35. **Artistic restoration.** (based on exam, 19.7.2015) (p.96) You have developed a process for restoring faded paintings. The process is slow and takes time. The annual cost is $\$A$ which is borrowed at annual interest i_c. There is no inflation. The curator is delighted, but would have preferred instantaneous restoration. The social benefit of the fully restored paintings will be R. The social benefit today is less if the painting is restored in the future. This can be approximated by discounting the social benefit at the rate i_b, compounded annually.

    (a) One version of the process acts simultaneously on all paintings that are fully restored after N years, and have no value until then. Derive expressions for:

    - The discounted present benefit of the paintings restored after N years.
    - The discounted present worth of the annual cost of the process over the N years.
    - The benefit cost ratio.

    (b) A different version of the process fully restores a fraction $1/N$ of the paintings each year, which obtain their full social value when restored. The annual cost of the process is $\$A$ as before. Derive expressions for:

    - The discounted benefit worth of the paintings restored yearly during N years.
    - The discounted present worth of the annual cost of the process over the N years.
    - The benefit cost ratio.

    (c) Compare the BCR's of the two options. Which is better, and what is the time-value explanation for the difference?

36. **Earnings and investments.** (based on exam, 15.10.2015) (p.97)

(a) You will earn F dollars at the end of 5 years. What must F equal if its present worth is \$100,000 and the annual interest rate is 0.07? There is no inflation.

(b) You will earn $[1 + 0.1(k - 1)]F$ dollars at the end of year k for $k = 1, 2, 3$. What must F equal if the sum of the present worth of this income stream is \$250,000? The annual interest rate is 0.03. There is no inflation.

(c) Your real earnings will be 50,000 peso at the end of each year for 3 years. You invest these earnings, at the end of each year, at 5% nominal annual interest. At the end of 3 years you withdraw the total peso balance and exchange it for dollars at 30 pesos/dollar. What is the dollar sum you obtain? The nominal annual interest rate for dollars is 0.06. The inflation rates for pesos and dollars are 0.07 and 0.03 respectively.

(d) We now generalize problem 36b as follows. You will earn $[1 + (k - 1)x]F$ (F is positive) at the end of year k, for $k = 1, \ldots, N$. The value of x is estimated at $\tilde{x}$, but this could err by $\pm s$ or more. Represent the uncertainty in x with this info-gap model:

$$\mathcal{U}(h) = \left\{ x : \ \left| \frac{x - \tilde{x}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{22}$$

We require that the present worth of the sum of the income stream must be no less than PW_c. The annual interest rate is i. There is no inflation. Derive an explicit algebraic expression for the robustness function.

(e) We continue part 36d for the special case that the parameter values are those of part 36b: $N = 3$, $i = 0.03$ and $\tilde{x} = 0.1$. Let F_{36b} denote the answer to part 36b, and choose $F = F_{36b}$. What is the numerical value of the robustness if $\text{PW}_\text{c} = 250,000$?

Solution to Problem 35, Artistic restoration (p.28).

(35a) The discounted PW of the benefit, B, and the cost, C, and the BCR, are:

$$B = (1+i_b)^{-N}R \tag{386}$$

$$C = \sum_{k=1}^{N}(1+i_c)^{-k}A = \delta_f(i_c)A \tag{387}$$

$$\text{BCR}_1 = \frac{B}{C} = \frac{(1+i_b)^{-N}R}{\delta_f(i_c)A} \tag{388}$$

(35b) The discounted PW of the benefit, B, and the cost, C, and the BCR, are:

$$B = \sum_{k=1}^{N}(1+i_b)^{-k}\frac{R}{N} = \delta_f(i_b)\frac{R}{N} \tag{389}$$

$$C = \sum_{k=1}^{N}(1+i_c)^{-k}A = \delta_f(i_c)A \tag{390}$$

$$\text{BCR}_2 = \frac{B}{C} = \frac{\delta_f(i_b)R}{\delta_f(i_c)AN} \tag{391}$$

(35c) The ratio of the BCR of the 2nd to the 1st option is:

$$\frac{\text{BCR}_2}{\text{BCR}_1} = \frac{\delta_f(i_b)}{N(1+i_b)^{-N}} = \frac{\sum_{k=1}^{N}(1+i_b)^{-k}}{N(1+i_b)^{-N}} > \frac{N(1+i_b)^{-N}}{N(1+i_b)^{-N}} = 1 \tag{392}$$

The economic or time-value reason that the 2nd option has greater BCR is that its benefit is discounted less due to benefits accruing throughout the restoration period, while its cost has the same time profile.

Solution to Problem 36, Earnings and investments (p.29).

(36a) Present and future worth are related as $\text{PW} = (1+i)^{-k}F$, so F must equal:

$$F = (1+i)^k\text{PW} = 1.07^5 \times 100{,}000 = \boxed{140{,}255.17} \tag{393}$$

(36b)

Year $k = 1$: F is earned at the end of year 1. The present worth of this is:

$$\text{PW}_1 = (1+i)^{-1}F = 1.03^{-1}F = 0.97087F \tag{394}$$

Year $k = 2$: $1.1F$ is earned at the end of year 2. The present worth of this is:

$$\text{PW}_2 = (1+i)^{-2}1.1F = 1.03^{-2} \times 1.1F = 1.036855F \tag{395}$$

Year $k = 3$: $1.2F$ is earned at the end of year 3. The present worth of this is:

$$\text{PW}_3 = (1+i)^{-3}1.2F = 1.03^{-3} \times 1.2F = 1.0981699F \tag{396}$$

Thus:

$$\text{PW} = \text{PW}_1 + \text{PW}_2 + \text{PW}_3 = 3.10589F = 250{,}000 \quad \Longrightarrow \quad \boxed{F = 80{,}492.10} \tag{397}$$

(36c) Consider year k. The real earnings at the end of the year are $F = 50{,}000$ pesos. The annual peso inflation rate is $f_p = 0.07$. Thus the nominal peso earnings at the end of year k are $(1+f_p)^kF$. This is invested for $N-k$ years, $N = 3$, at nominal annual interest of $i = 0.05$. Thus the nominal peso value of the year-k earnings, at the end of year N, is:

$$A_{k,p} = (1+i)^{N-k}(1+f_p)^kF \tag{398}$$

This is exchanged to dollars at the rate of r pesos/dollar, so the nominal dollar value, at the end of year N, of the earnings from year k, are:

$$A_{k,d} = A_{k,p}/r = (1+i)^{N-k}(1+f_p)^kF/r \tag{399}$$

These yearly nominal dollar values are:

$$A_{1,d} = 1.05^2 \times 1.07^1 \times 50{,}000/30 = 1{,}966.13 \tag{400}$$

$$A_{2,d} = 1.05^1 \times 1.07^2 \times 50{,}000/30 = 2{,}003.58 \tag{401}$$

$$A_{3,d} = 1.05^0 \times 1.07^3 \times 50{,}000/30 = 2{,}041.74 \tag{402}$$

The total nominal dollar value obtained at the end of year N is:

$$A_d = A_{1,d} + A_{2,d} + A_{3,d} = \boxed{6{,}011.45} \tag{403}$$

(36d) The present worth of the earnings of year k is:

$$\text{PW}_k = (1+i)^{-k}[1+(k-1)x]F \tag{404}$$

The present worth of the sum of the earnings is:

$$\text{PW} = \sum_{k=1}^{N}\text{PW}_k = \sum_{k=1}^{N}(1+i)^{-k}[1+(k-1)x]F = F\underbrace{\sum_{k=1}^{N}(1+i)^{-k}}_{P_0} + xF\underbrace{\sum_{k=1}^{N}(k-1)(1+i)^{-k}}_{P_1} \tag{405}$$

$$= P_0F + P_1Fx \tag{406}$$

The definition of the robustness is:

$$\widehat{h} = \max \left\{ h : \ \left(\min_{x \in \mathcal{U}(h)} \text{PW} \right) \geq \text{PW}_\text{c} \right\} \tag{407}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function, and occurs for $x = \widetilde{x} - sh$:

$$m(h) = P_0 F + P_1 F(\widetilde{x} - sh) \geq \text{PW}_\text{c} \quad \Longrightarrow \quad \boxed{\widehat{h} = \frac{P_0 F + P_1 F \widetilde{x} - \text{PW}_\text{c}}{P_1 F s} = \frac{\text{PW}(\widetilde{x}) - \text{PW}_\text{c}}{P_1 F s}} \tag{408}$$

or zero if this is negative.

(36e) With $F = F_{36b}$ we know that PW$(\widetilde{x}) = 250,000$. Hence $\widehat{h}(\text{PW}_\text{c} = 250,000) = 0$ by the zeroing theorem.