15. **Benefit-cost ratio of two design concepts.** (p.42) Consider two design concepts. For both concepts the benefit and maintenance cost at the end of each year is \$4,000 and \$1,500, respectively. The interest rate is $i = 0.05$.

    (a) Evaluate the benefit-cost ratios of the two design concepts, where the anticipated usable lifetimes of designs 1 and 2 are $N_1 = 3$ years and $N_2 = 5$ years. The initial investments in designs 1 and 2 are $S_1 = \$20,000$ $S_2 = \$33,333$. Which design has a better benefit-cost ratio (BCR)? Why?

    (b) For what ratio of initial investment is the BCR the same for the two designs? What does this imply about the initial costs of the two designs?

Solution to Problem 15, Benefit-cost ratio of two design concepts, (p.11).

(a) Present worth of the benefits of design j are:

$$\begin{aligned} B_{pw}(j) &= \sum_{n=1}^{N_j}(1+i)^{-n}B && (113)\\ &= \frac{1-(1+i)^{-N_j}}{i}B && (114)\\ &= \delta_{fj}(i)B && (115)\end{aligned}$$

Present worth of the initial investment and maintenance costs of design j are:

$$\begin{aligned} C_{pw}(j) &= S+\sum_{n=1}^{N_j}(1+i)^{-n}C && (116)\\ &= S_j+\frac{1-(1+i)^{-N_j}}{i}C && (117)\\ &= S+\delta_{fj}(i)C && (118)\end{aligned}$$

The BCR of design j is:

$$\begin{aligned} \text{BCR}(j) &= \frac{B_{pw}(j)}{C_{pw}(j)} && (119)\\ &= \frac{\delta_{fj}(i)B}{S_j+\delta_{fj}(i)C} && (120)\end{aligned}$$

The discount factors for the two designs are:

$$\delta_{f1}(i)=2.7232, \quad \delta_{f2}(i)=4.3295 \tag{121}$$

δ_{f1} is less than δ_{f2} because $N_1<N_2$. However, the ratio is greater:

$$\frac{\delta_{f1}}{\delta_{f2}}=0.6290>0.6=\frac{N_1}{N_2} \tag{122}$$

The reason: later periods get less weight than earlier periods.

The BCRs of the two designs are:

$$\begin{aligned} \text{BCR}(1) &= \frac{2.7232\times 4,000}{20,000+2.7232\times 1,500}=0.4523 && (123)\\ \text{BCR}(2) &= \frac{4.3295\times 4,000}{33,333+4.3295\times 1,500}=0.4348 && (124)\end{aligned}$$

Both the benefits and the costs are lower for design 1 than for design 2, but the BCRs are nearly the same. Nonetheless, design 1 has a better (higher) ratio even though the costs and benefits each period are the same and the initial investment per year is the same ($S_1/N_1=S_2/N_2$). The reason: The benefits over 5 years from design 2 are discounted disproportionately, as shown in eq.(122). Even though the initial investments look the same for both designs, they are not because future benefits are discounted.

(b) Equate the BCRs and solve for the ratio S_1/S_2:

$$\begin{aligned} \text{BCR}(1) &= \text{BCR}(2) \implies \frac{\delta_{f1}B}{S_1+\delta_{f1}C}=\frac{\delta_{f2}B}{S_2+\delta_{f2}C}\implies (S_2+\delta_{f2}C)\delta_{f1}=(S_1+\delta_{f1}C)\delta_{f2} && (125)\\ \implies \frac{S_1}{S_2} &= \frac{\delta_{f1}}{\delta_{f2}} && (126)\end{aligned}$$

The BCRs will be the same if the ratio of initial investments, S_1/S_2, equals the ratio δ_{f1}/δ_{f2}. However, the initial investments are proportional to the lifetimes, $S_1/S_2 = N_1/N_2 = 3/5 = 0.6$, and the discount functions are not: $\delta_{f1}/\delta_{f2} = 0.6290$. In other words, design 1 has a better BCR than design 2 because design 1 is under-priced relative to the initial cost of design 2. Its "discounted fair price", resulting in the same BCR for both designs, would be:

$$S_1 = \frac{\delta_{f1}}{\delta_{f2}} S_2 = 0.629 \times \$33,333 = \$20,966 > 20,000 \tag{127}$$

Design 1 has a higher benefit-to-cost ratio because its initial cost is low; it is a better buy: more benefit per dollar (of initial investment and discounted maintenance cost).