13. **Investment with uncertain probabilistic returns.** (p.39) Consider the following project. The initial cost is S. The income is R at the end of each year. The annual cost of operating is C at the end of each year. The company's minimal acceptable rate of return (MARR) is i. The system will operate for N years.

(a) Suppose that R is the same each year but that its value is a random variable with an exponential distribution: $p(R) = \lambda \mathrm{e}^{-\lambda R}$, for $R \geq 0$. Derive an expression for the probability of "failure": PW less than the critical value PW_{c}.

(b) Suppose that R is the same each year but that its value is a random variable with an exponential distribution as in part (a), and that λ is uncertain. The estimated value is $\widetilde{\lambda}$, but the fractional error of this estimate is unknown. We require that the probability of failure not exceed a critical value P_{fc}. Derive the robustness function.

(c)‡ Suppose that R is the same each year but that its value is a random variable whose distribution is thought to be exponential with coefficient $\widetilde{\lambda}$. However, the absolute error of this probability density is unknown. Consider the special case that the estimated probability of failure is much less than 1. Derive the robustness function for avoiding failure.

Solution to Problem 13, Investment with uncertain probabilistic returns, (p.9).

(a)

• $S =$ Initial cost of the project.
• $R =$ estimated revenue at the end of each period.
• $C =$ estimated operating cost at the end of each period.
• $N =$ number of periods.
• MARR = i.

The PW is:

$$PW(R) = -S + \sum_{k=1}^{N}(1+i)^{-k}R - \sum_{k=1}^{N}(1+i)^{-k}C \tag{87}$$

$$= -S + (R-C)\underbrace{\frac{1-(1+i)^{-N}}{i}}_{\eta} \tag{88}$$

The probability of failure is:

$$P_{\rm f} = \text{Prob}(\text{PW} \le \text{PW}_{\rm c}) \tag{89}$$

$$= \text{Prob}(-S + (R-C)\eta \le \text{PW}_{\rm c}) \tag{90}$$

$$= \text{Prob}\left(R \le \frac{\text{PW}_{\rm c} + S + \eta C}{\eta}\right) \tag{91}$$

$$= \text{Prob}\,(R \le R_{\rm c}) \quad \text{(which defines } R_{\rm c}) \tag{92}$$

$$= \int_0^{R_{\rm c}} p(R)\,{\rm d}R \tag{93}$$

$$= 1 - {\rm e}^{-\lambda R_{\rm c}} \tag{94}$$

(b) The info-gap model for uncertain exponential distribution is:

$$\mathcal{U}(h) = \left\{p(R) = \lambda{\rm e}^{-\lambda R}:\ \lambda \ge 0,\ \left|\frac{\lambda-\widetilde{\lambda}}{\widetilde{\lambda}}\right| \le h\right\},\quad h \ge 0 \tag{95}$$

The robustness is defined as:

$$\widehat{h} = \max\left\{h:\ \left(\max_{p\in\mathcal{U}(h)} P_{\rm f}(p)\right) \le P_{\rm fc}\right\} \tag{96}$$

Let $m(h)$ denote the inner maximum, which occurs when λ is as large as possible at horizon of uncertainty h: $\lambda = (1+h)\widetilde{\lambda}$. Thus:

$$m(h) = 1 - \exp\left[-(1+h)\widetilde{\lambda}R_{\rm c}\right] \tag{97}$$

We require:

$$1 - \exp\left[-(1+h)\widetilde{\lambda}R_{\rm c}\right] \le P_{\rm fc} \tag{98}$$

Solve for h to obtain the robustness:

$$\widehat{h}(P_{\rm fc}) = -1 - \frac{\ln(1-P_{\rm fc})}{\widetilde{\lambda}R_{\rm c}} \tag{99}$$

We see that $\widehat{h}(P_{\rm fc}) = 0$ when $P_{\rm fc} = P_{\rm f}(\widetilde{p})$. Also, $\widehat{h} > 0$ when $P_{\rm fc} > P_{\rm f}(\widetilde{p})$.

(c) Let $\mathcal{P}$ denote the set of all mathematically legitimate pdf's. The info-gap model is:

$$\mathcal{U}(h) = \{p(R):\ p(R)\in\mathcal{P},\ |p(R)-\widetilde{p}(R)| \le h\},\quad h \ge 0 \tag{100}$$

The robustness is the same as eq.(96) with the new info-gap model.

The estimated probability of failure, from eq.(94), is:

$$P_{\rm f}(\widetilde{p}) = 1 - {\rm e}^{-\widetilde{\lambda} R_{\rm c}} \tag{101}$$

We are told that $P_{\rm f}(\widetilde{p}) \ll 1$, so:

$$R_{\rm c} \ll 1/\widetilde{\lambda} \tag{102}$$

Thus, from eq.(93), we see that the inner maximum, $m(h)$, occurs when $p(R)$ is a large as possible in the interval $0 \le R \le R_{\rm c}$:

$$p(R) = \widetilde{p}(R) + h, \quad R \le R_{\rm c} \tag{103}$$

Because of eq.(102) we will be able to normalize this pdf by reducing the tail for $R > R_{\rm c}$.

Now we find that:

$$m(h) = \int_0^{R_{\rm c}} (\widetilde{p}(R) + h)\, {\rm d}R = \underbrace{1 - {\rm e}^{-\widetilde{\lambda} R_{\rm c}}}_{P_{\rm f}(\widetilde{p})} + h\, R_{\rm c} \tag{104}$$

Equating this to $P_{\rm fc}$ and solving for h yields the robustness:

$$P_{\rm f}(\widetilde{p}) + h\, R_{\rm c} = P_{\rm fc} \quad \Longrightarrow \quad \widehat{h}(P_{\rm fc}) = \frac{P_{\rm fc} - P_{\rm f}(\widetilde{p})}{R_{\rm c}} \tag{105}$$

or zero if this is negative.