12. **Investment with uncertain costs and returns.** (p.20) Consider the following project. The initial cost is S. The income is estimated to be $\widetilde{R}$ at the end of each year. The annual cost of operating the new system is estimated to be $\widetilde{C}$ at the end of each year. These estimates may err substantially. Use a fractional error info-gap model for costs and returns:

$$\mathcal{U}(h) \left\{ R, C : \left| \frac{R_k - \widetilde{R}_k}{\varepsilon_R \widetilde{R}_k} \right| \le h, \ \left| \frac{C_k - \widetilde{C}_k}{\varepsilon_C \widetilde{C}_k} \right| \le h, \ k = 1, \ldots, N \right\}, \quad h \ge 0 \tag{2}$$

The company's minimal acceptable rate of return (MARR) is i. The system will operate for N years.
(a) Derive the robustness function of the PW.
(b) Compare two realizations of this system with the following characteristics:

$$\mathrm{PW}(\widetilde{R}_1, \widetilde{C}_1) \ < \ \mathrm{PW}(\widetilde{R}_2, \widetilde{C}_2) \tag{3}$$

$$\varepsilon_{1,R}\widetilde{R}_1 + \varepsilon_{1,C}\widetilde{C}_1 \ < \ \varepsilon_{2,R}\widetilde{R}_2 + \varepsilon_{2,C}\widetilde{C}_2 \tag{4}$$

For what values of the PW do you prefer option 1? Provide an intuitive explanation of the results.

Solution to Problem 12, Investment with uncertain costs and returns, (p.4).

(a)

- S = Initial cost of the project.
- $\widetilde{R}$ = estimated revenue at the end of each period.
- $\widetilde{C}$ = estimated operating cost at the end of each period.
- N = number of periods.
- MARR = i.

The PW is:

$$PW(R,C) = -S + \sum_{k=1}^{N}(1+i)^{-k}R - \sum_{k=1}^{N}(1+i)^{-k}C \tag{64}$$

The robustness is defined as:

$$\widehat{h} = \max\left\{h : \left(\min_{R,C\in\mathcal{U}(h)} \mathrm{PW}(R,C)\right) \geq \mathrm{PW_c}\right\} \tag{65}$$

Let $m(h)$ denote the inner minimum, which occurs when:

$$R = \widetilde{R} - \varepsilon_R\widetilde{R}h = \widetilde{R}(1-\varepsilon_R h), \quad C = \widetilde{C} + \varepsilon_C\widetilde{C}h = \widetilde{C}(1+\varepsilon_C h) \tag{66}$$

Thus:

$$\begin{aligned} m(h) &= -S + \sum_{k=1}^{N}(1+i)^{-k}\widetilde{R}(1-\varepsilon_R h) - \sum_{k=1}^{N}(1+i)^{-k}\widetilde{C}(1+\varepsilon_C h) && (67)\\ &= \mathrm{PW}(\widetilde{R},\widetilde{C}) - (\varepsilon_R\widetilde{R} + \varepsilon_C\widetilde{C})h\sum_{k=1}^{N}(1+i)^{-k} && (68)\\ &= \mathrm{PW}(\widetilde{R},\widetilde{C}) - (\varepsilon_R\widetilde{R} + \varepsilon_C\widetilde{C})h\frac{1-(1+i)^{-N}}{i} && (69)\end{aligned}$$

Equate this to $\mathrm{PW_c}$ and solve for h to obtain the robustness:

$$\widehat{h} = \frac{\mathrm{PW}(\widetilde{R},\widetilde{C}) - \mathrm{PW_c}}{\varepsilon_R\widetilde{R} + \varepsilon_C\widetilde{C}}\,\frac{i}{1-(1+i)^{-N}} \tag{70}$$

or zero if this is negative.

(b) The conditions of eqs.(3) and (4) cause the robustness curves of the two options to cross, where option 1 is nominally worse but with lower cost of robustness (steeper robustness curve). Thus, the robustness criterion prefers option 1 when $\mathrm{PW_c}$ is less than the value, $\mathrm{PW}_\times$, at which the robustness curves cross one another. This is obtained by equating the robustness functions:

$$\widehat{h}_1 = \widehat{h}_2 \implies \frac{\mathrm{PW}(\widetilde{R}_1,\widetilde{C}_1) - \mathrm{PW}_\times}{\varepsilon_{1,R}\widetilde{R}_1 + \varepsilon_{1,C}\widetilde{C}_1} = \frac{\mathrm{PW}(\widetilde{R}_2,\widetilde{C}_2) - \mathrm{PW}_\times}{\varepsilon_{2,R}\widetilde{R}_2 + \varepsilon_{2,C}\widetilde{C}_2} \tag{71}$$

$$\implies \mathrm{PW}_\times = \frac{\widetilde{\mathrm{PW}}_1 E_2 - \widetilde{\mathrm{PW}}_2 E_1}{E_2 - E_1} \tag{72}$$

where we have defined:

$$E_i = \varepsilon_{i,R}\widetilde{R}_i + \varepsilon_{i,C}\widetilde{C}_i, \quad \widetilde{\mathrm{PW}}_i = \mathrm{PW}(\widetilde{R}_i,\widetilde{C}_i) \tag{73}$$

In summary:

$$\text{option 1} \succ \text{option 2} \quad \text{if} \quad \mathrm{PW_c} < \mathrm{PW}_\times \tag{74}$$