' (SotA) and the other called 'New and Innovative' (NaI). These systems are subject to generic loads. We require that the time to recovery of basic functions of the system, after an adverse event, not exceed a critical value, $t_{\rm c}$. Let $t_q(a)$ denote the time to recovery after an event whose load amplitude is a, where $q=0$ denotes SotA and $q=1$ denotes NaI.

(a) The recovery-time functions for the two designs, SotA and NaI respectively, are:

$$t_0(a) = \left\{ \begin{array}{ll} 0, & \text{if } a < \beta_0 \\ \alpha_0(a-\beta_0), & \text{if } a \geq \beta_0 \end{array} \right. \tag{426}$$

$$t_1(a) = \left\{ \begin{array}{ll} 0, & \text{if } a < \beta_1 \\ \alpha_1(a-\beta_1), & \text{if } \beta_1 \leq a < \delta_1 \\ \gamma_1(a-\delta_1)+\alpha_1(\delta_1-\beta_1), & \text{if } \delta_1 \leq a \end{array} \right. \tag{427}$$

Show that these recovery-time functions cross one another, and discuss the meaning and significance of this, assuming the following relations among the coefficients:

$$\gamma_1 > \alpha_0 = \alpha_1 > 0, \quad \delta_1 > \beta_1 > \beta_0 > 0 \tag{428}$$

The load amplitude is estimated to be $\widetilde{a}$ with an error estimate of s, but the error may be greater. The uncertainty in the load is represented with this info-gap model:

$$\mathcal{U}(h) = \left\{ a: \ a \geq 0, \ \left| \frac{a-\widetilde{a}}{s} \right| \leq h \right\}, \quad h \geq 0 \tag{429}$$

Assume that $\widetilde{a} \geq \beta_1$. Derive an explicit algebraic expression for the inverse of the robustness function for each design. Show that these robustness curves cross one another if eqs.(428) hold. What is the design implication of this?

(b) Instead of eqs.(426) and (427), the estimated recovery-time functions for the two designs are:

$$\widetilde{t}_q(a) = \left\{ \begin{array}{ll} 0 & \text{if } a < \beta_q \\ \alpha(a-\beta_q)^2, & \text{else} \end{array} \right. \tag{430}$$

Assume that $\beta_1 > \beta_0 > 0$ and $\alpha > 0$.

The load amplitude is **not** uncertain and is known to equal $\widetilde{a}$, where $\widetilde{a} > \beta_1$. However, the true recovery-time functions are fractionally uncertain, so the info-gap model is:

$$\mathcal{U}(h) = \left\{ \ t_q(a): \ \ t_q(a) \geq 0, \ \ \left| t_q(a) - \widetilde{t}_q(a) \right| \leq h w_q \widetilde{t}_q(a), \ \ q = 0,\, 1. \ \right\}, \ \ h \geq 0 \tag{431}$$

where each w_q is known and positive and $w_1\widetilde{t}_1(\widetilde{a}) > w_0\widetilde{t}_0(\widetilde{a})$. Show that the nominal recovery-time functions do not intersect, but that the robustness curves do intersect. Discuss the different origin of this preference reversal from part 90a. Specifically, compare the robustness of a known system vulnerability to an uncertain externality (part 90a), to the robustness of an uncertain system to a known externality (part 90b).

Based on \people\kanno\resilience2016\rsl003.tex.

(c) Continue part 90b but now assume that the load amplitude is uncertain as in eq.(429). The true recovery-time functions are also uncertain so the overall info-gap model is:

$$\mathcal{U}(h) = \Big\{ \, a, t_q(a) : \;\; t_q(a) \ge 0, \;\; \Big| t_q(a) - \widetilde{t}_q(a) \Big| \le h w_q \widetilde{t}_q(a), \;\; q = 0, \, 1.$$

$$a > 0, \; \left| \frac{a - \widetilde{a}}{s} \right| \le h \, \Big\}, \quad h \ge 0 \qquad (432)$$

where each w_q is known and positive and $w_1 \widetilde{t}_1(\widetilde{a}) > w_0 \widetilde{t}_0(\widetilde{a})$. Assume that $\widetilde{a} > \beta_1$ as before. Derive the inverse of the robustness curves and show that they intersect. Derive the inverse of the opportuneness curves.

91. **Innovation dilemma of rural poverty.** (p.321). Consider a subsistence level rural society. The farmers use traditional tools and methods and the agricultural productivity is low. We will refer to this system as the State of the Art (SotA) of this traditional society. An international aid organization offers innovative new methods and tools. This innovative approach could greatly increase the productivity. However, the new strains of plants have not been grown in this region, the new production methods could entail social changes and upheavals, and the innovations could result in much worse than anticipated outcomes, even endangering the survival of the traditional society.

   (a) Let x denote the agricultural productivity of wheat, in units of kg/ha. Anthropologists have studied this traditional society and reported reliably that the traditional SotA methods have yearly agricultural productivity of wheat that is normally distributed with an average of $\mu_0 = 1500$ kg/ha, with a standard deviation of $\sigma_0 = 200$ kg/ha. Denote this pdf $\widetilde{p}_0(x)$. Survival of the community requires yearly wheat productivity no less than $x_{\rm s} = 1171$ kg/ha. Calculate the probability of survival, $P_{\rm s}(\widetilde{p}_0)$, with the traditional SotA. What is the average interval between survival catastrophes?

   (b) The innovative methods are estimated to have agricultural wheat productivity that is normally distributed with mean $\mu_1 = 1850$ kg/ha and standard deviation $\sigma_1 = 250$ kg/ha, whose pdf is denoted $\widetilde{p}_1(x)$. Calculate the probability of survival, $P_{\rm s}(\widetilde{p}_1)$, with this estimated pdf. What is the average interval between survival catastrophes with innovative production methods?

   (c) The estimated pdf for innovative methods, $\widetilde{p}_1(x)$, is highly uncertain for values of x less than the survival level, $x_{\rm s}$. Specifically uncertainty in the correct pdf of the innovative agricultural productivity, $p_1(x)$, is described by this info-gap model:

$$\mathcal{U}(h) = \left\{ p_1(x) : \ p_1(x) \ge 0, \ \int_0^\infty p_1(x)\,{\rm d}x = 1, \ |p_1(x) - \widetilde{p}_1(x)| \le h\widetilde{p}_1(x_{\rm s}) \ \text{ for } \ x \le x_{\rm s} \right\}, \quad h \ge 0 \tag{433}$$

   We require that the probability of survival be no less than $P_{\rm s,c}$. Derive an explicit algebraic expression for the robustness of satisfying the survival-probability requirement, using innovative methods, as a function of $P_{\rm s,c}$.

   (d) We can assume that the anthropologists' determination of $\widetilde{p}_0(x)$ is accurate. Derive an explicit algebraic expression for the robustness of satisfying the survival-probability requirement, using traditional SotA, as a function of $P_{\rm s,c}$.

   (e) For what values of $P_{\rm s,c}$ are the innovative methods robust-preferred over the traditional SotA? Explain the innovation dilemma that confronts the traditional farmers.

92. **Elastic system with uncertainties.** (p.323).

(a) Consider an elastic system subject to a distributed load $f(x)$, for $0 \le x \le L$ that results in angular rotation given by:

$$\theta = \frac{1}{k} \int_0^L x f(x) \, \mathrm{d}x \tag{434}$$

The known nominal load is $\widetilde{f}(x)$ which is positive everywhere, but the true load is uncertain as described by this info-gap model:

$$\mathcal{U}(h) = \left\{ f(x) : \ \left| \frac{f(x) - \widetilde{f}(x)}{\widetilde{f}(x)} \right| \le h \right\}, \quad h \ge 0 \tag{435}$$

We require that the angle of rotation not exceed the critical value θ_{c}. Derive an explicit algebraic expression for the robustness.

(b) Continue part 92a but now derive an explicit algebraic expression for the opportuneness function, where the aspiration is that the angle of rotation not exceed the desirable value θ_{w}. Are the robustness and opportuneness functions sympathetic or antagonistic, with respect to change in the nominal angle of rotation? Explain.

(c) Repeat part 92a but with this info-gap model:

$$\mathcal{U}(h) = \left\{ f(x) : \ \left| \frac{f(x) - \widetilde{f}(x)}{s} \right| \le h \right\}, \quad h \ge 0 \tag{436}$$

where s is a known positive uncertainty weight. Consider two alternative designs with known nominal positive loads $\widetilde{f}_1(x)$ and $\widetilde{f}_2(x)$ and known positive uncertainty weighs s_1 and s_2, respectively, where:

$$\widetilde{f}_1(x) > \widetilde{f}_2(x) > 0, \quad 0 < s_1 < s_2 \tag{437}$$

For what values of the critical rotation angle, θ_{c}, is design 1 robust-preferred?

(d) Repeat part 92a but now assume that the load function is a truncated Fourier series:

$$f(x) = \sum_{n=0}^{K} c_n \cos n\pi x = c^T \gamma(x) \tag{438}$$

where c is the vector of Fourier coefficients and $\gamma(x)$ is the vector of corresponding cosine functions. Uncertainty in the load function is represented with this info-gap model:

$$\mathcal{U}(h) = \left\{ f(x) = c^T \gamma(x) : \ c^T W c \le h^2 \right\}, \quad h \ge 0 \tag{439}$$

where W is a known, real, symmetric, positive definite matrix. Derive an explicit algebraic expression for the robustness.

(e) The angle of rotation of an elastic system is measured 5 times with a sensor that is corrupted by zero-mean normal noise. The mean and variance of this statistical sample are 0.074 radian and 0.0001 radian^2, respectively. Sam claims that the true angle of rotation is 0.062, while Sally claims that the true angle is greater than 0.062. Formulate and evaluate a statistical test. Do you accept or reject Sam's claim at 0.01 level of significance?

(f) An elastic system displays random angles of rotation that are normally distributed with mean and variance of 0.06 radian and 0.0002 radian^2, respectively. What is the probability that the angle of rotation will exceed 0.07?

(g) An elastic system displays random angles of rotation. Damage results only if the angle exceeds a critical value. The probability that any single rotation angle exceeds the critical value is 0.01. Given a random sample of M rotation angles, what is the probability that exactly n rotation angles will exceed the critical value, if $M = 6$ and $n = 2$?

(h) A motion sensor classifies rotations of the elastic system as either "small" or "large". The probability of a small rotation is 0.3 under normal conditions. A random sample of detections contained 39 small rotations and 61 large rotations. Do you accept the claim that the system is normal, at level of significance of 0.1?

93. **Stress-strain relation.** (p.326).

(a) Consider the linear-elastic relation between stress, σ, and strain, ε:

$$\sigma = E\varepsilon \tag{440}$$

The Young's modulus, E, is uncertain according to this fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ E : \ E \geq 0, \ \left| \frac{E - \widetilde{E}}{w_y} \right| \leq h \right\}, \quad h \geq 0 \tag{441}$$

where $\widetilde{E}$ and w_y are known positive constants.

i. (Problem 1, moed bet, August 2016) We require that the stress not exceed the critical value $\sigma_{\rm c}$. Assume that the strain is positive, and derive an explicit algebraic expression for the robustness function.

ii. (Problem 1, moed bet, September 2016) We require that the stress not be less than the positive critical value $\sigma_{\rm c}$. Assume that the strain is positive, and derive an explicit algebraic expression for the robustness function.

(b) (Problem 2, moed bet, August 2016) Use the linear-elastic relation between stress, σ, and strain, ε in eq.(440) and assume that both E and ε are uncertain according to this info-gap model:

$$\mathcal{U}(h) = \left\{ E, \varepsilon : \ E \geq 0, \ \left| \frac{E - \widetilde{E}}{w_y} \right| \leq h, \ \left| \frac{\varepsilon - \widetilde{\varepsilon}}{w_\varepsilon} \right| \leq h \right\}, \quad h \geq 0 \tag{442}$$

where $\widetilde{E}$, w_y, $\widetilde{\varepsilon}$ and w_ε are known positive constants. We require that the stress not exceed the critical value $\sigma_{\rm c}$. Derive an explicit algebraic expression for the inverse of the robustness function. What is the value of the robustness if $\sigma_{\rm c} = 0$? If $\sigma_{\rm c} = \infty$? Derive an explicit algebraic expression for the robustness if:

$$w_\varepsilon = \widetilde{\varepsilon}, \quad w_y = \widetilde{E} \tag{443}$$

(c) (Problem 2, moed bet, September 2016) Use the relation in eq.(440) and consider σ and E to be uncertain according to this info-gap model:

$$\mathcal{U}(h) = \left\{ E, \sigma : \ E \geq 0, \ \left| \frac{E - \widetilde{E}}{w_y} \right| \leq h, \ \left| \frac{\sigma - \widetilde{\sigma}}{w_\sigma} \right| \leq h \right\}, \quad h \geq 0 \tag{444}$$

where $\widetilde{E}$, w_y, $\widetilde{\sigma}$ and w_σ are known positive constants. We require that the strain exceed the critical value $\varepsilon_{\rm c}$. Derive an explicit algebraic expression for the inverse of the robustness function. What is the value of the robustness if $\varepsilon_{\rm c} = 0$? If $\varepsilon_{\rm c} = \widetilde{\sigma}/\widetilde{E}$? If $\varepsilon_{\rm c} = -w_\sigma/w_y$?

(d) Use the relation in eq.(440) and consider E to be known.

i. (Problem 3, moed bet, August 2016) Let ε be a uniformly distributed random variable:

$$p(\varepsilon) = \frac{1}{\varepsilon_1}, \quad 0 \leq \varepsilon \leq \varepsilon_1 \tag{445}$$

and zero otherwise, where ε_1 is a known positive constant. The system fails if the stress, σ, exceeds the critical value $\sigma_{\rm c}$. Derive an explicit algebraic expression for the probability of failure.

ii. (Problem 4, moed bet, August 2016) Continue from part 93(d)i with uncertainty in the constant of the probability distribution:

$$\mathcal{U}(h) = \left\{ p(\varepsilon) = \frac{1}{\varepsilon_1} : \ \varepsilon_1 \geq 0, \ \left| \frac{\varepsilon_1 - \widetilde{\varepsilon}_1}{s} \right| \leq h \right\}, \quad h \geq 0 \tag{446}$$

We require that the probability of failure not exceed the critical value P_c. Derive an explicit algebraic expression for the inverse of the robustness function, and from that derive an explicit algebraic expression for the robustness function.

(e) Use the relation in eq.(440) and consider E to be known.

i. (Problem 3, moed bet, September 2016) Let ε be an exponentially distributed random variable:

$$p(\varepsilon) = \lambda \mathrm{e}^{-\lambda\varepsilon}, \quad \varepsilon \geq 0 \tag{447}$$

The system fails if the stress, σ, exceeds the critical value σ_c. Derive an explicit algebraic expression for the probability of failure.

ii. (Problem 4, moed bet, September 2016) Continue from part 93(e)i with uncertainty in the exponential coefficient of the probability distribution:

$$\mathcal{U}(h) = \left\{ p(\varepsilon) = \lambda \mathrm{e}^{-\lambda\varepsilon} : \ \lambda \geq 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{s} \right| \leq h \right\}, \quad h \geq 0 \tag{448}$$

We require that the probability of failure not exceed the critical value P_c. Derive an explicit algebraic expression for the inverse of the robustness function, and from that derive an explicit algebraic expression for the robustness function.

(f) Consider the following non-linear modification of the stress-strain relation:

$$\sigma = \sum_{n=1}^{N} E_n \varepsilon^n = x^T y \tag{449}$$

where x^T is the vector of coefficients $(E_1, \ldots, E_N)$ and y^T is the vector of powers of the strain $(\varepsilon^1, \ldots, \varepsilon^N)$. The coefficients x are uncertain according to this ellipsoidal-bound info-gap model:

$$\mathcal{U}(h) = \left\{ x : \ (x - \widetilde{x})^T W^{-1} (x - \widetilde{x}) \leq h^2 \right\}, \quad h \geq 0 \tag{450}$$

where $\widetilde{x}$ is a known vector and W is a known, real, symmetric, positive definite matrix.

i. (Problem 5, moed bet, August 2016) We require that the stress exceed the critical value σ_c. Derive an explicit algebraic expression for the robustness function.

ii. (Problem 5, moed bet, September 2016) We require that the stress not exceed the critical value σ_c. Derive an explicit algebraic expression for the robustness function.

(g) A random sample of the strain at a particular point in a structure has 7 observations with sample mean and variance of 6×10^{-4} and 1×10^{-7}, respectively.

i. (Problem 6, moed bet, August 2016) Joe claims that the true strain is 1×10^{-4} while Jill claims that the true strain is greater than 1×10^{-4}. Formulate a statistical hypothesis test. Do you accept or reject Joe's claim at the 0.01 level of significance? Explain.

ii. (Problem 6, moed bet, September 2016) Joe claims that the true strain is 1×10^{-4} while Jill claims that the true strain is **less**, not greater, than 1×10^{-4}. Formulate a statistical hypothesis test. Do you accept or reject Joe's claim at the 0.01 level of significance? Explain.

(h) The strain, ε, is a random variable.

i. (Problem 7, moed bet, August 2016) The pdf of ε is:

$$p(\varepsilon) = -\frac{2}{\varepsilon_0^2}\varepsilon + \frac{2}{\varepsilon_0}, \quad 0 \leq \varepsilon \leq \varepsilon_0 \tag{451}$$

and zero otherwise. What is the probability that the strain will not exceed $\varepsilon_0/4$?

ii. (Problem 7, moed bet, September 2016) The pdf of ε is:

$$p(\varepsilon) = \frac{2}{\varepsilon_0^2}\varepsilon, \quad 0 \le \varepsilon \le \varepsilon_0 \tag{452}$$

and zero otherwise. What is the probability that the strain will exceed $\varepsilon_0/2$?

(i) (Problem 8, moed bet, August 2016) A sample is tested and yields one of two possible results: either "yes" or "no". In normal circumstances the probability of "yes" is $p_y = 1/2$. A random sample of $N = 100$ observations has exactly N_y "yes" observations. Ted claims that circumstances are normal. Formulate a statistical hypothesis test to test Ted's assertion. What is the smallest value of N_y that **does not reject** Ted's assertion at 0.025 level of significance?

(j) (Problem 8, moed bet, September 2016) The strain of a mechanical element is observed in a random sample with 90 observations, where 54 observations show "large" strain and 36 observations show "small" strain. In normal conditions the probability of "large" strain is 0.75. Susie claims that the observations were made on an element in normal conditions, while Sam denies this. Formulate and implement a statistical hypothesis test to test Susie's claim. Do you accept or reject her claim at 0.02 level of significance?

94. **Allocation of scarce resource** (based on exam in 036057, 16.1.2017), (p.330). Consider allocation of a scarce resource, such as time or money, among a number of different items. Given $N > 1$ items and a total resource budget R, let r_n denote the allocation to item n, for $n = 1, \ldots, N$, where $r_n \ge 0$. The benefit resulting from allocating r_n to item n is $r_n b_n$ where the benefit per unit allocation, b_n, is uncertain. The total benefit is $B = \sum_{n=1}^{N} r_n b_n$, and we require that the total benefit be no less than the critical value B_{c}.

(a) The benefit per unit allocation is estimated as $\widetilde{b}_n \pm s_n$, but it may be either less or more, where $\widetilde{b}_n > 0$ and $s_n > 0$ are known. The info-gap model for uncertainty is:

$$\mathcal{U}(h) = \left\{ b : \left| \frac{b_n - \widetilde{b}_n}{s_n} \right| \le h, \ n = 1, \ldots, N \right\}, \quad h \ge 0 \tag{453}$$

Derive an explicit algebraic expression for the robustness function.

(b) Let $\widetilde{b}$ and s denote the vectors of estimated benefits per unit allocation, $\widetilde{b}_n$, and error weights, s_n, respectively. Consider two different vectors of allocations $r = (r_1, \ldots, r_N)$ and $\rho = (\rho_1, \ldots, \rho_N)$. These allocations satisfy the following relations:

$$r^T \widetilde{b} \ > \ \rho^T \widetilde{b} \tag{454}$$

$$\frac{r^T \widetilde{b}}{r^T s} \ < \ \frac{\rho^T \widetilde{b}}{\rho^T s} \tag{455}$$

What is an intuitive interpretation of these relations? Specifically, how do they reflect a dilemma facing the decision maker? Using the answer to part 94a, derive an explicit algebraic expression for the values of critical benefit, B_{c}, for which allocation r is robust-preferred over allocation ρ.

(c) Return to the basic formulation of the problem, prior to part 94a, and consider two different programs within which the resource can be allocated. Program 1 has nominal predicted total benefit B_1 which is a known positive number. However, the actual benefits are uncertain and the robustness function for allocation vector r in program 1 is known and finite for all values of B_{c}. Program 2 has exactly known benefits, and the total benefit is guaranteed to be B_2 for the same allocation vector, r. However, $B_2 < B_1$. Derive an explicit algebraic expression for the values of critical benefit, $B_{\rm c}$, for which program 1 is robust-preferred over program 2.

(d) Return to the basic formulation of the problem, prior to part 94a, and consider the following ellipsoid-bound info-gap model for uncertainty in the benefit vector:

$$\mathcal{U}(h) = \left\{ b : \ (b - \widetilde{b})^T W^{-1} (b - \widetilde{b}) \le h^2 \right\}, \quad h \ge 0 \tag{456}$$

where W is a real, symmetric, positive definite $N \times N$ matrix. Derive an explicit algebraic expression for the robustness function.

(e) Suppose that the total benefit, B, is an exponentially distributed random variable, whose probability density function is:

$$p(B) = \lambda \mathrm{e}^{-\lambda B}, \quad B \ge 0 \tag{457}$$

What is the probability that the total benefit exceeds the critical value $B_{\rm c}$?

(f) Continuing part 94e, suppose that you require that the probability of exceeding the critical benefit, $B_{\rm c}$, must be no less than the critical probability $P_{\rm c}$. However, the critical benefit, $B_{\rm c}$, is uncertain (you don't really know what you need). Use the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ B_{\rm c} : \ \left| \frac{B_{\rm c} - \widetilde{B}_{\rm c}}{\widetilde{B}_{\rm c}} \right| \le h \right\}, \quad h \ge 0 \tag{458}$$

Derive an explicit algebraic expression for the robustness function for satisfying the probabilistic requirement.

(g) Repeat part 94a with the following info-gap model:

$$\mathcal{U}(h) = \left\{ b : \ \left(b - \widetilde{b} \right)^T W^{-1} \left(b - \widetilde{b} \right) \le h^2 \right\}, \quad h \ge 0 \tag{459}$$

where W is a real, symmetric positive definite matrix. W and $\widetilde{b}$ are known. Derive an explicit algebraic expression for the robustness function.

95. **Continuous linear system** (based on exam in 036057, 7.2.2017), (p.334). **Background.** The response, y, of a linear system is described by:

$$y = \int_{-1}^{+1} u(x) f(x)\, \mathrm{d}x \tag{460}$$

where $f(x)$ is a known function and $u(x)$ is uncertain. We require that the response be no greater than the critical value y_{c}.

(a) The function $f(x)$ is:

$$f(x) = \begin{cases} f_0 & \text{if } x \geq 0 \\ -f_0 & \text{else} \end{cases} \tag{461}$$

where f_0 is a known positive value. Uncertainty in the function $u(x)$ is represented by the fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ u(x) : \left| \frac{u(x) - \widetilde{u}}{\widetilde{u}} \right| \leq h \right\}, \quad h \geq 0 \tag{462}$$

where $\widetilde{u}$ is a known and positive constant. Derive an explicit algebraic expression for the robustness function.

(b) Return to the background, prior to part 95a. Uncertainty in the function $u(x)$ is represented by the energy-bound info-gap model:

$$\mathcal{U}(h) = \left\{ u(x) : \int_{-1}^{+1} [u(x) - \widetilde{u}(x)]^2 \, \mathrm{d}x \leq h^2 \right\}, \quad h \geq 0 \tag{463}$$

where $\widetilde{u}(x)$ is a known finite-valued function. Derive an explicit algebraic expression for the robustness function.

(c) Return to the background, prior to part 95a. The function $u(x)$ is a truncated Taylor series:

$$u(x) = \sum_{m=1}^{m=N} a_m \sin m\pi x = a^T \sigma(x) \tag{464}$$

which defines the vector a of coefficients and the vector $\sigma(x)$ of sine functions. Uncertainty in the function $u(x)$ is represented by the ellipsoid-bound info-gap model:

$$\mathcal{U}(h) = \left\{ u(x) = a^T \sigma(x) : \ (a - \widetilde{a})^T W^{-1} (a - \widetilde{a}) \leq h^2 \right\}, \quad h \geq 0 \tag{465}$$

where W is a known, symmetric, positive definite, real matrix, and $\widetilde{a}$ is a known real vector. Derive an explicit algebraic expression for the robustness function.

(d) Return to the background, prior to part 95a and consider two different designs of the system, represented by two different choices of the known function, either $f_1(x)$ or $f_2(x)$, where:

$$0 < f_1(x) < f_2(x) \qquad \text{for all } x \in [0,\ 1] \tag{466}$$

$$f_1(x) = 0 = f_2(x) \qquad \text{for all } x \in [-1,\ 0) \tag{467}$$

The uncertainty in $u(x)$ is different in the two different designs. The info-gap model for design i is the following fractional-error model:

$$\mathcal{U}_i(h) = \left\{ u(x) : \left| \frac{u(x) - \widetilde{u}}{s_i} \right| \leq h \right\}, \quad h \geq 0, \quad i = 1, 2 \tag{468}$$

where $\widetilde{u}$, s_1 and s_2 are known positive constants satisfying the relation:

$$s_1 \int_0^1 f_1(x)\, \mathrm{d}x \ > \ s_2 \int_0^1 f_2(x)\, \mathrm{d}x \tag{469}$$

Derive an explicit algebraic expression for the range of y_{c}-values for which design 2 is preferred according to the robustness criterion.

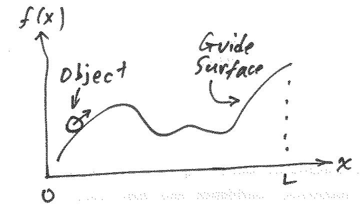

Figure 20: Guide surface for problem 96.

96. **Guide surface** (based on exam in 035018, 6.6.2018), (p.336). An object moves along a guide surface as shown in fig. 20. This object determines the motion of other objects, e.g. optically linked milling tools. The desired shape of the surface is specified by the function $\widetilde{f}(x)$. However the actual shape, $f(x)$, may differ in unknown ways. Assume that $\widetilde{f}(x) > 0$. The performance is assessed as:

$$B(f) = \int_0^L f(x) g(x) \, \mathrm{d}x \tag{470}$$

where $g(x)$ is a known function that takes both positive and negative values. We require that the performance with the actual surface, $f(x)$, deviate from the anticipated performance based on the specified surface, $\widetilde{f}(x)$, by no more than ε:

$$\left| B(f) - B(\widetilde{f}) \right| \le \varepsilon \tag{471}$$

(a) The uncertainty in the surface shape is specified by this info-gap model:

$$\mathcal{U}(h) = \left\{ f(x) : \ \left| \frac{f(x) - \widetilde{f}(x)}{\widetilde{f}(x)} \right| \le h \right\}, \quad h \ge 0 \tag{472}$$

Derive an explicit algebraic expression for the robustness function.

(b) Derive an explicit algebraic expression for the robustness function with this info-gap model:

$$\mathcal{U}(h) = \left\{ f(x) = c^T \sigma(x) : \ (c - \widetilde{c})^T W (c - \widetilde{c}) \le h^2 \, \widetilde{c}^T W \widetilde{c} \right\}, \quad h \ge 0 \tag{473}$$

where c is a vector of uncertain Fourier coefficients, $\widetilde{c}$ is a known estimate of c, W is a known, real, symmetric, positive definite matrix, and $\sigma(x)$ is a vector of trigonometric functions. Note that $\widetilde{c}^T W \widetilde{c}$ is a normalization constant that assures that the horizon of uncertainty, h, is dimensionless.

(c) Derive an explicit algebraic expression for the robustness function with this info-gap model:

$$\mathcal{U}(h) = \left\{ f(x) : \ \int_0^L \left(f(x) - \widetilde{f}(x) \right)^2 \mathrm{d}x \le h^2 \int_0^L \widetilde{f}(x)^2 \, \mathrm{d}x \right\}, \quad h \ge 0 \tag{474}$$

Note that $\int_0^L \widetilde{f}(x)^2 \, \mathrm{d}x$ is a normalization constant that assures that the horizon of uncertainty, h, is dimensionless.

97. **Failure detection** (based on exam in 035018, 17.7.2018), (p.338).

(a) The condition of a system will be assessed with a vector, c, of N measurements. In the functional state the measurement vector should be $\widetilde{c}$, which is a known vector. However, the actual measured vector in the functional state varies uncertainly as described by this info-gap model:

$$\mathcal{U}(h) = \left\{ c : \left| \frac{c_i - \widetilde{c}_i}{s_i} \right| \le h, \; i = 1, \ldots, N \right\}, \quad h \ge 0 \tag{475}$$

where the s_i's are known and positive. For any measurement, c, we will declare the system "functional" if:

$$\| c - \widetilde{c} \|^2 \le \varepsilon^2 \tag{476}$$

where, for any vector, $\|x\|^2 = x^T x$ is the square of the Euclidean norm. Derive an explicit algebraic expression for the robustness of correctly declaring a functional system to be functional.

(b) Repeat part 97a with this info-gap model:

$$\mathcal{U}(h) = \left\{ c : \; (c - \widetilde{c})^T W^{-1} (c - \widetilde{c}) \le h^2 \right\}, \quad h \ge 0 \tag{477}$$

where W is a known, real, symmetric, positive definite matrix. Unlike eq.(476), we will declare the system "functional" if:

$$\sum_{i=1}^{N} c_i \le \varepsilon \tag{478}$$

Derive an explicit algebraic expression for the robustness of correctly declaring a functional system to be functional.

(c) The condition of a system is assessed with a measurement, c, which is a random variable. If the system is "functional" then the pdf of c is:

$$p_1(c) = \lambda \mathrm{e}^{-\lambda c}, \quad c \ge 0 \tag{479}$$

If the system is "failed" then the pdf of c is:

$$p_2(c) = \mu \mathrm{e}^{-\mu (c - c_{\mathrm{f}})}, \quad c \ge c_{\mathrm{f}} \tag{480}$$

where c_{f} is a known positive value.

Given a measurement, c_{m}, we declare the system "functional" if:

$$c_{\mathrm{m}} \le c_1 \tag{481}$$

Likewise, we declare the system "failed" if:

$$c_{\mathrm{m}} \ge c_2 \tag{482}$$

Assume that $c_1 > c_{\mathrm{f}}$.

Derive an explicit algebraic expression for the probability of declaring a functional system to be functional. Derive an explicit algebraic expression for the probability of declaring a failed system to be functional.

(d) The condition of a system is specified by the value of c. When the system is functional:

$$c = 3.7 \tag{483}$$

When the system is failed:

$$c \neq 3.7 \tag{484}$$

Measurement of c is corrupted by random normal noise with zero mean and unknown variance. A random sample of c values of size $N = 5$ is taken. The state of the system is constant throughout the sampling. The sample mean and sample variance are $\overline{x} = 3.2$, and $s^2 = 0.5$. Select a statistical test to determine if the system is functional or failed. Briefly explain your selection. Do you reject the hypothesis that the system is functional at the 0.02 level of significance?

(e) Repeat part 97d with a single difference. Instead of eq.(484) we have:
When the system is failed:

$$c < 3.7 \tag{485}$$

Do you reject the hypothesis that the system is functional at the 0.02 level of significance?

(f) Repeat part 97d with a single difference. Instead of eq.(484) we have:
When the system is failed:

$$c > 3.7 \tag{486}$$

Do you reject the hypothesis that the system is functional at the 0.02 level of significance?

98. **Fire suppression** (based on exam in 035018, 7.10.2018), (p.340). You are responsible for allocating fire-suppression units for managing a major fire raging at N distinct locations. The area, A, in which fire has been extinguished t hours after initiation of response, is:

$$A(a, f) = t \sum_{i=1}^{N} a_i f_i = t\, a^T f \tag{487}$$

where f_i is the number of units allocated to location i and a_i is a coefficient expressing the relative ease with which location i is controlled. a and f are vectors of the corresponding quantities a_i and f_i. You must choose the vector f, where the coefficients a are uncertain. Your goal is to extinguish the fire in an area, A, no less than A_c within a specified duration t.

(a) The uncertainty in the coefficients a is expressed by this fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ a : \left| \frac{a_i - \widetilde{a}_i}{w_i} \right| \le h,\ i = 1, \ldots, N \right\}, \quad h \ge 0 \tag{488}$$

where $\widetilde{a}_i$ and w_i are known positive constants. Derive an explicit algebraic expression for the robustness function.

(b) Continue part 98a and compare two alternative allocations of fire-fighting units, denoted f and f', where:

$$\widetilde{a}^T f > \widetilde{a}^T f', \quad \frac{\widetilde{a}^T f}{w^T f} < \frac{\widetilde{a}^T f'}{w^T f'} \tag{489}$$

The left hand relation asserts that allocation f is predicted to achieve control of a greater area than allocation f'. However, predictions have zero robustness to uncertainty, and the righthand relation asserts that allocation f has greater relative error than allocation f'. Using the robustness function from part 98a, derive an explicit algebraic expression for the range of critical values, A_c, for which allocation f is robust-preferred over allocation f'.

(c) Repeat part 98a with this info-gap model:

$$\mathcal{U}(h) = \left\{ a :\ (a - \widetilde{a})^T W^{-1} (a - \widetilde{a}) \le h^2 \right\}, \quad h \ge 0 \tag{490}$$

where W is a known, real, symmetric, positive definite matrix and $\widetilde{a}$ is the vector of estimated coefficients.

(d) Let θ denote the time it takes to extinguish a fire. For naturally occurring fires, the probability density function (pdf) of θ is:

$$p(\theta) = \begin{cases} \dfrac{\theta_1}{\theta^2} & \text{if } \theta \ge \theta_1 \\ 0 & \text{else} \end{cases} \tag{491}$$

where θ_1 is a known positive constant.

Let θ_2 be a duration greater than θ_1. Derive an explicit algebraic expression for the probability that the time required to extinguish a fire will exceed θ_2.

(e) Continuing part 98d, suppose a specific fire was extinguished after time $\theta_2 = 100\theta_1$. Can you confidently assert that this was a naturally occurring fire? Explain briefly.

(f) We now extend and modify part 98d. The pdf in eq.(491) is an estimate, which we denote $\widetilde{p}(\theta)$. The true pdf, $p(\theta)$, is unknown. The uncertainty in the estimate is represented with this info-gap model:

$$\mathcal{U}(h) = \left\{ p(\theta) :\ p(\theta) \ge 0,\ \int_0^\infty p(\theta) \mathrm{d}\theta = 1,\ |p(\theta) - \widetilde{p}(\theta)| \le h \widetilde{p}(\theta) \right\}, \quad h \ge 0 \tag{492}$$

Let θ_2 be a duration greater than θ_1. Let $P(\theta \geq \theta_2|p)$ denote the probability that the time required to extinguish a fire exceeds the value θ_2, based on the pdf $p(\theta)$. We don't know the function $p(\theta)$, so we will estimate $P(\theta \geq \theta_2|p)$ as $P(\theta \geq \theta_2|\widetilde{p})$. We require that this estimate not err more than ε. That is, we require:

$$|P(\theta \geq \theta_2|p) - P(\theta \geq \theta_2|\widetilde{p})| \leq \varepsilon \tag{493}$$

Derive an explicit algebraic expression for the robustness function of this uncertain estimate. Assume that θ_2 is much much greater than θ_1.

(g) The average duration of naturally-occurring forest fires in a specific region is 30 hours. The mean and variance of a normally distributed random sample of 6 fires from that region are $\overline{x} = 33$ hours and $s^2 = 3$ hours^2, respectively. Choose a statistical test to test the hypothesis that this is a sample of naturally-occurring forest fires. Do you accept or reject this hypothesis at 0.05 level of significance? Explain.

99. **Elastic deflection with uncertain stiffness** (based on exam in 036057, 15.1.2019), (p.342).

Background: The force-displacement relation of a linear elastic system is described by $F = kx$. The estimated value of the stiffness coefficient is $\widetilde{k}$, but this is uncertain. We explore some design implications.

(a) Uncertainty in k is specified by the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ k : \ k \ge 0, \ \left| \frac{k - \widetilde{k}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{494}$$

where w is a known positive constant. We will apply a positive force, F, and we require that the resulting deflection be no less than the critical value x_{c}. Derive an explicit algebraic expression for the robustness.

(b) Now suppose that k is a random variable. We will apply a positive force F, and require that the resulting displacement be no less than the critical value x_{c}. The probability of failing to achieve this requirement is:

$$P_{\mathrm{f}}(F, x_{\mathrm{c}}) = \frac{1}{1 + (F/F_0)}, \quad F \ge 0 \tag{495}$$

where F_0 is estimated as $\widetilde{F}_0$, which is positive, but F_0 is uncertain. The uncertainty in F_0 is represented by the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ F_0 : \ F_0 > 0, \ \left| \frac{F_0 - \widetilde{F}_0}{\widetilde{F}_0} \right| \le h \right\}, \quad h \ge 0 \tag{496}$$

We require that the probability of failure be no greater than P_{c}. Derive an explicit algebraic expression for the robustness.

(c) We now extend the problem to a vector of forces, $F^T = (F_1, \ldots, F_N)$ applied to an elastic system. The deflection at a specific location is:

$$x = \sum_{i=1}^{N} c_i F_i \tag{497}$$

where the vector of flexibility coefficients, c_i, is uncertain according to this ellipsoid-bound info-gap model:

$$\mathcal{U}(h) = \left\{ c : \ (c - \widetilde{c})^T \, W \, (c - \widetilde{c}) \le h^2 \right\}, \quad h \ge 0 \tag{498}$$

where $\widetilde{c}$ is a known vector and W is a known, positive definite, symmetric, real matrix. We require that the deflection be no less than the critical value x_{c}. Derive an explicit algebraic expression for the robustness.

(d) Return to part 99c but now consider opportuneness rather than robustness. We aspire (but do not require) that the deflection be at least as large as x_{w}. Derive an explicit algebraic expression for the opportuneness function.

100. **Attrition time** (based on exam in 036057, 18.2.2019), (p.344).

Background. Attrition of a system occurs over time. This may be due to mechanical wear, or environmental erosion, or enemy fire, etc. The time required for complete attrition is:

$$t = \frac{1}{\rho} \tag{499}$$

The estimated value of ρ is $\widetilde{\rho}$, but this is uncertain. We require that complete attrition be reached no later than time $t_{\rm c}$.

(a) The uncertainty in ρ is expressed by the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ \rho: \ \rho > 0, \ \left| \frac{\rho - \widetilde{\rho}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{500}$$

Derive an explicit algebraic expression for the robustness function.

(b) It would be wonderful if complete attrition is achieved within time $t_{\rm w}$, which is less than $t_{\rm c}$. Using the info-gap model of eq.(500), derive an explicit algebraic expression for the opportuneness function.

(c) We now change the story a bit, and consider ρ to be a random variable. The attrition does not complete in time if $\rho \le 1/t_{\rm c}$. In other words, the probability of failure, $P_{\rm f}$, is:

$$\mathrm{Prob}\left(\rho \le \frac{1}{t_{\rm c}} \right) = 1 - \mathrm{e}^{-\lambda t_{\rm c}} \tag{501}$$

However, λ is uncertain according to this fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda: \ \lambda > 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{\widetilde{\lambda}} \right| \le h \right\}, \quad h \ge 0 \tag{502}$$

We require that the probability of failure be no greater than the critical value $P_{\rm c}$. Derive an explicit algebraic expression for the robustness function.

(d) Generalizing eq.(499), the time required to complete the attrition is:

$$t = a^T y \tag{503}$$

where y is a known vector and a is uncertain according to this ellipsoidal-bound info-gap model:

$$\mathcal{U}(h) = \left\{ a: \ (a - \widetilde{a})^T W (a - \widetilde{a}) \le h^2 \right\}, \quad h \ge 0 \tag{504}$$

where $\widetilde{a}$ is a known vector and W is a known, real, symmetric, positive definite matrix. We require that complete attrition be reached no later than time $t_{\rm c}$. Derive an explicit algebraic expression for the robustness function.

101. **Energy conservation by feedback** (based on exam in 035018, 22.5.2019), (p.345). People change their energy consumption in response to feedback about their prior energy use. Define:

$n(c)\,\mathrm{d}c =$ number of consumers whose prior energy consumption was in the interval $[c,\ c+\mathrm{d}c]$.

The estimated consumption in the next time interval, for a consumer whose prior consumption was in the interval $[c,\ c+\mathrm{d}c]$, is denoted $\widetilde{f}(c,\rho)$, where ρ is a parameter expressing the intensity of the feedback; greater ρ implies greater intensity.

The true consumption function is $f(c,\rho)$, whose uncertainty is represented by an info-gap model, $\mathcal{U}(h)$. The response of the entire population to feedback at intensity ρ is:

$$R(\rho, f) = \int_0^{\infty} f(c,\rho) n(c)\,\mathrm{d}c \tag{505}$$

We require that the population response be no greater than the critical value, R_{c}:

$$R(\rho, f) \le R_{\mathrm{c}} \tag{506}$$

(a) Derive an explicit algebraic expression for the robustness function for the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ f(c,\rho) :\ f(c,\rho) \ge 0,\ \left| \frac{f(c,\rho) - \widetilde{f}(c,\rho)}{\widetilde{f}(c,\rho)} \right| \le h \right\}, \quad h \ge 0 \tag{507}$$

(b) Derive an explicit algebraic expression for the robustness function for the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ f(c,\rho) :\ f(c,\rho) \ge 0,\ \left| \frac{f(c,\rho) - \widetilde{f}(c,\rho)}{w} \right| \le h \right\}, \quad h \ge 0 \tag{508}$$

where w is a known positive constant.

(c) Continuing from part 101b, consider two different situations, $(\rho_1,\ w_1)$ and $(\rho_2,\ w_2)$, where:

$$\rho_2 < \rho_1 \quad \text{and} \quad 0 < w_2 < w_1 \tag{509}$$

That is, the feedback in situation 1 is more intensive, but the uncertainty in this situation is greater. For what values of R_{c} is situation 2 robust-preferred? Assume that $\widetilde{f}(c,\rho) = (1-\rho)c$.

(d) For a particular info-gap model, the robustness function takes this form:

$$\widehat{h}(R_{\mathrm{c}}, \rho) = (R_{\mathrm{c}} - w)\rho \tag{510}$$

or zero if this is negative, where ρ and w are positive constants. Consider two different situations, $(\rho_1,\ w_1)$ and $(\rho_2,\ w_2)$, where:

$$0 < \rho_2 < \rho_1 \quad \text{and} \quad 0 < w_2 < w_1 \tag{511}$$

For what values of R_{c} is situation 1 robust-preferred?

(e) The true and estimated consumption functions are related as:

$$\begin{aligned} f(c,\rho) &= \widetilde{f}(c,\rho) + \sum_{j=1}^{J} a_j \sin\frac{j\pi c}{c_{\max}} && (512)\\ &= \widetilde{f}(c,\rho) + a^T \sigma(c) && (513) \end{aligned}$$

where $c_{\max}$ is a known positive number, and a and $\sigma(c)$ are the vectors of Fourier coefficients and sine functions in eq.(512). The uncertainty in $f(c, \rho)$ is represented by this Fourier-ellipsoid info-gap model:

$$\mathcal{U}(h) = \left\{ f(c, \rho) = \widetilde{f}(c, \rho) + a^T \sigma(c) : \ a^T W a \leq h^2 \right\}, \quad h \geq 0 \tag{514}$$

where W is a known, positive definite, real, symmetric matrix. Derive an explicit algebraic expression for the robustness function.

102. **Quality control with weighted means** (based on exam in 035018, 14.7.2019), (p.347). N attributes of a system are measured, yielding the values $y_1^{(m)}, \ldots, y_N^{(m)}$. The true values are $y_1, \ldots, y_N$. The system is accepted if and only if the weighted mean of the true values is no less than a critical value, y_c:

$$\sum_{i=1}^{N} w_i y_i \ge y_\mathrm{c} \tag{515}$$

where $w_i \ge 0$ and $\sum_{i=1}^{N} w_i = 1$.

(a) The measurements are uncertain over-estimates of the true values, $y_1, \ldots, y_N$. Given known values of the measurements, the uncertainty in the true values is represented by this info-gap model:

$$\mathcal{U}(h) = \left\{ y : \ 0 \le y_i^{(m)} - y_i \le h, \quad i = 1, \ldots, N \right\}, \quad h \ge 0 \tag{516}$$

Suppose that the measurements indicate acceptability according to eq.(515). Derive an explicit algebraic expression for the robustness of this decision.

(b) Repeat part 102a with this info-gap model:

$$\mathcal{U}(h) = \left\{ y : \ 0 \le \frac{y_i^{(m)} - y_i}{s_i} \le h, \quad i = 1, \ldots, N \right\}, \quad h \ge 0 \tag{517}$$

where the values s_i are known positive uncertainty weights of the measurements.

(c) z is a random variable whose value assesses a system, which is considered acceptable if and only if:

$$z \ge z_\mathrm{c} \tag{518}$$

where z_c is a positive value less than 1. The probability density function of z, for an acceptable system, is:

$$p(z|\text{acc}) = 6z(1-z), \quad 0 \le z \le 1 \tag{519}$$

The probability density function of z, for an unacceptable system, is:

$$p(z|\text{unacc}) = 2\mathrm{e}^{-2z}, \quad z \ge 0 \tag{520}$$

Given a single measurement of z, derive explicit algebraic expressions for:

(1) The probability that an acceptable system will be rejected.

(2) The probability that an unacceptable system will be accepted.

(d) z is a random variable whose value assesses a system, which is considered acceptable if eq.(518) holds. When the system is acceptable, the probability density function of z is eq.(519), which we denote $\widetilde{p}(z)$. However, the pdf when the system is unacceptable has an uncertain “contamination” for large values of z:

$$p(z) = \alpha \widetilde{p}(z) + (1-\alpha) q(z), \quad z \ge 0 \tag{521}$$

where α is an unknown number between 0 and 1 whose estimated value is $\widetilde{\alpha}$, and $q(z)$ is an unknown probability density function which is non-zero only for $z > 1$.

Derive an explicit algebraic expression for the probability that an unacceptable system will be accepted. Denote this probability P_f. Let P_fc denote the greatest probability of false acceptance that we can tolerate. That is, we require:

$$P_\mathrm{f} \le P_\mathrm{fc} \tag{522}$$

when measuring an unacceptable system. Derive an explicit algebraic expression for the robustness for satisfying eq.(522).

(e) z is a normally distributed random variable whose value assesses a system. Let μ denote the mean of z. When the system is acceptable, $\mu = \mu_0$ where μ_0 is a known value. When the system is unacceptable, $\mu > \mu_0$. That is:

$$\text{Acceptable:} \qquad \mu = \mu_0 \tag{523}$$

$$\text{Unacceptable:} \qquad \mu > \mu_0 \tag{524}$$

We have a random sample of z values from a specific system: 3, 5, 2, 4, 6. The status of the system was constant during the sample. The value of μ_0 is 2. Use a statistical test to decide whether this random sample supports the hypothesis that the system is acceptable. Do you reject this hypothesis at the 0.05 level of significance?

(f) z is a categorical random variable for assessing the acceptability of a system. z can take 3 values: yes, no, maybe. For an acceptable system, the probabilities that z takes the values yes, no, maybe are $p_1 = 0.90$, $p_2 = 0.03$, $p_3 = 0.07$, respectively. We have a random sample of z values on a system whose status is constant during the sample. The number of yes, no, and maybe values of z in the sample are $n_1 = 86$, $n_2 = 5$, and $n_3 = 9$, respectively. Use a statistical test to decide whether this random sample supports the hypothesis that the system is acceptable. Do you reject this hypothesis at the 0.05 level of significance?

(g) A system can be in one of two states: either normal or failed. z is a random variable in the interval $[0,\ 1]$ obtained by measuring the system.

If the system is normal then the pdf of z is:

$$p(z|N) = 2(1-z) \tag{525}$$

If the system has failed then the pdf of z is:

$$p(z|F) = 2z \tag{526}$$

Note that $p(z|N)$ is biased to small values of z, while $p(z|F)$ is biased to large values.

A large value of z (in the interval $[0,\ 1]$) suggests that the system probably failed.

A small value of z (in the interval $[0,\ 1]$) suggests that the system probably is normal.

We have obtained a single measurement, whose value is z_1. We want to use this value to decide between two hypotheses: normal (H_0) and failed (H_1). Derive an explicit algebraic expression for the level of significance for the null hypothesis, H_0. If $z_1 = 0.8$, do you reject H_0 at the 0.025 level of significance?

(h) The uncertainty model for a real N-dimensional vector, c, is:

$$\mathcal{U}(h) = \left\{ c: \ (c - \widetilde{c})^T W^{-1} (c - \widetilde{c}) \le h^2 \right\}, \quad h \ge 0 \tag{527}$$

where W is a known, real, symmetric, positive definite matrix and $\widetilde{c}$ is a known vector. What is the maximum value of $c^T \widetilde{c}$ at horizon of uncertainty h?

103. **Multi-site adverse events** (based on exam in 035018, 26.9.2019), (p.350). Adverse events (fires, earthquakes, terror attacks, etc.) can occur simultaneously at N distinct sites (residential, industrial, commercial, etc.). Each site requires teams made up of a combination of various resources (fire fighters, policemen, psychologists, etc.). Historical data indicate that, on average, the number of teams required at site i is $\overline{q}_i$, for $i = 1, \ldots, N$. However, the actual average requirement at site i, q_i, is uncertain. You work in the emergency-response planning office. You are responsible for choosing the average number of teams, $q_i^\star$, that should be allocated to site i, for $i = 1, \ldots, N$. The uncertainty in the vector, q, of average allocation requirements is represented by this fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ q : \ q_i \ge 0, \ \left| \frac{q_i - \overline{q}_i}{s_i} \right| h, \ i = 1, \ldots, N \right\}, \quad h \ge 0 \tag{528}$$

where the $\overline{q}_i$'s and s_i's are known positive values.

(a) The performance requirement is:

$$\sum_{i=1}^{N} (q_i^\star - q_i) w_i \ge \delta \tag{529}$$

where the w_i's are known positive values and δ is a specified safety margin. Derive an explicit algebraic expression for the robustness to uncertainty in the required average allocation.

(b) Consider a different allocation requirement:

$$q_i^\star - q_i \ge \delta \quad \text{for each } i = 1, \ldots, N \tag{530}$$

where δ is a specified safety margin. Derive an explicit algebraic expression for the robustness to uncertainty in the required average allocation.

(c) Consider a different info-gap model for uncertainty:

$$\mathcal{U}(h) = \left\{ q : \ (q - \overline{q})^T V (q - \overline{q}) \le h^2 \right\}, \quad h \ge 0 \tag{531}$$

where V is a known, symmetric, real, positive definite matrix. Using the performance requirement in eq.(529), derive an explicit algebraic expression for the robustness to uncertainty in the required average allocation.

(d) ‡ Repeat part 103a with a single change. The w_i are known values, but some are positive (penalty for under-allocation) and some are negative (penalty for over-allocation). Derive an explicit algebraic expression for the robustness to uncertainty in the required average allocation.

(e) We now introduce a loss function, $\ell(q_i^\star)$, which represents the loss at site i from average allocation $q_i^\star$. We require that the total loss be no greater than a critical value, ℓ_{c}:

$$\sum_{i=1}^{N} \ell(q_i^\star) \le \ell_{\mathrm{c}} \tag{532}$$

Furthermore, there are two different policy options. For example, option 1 has more police but fewer psychologists, while option 2 is the reverse. However, the loss function for each policy is uncertain, according to this info-gap model for policy option j:

$$\mathcal{U}_j(h) = \left\{ \ell(q_i^\star) : \ \ell(q_i^\star) \ge 0, \ \left| \frac{\ell(q_i^\star) - \widetilde{\ell}_j(q_i^\star)}{v_j \widetilde{\ell}_j(q_i^\star)} \right| \le h \right\}, \quad h \ge 0, \quad j = 1, \ 2 \tag{533}$$

where $\widetilde{\ell}_j(q_i^\star)$ and v_j are known and positive. Also:

$$\sum_{i=1}^{N} \widetilde{\ell}_1(q_i^\star) \quad < \quad \sum_{i=1}^{N} \widetilde{\ell}_2(q_i^\star) \tag{534}$$

$$v_1 \sum_{i=1}^{N} \widetilde{\ell}_1(q_i^\star) \quad > \quad v_2 \sum_{i=1}^{N} \widetilde{\ell}_2(q_i^\star) \tag{535}$$

For what values of $\ell_{\rm c}$ is option 2 preferred, according to the method of robust-satisficing?

(f) The severity of fires is measured with a non-negative scalar variable s. For residential fires the probability density function (pdf) for s is exponential:

$$p(s) = \lambda \mathrm{e}^{-\lambda s}, \quad s \geq 0 \tag{536}$$

where $\lambda = 0.01$. For non-residential fires (e.g. industrial fires) s tends to take larger values than for residential fires. A specific fire had an observed positive value of severity, $s_{\rm o} = 500$. Formulate an explicit algebraic expression for the level of statistical significance to decide between the following two hypotheses regarding this fire:

$$H_0: \quad \text{Residential fire} \tag{537}$$

$$H_1: \quad \text{Non-residential fire} \tag{538}$$

Do you accept H_0 at the 0.025 level of significance?

104. **Targeting an evasive object** (Exam in 036057, 21.1.2020) (p.352). A target moves in 3 dimensions, and the vector of its coordinates is $\theta(t, c)$ where t is time and $c \in \Re^J$ is a vector of parameters.

(a) The best available estimate of the parameter vector is $\widetilde{c}$, whose elements are all positive. The uncertainty in this estimate is represented by this info-gap model:

$$\mathcal{U}(h) = \left\{ c: \ c_j > 0, \ \left| \frac{c_j - \widetilde{c}_j}{w_j} \right| \le h, \ \ j = 1, \ldots, J \right\}, \quad h \ge 0 \tag{539}$$

where the w_j's are known and positive. The location error function is:

$$L(c) = g^T c \tag{540}$$

where g is a known vector whose elements are all positive. We require that the location error function be no greater than ε. Explicitly, we require:

$$L(c) \le \varepsilon \tag{541}$$

Derive an explicit algebraic expression for the robustness function.

(b) We repeat part 104a with a different info-gap model. Instead of the fractional error info-gap model of eq.(539), we use the following ellipsoid-bound info-gap model:

$$\mathcal{U}(h) = \left\{ c: \ (c - \widetilde{c})^T \, W^{-1} \, (c - \widetilde{c}) \le h^2 \right\}, \quad h \ge 0 \tag{542}$$

where W is a real, symmetric, positive definite matrix. Derive an explicit algebraic expression for the robustness function. (We make a simplifying assumption, namely, that $\widetilde{c}$ is far enough from the origin so that $g^T c$ is non-negative for all relevant values of c. Recall that the elements of g are all positive.)

(c) We now modify the previous formulation, and assume that the location error function is a random variable, x:

$$L(c) = x \tag{543}$$

The estimated probability density function (pdf) of x is uniform:

$$\widetilde{p}(x) = \left\{ \begin{array}{ll} p_0, & \text{if } 0 \le x \le x_{\max} \\ 0, & \text{else} \end{array} \right. \tag{544}$$

where p_0 is a known positive number. The true pdf is uncertain, as specified by this info-gap model:

$$\mathcal{U}(h) = \left\{ p(x): \ p(x) \ge 0, \ \ \int_0^{x_{\max}} p(x) \, \mathrm{d}x = 1, \ \ \left| \frac{p(x) - \widetilde{p}(x)}{p_0} \right| \le h \right\}, \quad h \ge 0 \tag{545}$$

The system fails if the location error function exceeds a critical value, x_{c}:

$$L(c) > x_{\mathrm{c}} \tag{546}$$

where $\frac{x_{\max}}{2} < x_{\mathrm{c}} < x_{\max}$. Let $P_{\mathrm{f}}(p)$ denote the probability of failure if the pdf of x is $p(x)$. We require that $P_{\mathrm{f}}(p)$ be no greater than a critical value:

$$P_{\mathrm{f}}(p) \le P_{\mathrm{fc}} \tag{547}$$

Derive an explicit algebraic expression for the robustness to satisfy the requirement in eq.(547).

(d) We modify problem 104c as follows. The location error function is a random variable, x, as in eq.(543). The pdf of x is exponential:

$$\widetilde{p}(x|\lambda) = \lambda \mathrm{e}^{-\lambda x}, \quad x \geq 0 \tag{548}$$

However, the value of λ is uncertain, its best estimate is $\widetilde{\lambda}$, and the info-gap model for λ is:

$$\mathcal{U}(h) = \left\{ \lambda : \ \lambda \geq 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{\widetilde{\lambda}} \right| \leq h \right\}, \quad h \geq 0 \tag{549}$$

The system fails if the location error function exceeds a critical value, x_{c}, as in eq.(546). We require that the probability of failure be no greater than a critical value, P_{fc}:

$$P_{\mathrm{f}}(\lambda) \leq P_{\mathrm{fc}} \tag{550}$$

Derive an explicit algebraic expression for the robustness to satisfy the requirement in eq.(550).

(e) The best available estimate of the parameter vector is $\widetilde{c}$, and its uncertainty is represented by this info-gap model:

$$\mathcal{U}(h) = \left\{ c : \ (c - \widetilde{c})^T \, W^{-1} \, (c - \widetilde{c}) \leq h^2 \right\}, \quad h \geq 0 \tag{551}$$

where W is a known, real, symmetric, positive definite matrix. The true location at time t, $\theta(t, c)$, is related to the estimated location at time t, $\theta(t, \widetilde{c})$, by:

$$\theta(t, c) = \theta(t, \widetilde{c}) + G \, (c - \widetilde{c}) \tag{552}$$

where G is a known $3 \times J$ matrix. We require that the error of the estimate be no greater than ε. Explicitly, we require:

$$\| \theta(t, c) - \theta(t, \widetilde{c}) \| \leq \varepsilon \tag{553}$$

where $\| \cdot \|$ is the Euclidean norm. Derive an explicit algebraic expression for the robustness function.