23. **Project financing.** (p.82)

(a) You are analyzing the financing of a project with 3 stages: (1) development, (2) implementation, and (3) operation. The 3 stages are performed in sequence: stage $n+1$ begins when stage n ends. The project starts at time $t=0$. The known duration (years) and known cost (in $t=0$ shekels) of stage n are t_n and c_n. The annual inflation rate is f. You will take a loan at the start of each stage to cover the cost (in current shekels) of that stage. The nominal interest rate of each loan is i_{nom}. You will pay back the entire cumulative loan t_4 years after completion of stage 3. Derive an explicit algebraic expression for the nominal shekel amount you will pay. What is the real shekel value of this payment (in the value of the shekel at the start of the project, $t=0$)?

(b) We now modify part (23a). At the end of each year during the 3rd stage—operation—you will receive income I_3 nominal shekels per year. This income is invested at interest i_{nom} until it is time to pay back the loans. There is no income during the first two stages. Everything else is the same as in part (23a). What is the nominal value of the cumulative income when you repay the loans? Assume all durations are integer years.

(c) We now return to part (23a). You could out-source the first stage—development—to a European country with currency in euros. The known duration (years) and cost (euros) of this first stage would be τ_1 and γ_1. The exchange rate is r shekels/euro. The inflation rates in euros and in shekels are f_{euro} and f_{nis}. Financing is done in shekels. You will take a loan in Israel, in current nominal shekels, at the start of each stage to cover the cost of that stage. The nominal shekel interest rate of each loan is i_{nom}. You will pay back the entire cumulative loan t_4 years after completion of stage 3. Derive an explicit algebraic expression for the nominal shekel amount you pay. What is the real shekel value of this payment (in the value of the shekel at the start of the project)?

(d) We now return to part (23a) and suppose that the costs, c_n, are uncertain. The estimated costs are $\widetilde{c}_n$ and the errors of these estimates are w_n. Use the following info-gap model:

$$\mathcal{U}(h)=\left\{c:\ \left|\frac{c_n-\widetilde{c}_n}{w_n}\right|\le h,\ n=1,\,2,\,3\right\},\quad h\ge 0 \tag{7}$$

We require that the real shekel value of the loan payment, in $t=0$ shekels, be no greater than the critical value R_{c}. Derive an explicit algebraic expression for the robustness to uncertainty in the project costs.

(e) We now return to part (23a) and suppose that the inflation rate, f, is constant but uncertain. The estimated inflation rate is $\widetilde{f}$ and the error of this estimate is s. Use the following info-gap model:

$$\mathcal{U}(h)=\left\{f:\ \left|\frac{f-\widetilde{f}}{s}\right|\le h\right\},\quad h\ge 0 \tag{8}$$

We require that the real shekel value of the loan payment, in $t=0$ shekels, be no greater than the critical value R_{c}. The robustness to uncertainty in the inflation is a function of the critical value: $\widehat{h}(R_{\text{c}})$. Derive an explicit algebraic expression for the inverse of the robustness function.

(f) We now return to part (23a) and suppose that the cost of stage 1, the development stage, is a normal random variable with mean $\widetilde{c}_1$ and variance σ_1^2. All other variables are known. The project owner will default on the loan if the real shekel value of the loan payment, in

$t = 0$ shekels, is greater than the critical value R_c. Derive an explicit analytical expression for the probability of default. Let $\Phi(z)$ denote the cumulative probability distribution function for a standard normal random variable z (zero-mean; unit variance).

Solution to Problem 23, Project financing (p.20).

(23a) Define the following quantities:

$t_0 = 0 =$ start time.

$t_n =$ duration of the nth stage, for $n = 1,\, 2,\, 3$.

$\theta_n = \sum_{j=1}^{n-1} t_j =$ the year in which stage n starts. Thus: $\theta_1 = 0,\ \ \theta_2 = t_1,\ \ \theta_3 = t_1 + t_2$.

$T_n = \sum_{j=n}^{4} t_j =$ duration from start of stage n to t_4 years after the end of the project. Thus:

$T_1 = t_1 + t_2 + t_3 + t_4,\ \ T_2 = t_2 + t_3 + t_4,\ \ T_3 = t_3 + t_4,\ \ T_4 = t_4$

Note that $T_1 = \theta_n + T_n$.

Stage n starts θ_n years after the beginning of the project. The cost of stage n, in $t = 0$ shekels, is c_n. This cost inflates at the annual rate f. Thus at the start of stage n we take a loan of $(1+f)^{\theta_n} c_n$ shekels. We pay this loan at nominal interest rate $i_{\rm nom}$ after T_n years. This nominal payment is:

$$A_n = (1 + i_{\rm nom})^{T_n} (1 + f)^{\theta_n} c_n \tag{278}$$

The total nominal payment is:

$$A = \sum_{n=1}^{3} A_n = \sum_{n=1}^{3} (1 + i_{\rm nom})^{T_n} (1 + f)^{\theta_n} c_n \tag{279}$$

The real shekel value, in project-start-time ($t = 0$) shekels, is:

$$R = (1 + f)^{-T_1} A \tag{280}$$

$$= (1 + f)^{-T_1} \sum_{n=1}^{3} (1 + i_{\rm nom})^{T_n} (1 + f)^{\theta_n} c_n \tag{281}$$

$$= \sum_{n=1}^{3} (1 + i_{\rm nom})^{T_n} (1 + f)^{\theta_n - T_1} c_n \tag{282}$$

$$= \sum_{n=1}^{3} (1 + i_{\rm nom})^{T_n} (1 + f)^{-T_n} c_n \tag{283}$$

$$= \sum_{n=1}^{3} \left(\frac{1 + i_{\rm nom}}{1 + f} \right)^{T_n} c_n \tag{284}$$

(23b) Nominal income I_3 is received at the end of each of t_3 years, invested at interest $i_{\rm nom}$ and held an additional t_4 years after completion:

End of year 1: I_3 invested for $t_3 - 1 + t_4$ years $\Longrightarrow$ $(1 + i_{\rm nom})^{t_3 - 1 + t_4} I_3$ when loan repaid.

End of year 2: I_3 invested for $t_3 - 2 + t_4$ years $\Longrightarrow$ $(1 + i_{\rm nom})^{t_3 - 2 + t_4} I_3$ when loan repaid.

...

End of year t_3: I_3 invested for $t_3 - t_3 + t_4$ years $\Longrightarrow$ $(1 + i_{\rm nom})^{t_3 - t_3 + t_4} I_3$ when loan repaid.

Hence the total nominal cumulative income at repayment is:

$$I = I_3 \sum_{j=1}^{t_3} (1 + i_{\rm nom})^{t_3 - j + t_4} \tag{285}$$

(23c) The solution is the same as part (23a) **except** that we must calculate c_1 from the euro cost γ_1 as: $c_1 = r\gamma_1$. Now the solution is the same eqs.(279) and (284) with t_1 replaced by τ_1.

(23d) The definition of the robustness is:

$$\widehat{h}(R_{\rm c}) = \max \left\{ h: \ \left(\max_{c \in \mathcal{U}(h)} R(c) \right) \le R_{\rm c} \right\} \tag{286}$$

where $R(c)$ is the real shekel cost of the project from eq.(284). Let $m(h)$ denote the inner maximum, which is the inverse of the robustness. This maximum occurs when $c_n = \widetilde{c}_n + hw_n$. From eq.(284), where we define the coefficient of c_n in that expression as b_n, we find:

$$m(h) = \sum_{n=1}^{3} b_n(\widetilde{c}_n + hw_n) = R(\widetilde{c}) + hR(w) \le R_{\rm c} \quad \Longrightarrow \quad \widehat{h}(R_{\rm c}) = \frac{R_{\rm c} - R(\widetilde{c})}{R(w)} \tag{287}$$

or zero if this is negative.

(23e) The definition of the robustness is:

$$\widehat{h}(R_{\rm c}) = \max \left\{ h: \ \left(\max_{f \in \mathcal{U}(h)} R(f) \right) \le R_{\rm c} \right\} \tag{288}$$

where $R(f)$ is the real shekel cost of the project from eq.(284). Let $m(h)$ denote the inner maximum, which is the inverse of the robustness. This maximum occurs when $f = \widetilde{f} - hs$. From eq.(284) we find:

$$m(h) = \sum_{n=1}^{3} \left(\frac{1 + i_{\rm nom}}{1 + \widetilde{f} - hs} \right)^{T_n} c_n \tag{289}$$

This is the inverse of the robustness function, $\widehat{h}(R_{\rm c})$.

(23f) The probability of default is defined as:

$$P_{\rm d} = \text{Prob}(R \ge R_{\rm c}) \tag{290}$$

From eq.(284), where we define the coefficient of c_n in that expression as b_n, we find:

$$R = \sum_{n=1}^{3} b_n c_n = b_1 c_1 + g \tag{291}$$

where defines g whose value is known. Thus the probability of default becomes:

$$P_{\rm d} = \text{Prob}(b_1 c_1 + g \ge R_{\rm c}) = \text{Prob}\left(c_1 \ge \frac{R_{\rm c} - g}{b_1} \right) = \text{Prob}\left(\frac{c_1 - \widetilde{c}_1}{\sigma_1} \ge \frac{R_{\rm c} - g}{b_1 \sigma_1} - \frac{\widetilde{c}_1}{\sigma_1} \right) \tag{292}$$

$$= 1 - \Phi\left(\frac{R_{\rm c} - g}{b_1 \sigma_1} - \frac{\widetilde{c}_1}{\sigma_1} \right) \tag{293}$$