Problem Set on

Info-Gap Risks in Project Management

List of problems:

[0] \lectures\p_mgt\p-mgt-hw02.tex 19.6.2022

1. **Planning for uncertain costs.** Consider a project with N tasks whose costs are $c = (c_1, \ldots, c_N)^T$ and which are initiated at times $t = (t_1, \ldots, t_N)^T$. The total project duration is T. The money needed for each task must be available when that task is initiated, and will be funded by a loan at interest rate r. The loan will be repaid at the end of the project. The total project cost, including interest, is:

$$C(c,t) = \sum_{i=1}^{N} c_i \mathrm{e}^{r(T-t_i)} \tag{1}$$

We have very limited information about the task costs. We have estimates from the past of the mean costs and the variances and co-variances of the costs. The historical mean vector and covariance matrix are $\widetilde{c}$ and V. We will consider two different info-gap models for uncertainty in the task costs.

1st info-gap model. We will use an ellipsoid-bound info-gap model to represent uncertain future realizations of the cost vector:

$$\mathcal{U}(h,\widetilde{c}) = \left\{ c: \ (c-\widetilde{c})^T V^{-1} (c-\widetilde{c}) \le h^2 \right\}, \quad h \ge 0 \tag{2}$$

2nd info-gap model. We assume that the covariances are all zero (or we simply ignore covariance information), so V is a diagonal matrix of variances v_i. The fractional-error info-gap model is:

$$\mathcal{U}(h,\widetilde{c}) = \left\{ c: \ |c_i - \widetilde{c}_i| \le h\sqrt{v_i}, \ i = 1, \ldots, N \right\}, \quad h \ge 0 \tag{3}$$

(a) Use the info-gap model of eq.(2) to derive an expression for the robustness, to uncertainty in task costs, of task-initiation times t. The manager aspires to keep the total project cost below the critical value C_{f}.

(b) Now suppose that the covariance matrix V is diagonal with elements $v_1, \ldots, v_N$. Use the results of question 1a to indicate how one might wish to choose the task-initiation times t.

(c) Repeat question 1a with the info-gap model of eq.(3).

(d) Repeat question 1b with the info-gap model of eq.(3).

2. **Adaptive planning for uncertain costs.** In problem 1 we considered planning task-initiation times given task-cost uncertainty and budget constraint. We included the effect of time by considering the net present value (at the time of planning) of future expenditures. Now modify that problem to consider the adaptive determination, “on-line” or in “real time”, of the task-initiation times. That is, at any moment of time *during* implementation of the project, a certain number of tasks have already been funded, a certain amount of the total budget has already been spent, a certain fraction of the total project time T remains, and the remaining tasks must be implemented with the remaining budget. Using the info-gap models of eqs.(2) or (3), formulate a robustness function for the adaptive determination of task-initiation time.

3. **Budget allocation.** Consider a project with N tasks. After budget allocation for basic operational needs of the N tasks, a quantity Q of money remains for allocation among the tasks, on the basis of their anticipated contribution to project profitability. The anticipated rate of return of each additional dollar allocated to task i is $\widetilde{u}_i$, for $i = 1, \ldots, N$. Thus an allocation q is anticipated to result in added revenue:

$$R(q, \widetilde{u}) = \sum_{i=1}^{N} q_i \widetilde{u}_i \tag{4}$$

The allocation q has succeeded if the added revenue is no less than R_c. However, the anticipated return vector $\widetilde{u}$ is highly uncertain. The actual rates of return, u_i, are unknown, and the uncertainty in the vector of rates of return u is denoted by an info-gap model, $\mathcal{U}(h, \widetilde{u})$. Consider two info-gap models:

$$\mathcal{U}(h, \widetilde{u}) = \{u : \ |u_i - \widetilde{u}_i| \le h w_i, \ i = 1, \ldots, N\}, \quad h \ge 0 \tag{5}$$

where the uncertainty weights w_i are known positive numbers, and:

$$\mathcal{U}(h, \widetilde{u}) = \left\{u : \ (u - \widetilde{u})^T V^{-1} (u - \widetilde{u}) \le h^2 \right\}, \quad h \ge 0 \tag{6}$$

where V^{-1} is a known, positive definite, real, symmetric matrix.

Assume that no allocations are negative, so $q_i \ge 0$ for all i.

(a) Use the info-gap model of eq.(5) to suggest how one might go about choosing an allocation of Q among the tasks, based on the robustness function.

(b) Repeat question 3a based on the opportuneness function.

(c) Repeat question 3a based on the info-gap model of eq.(6).

(d) ‡ Now suppose that we have an estimated joint probability density (pdf) for the rates of return, $\widetilde{p}(u)$. Specifically, suppose that the estimated pdf of the revenue from the ith task is normal with mean μ_i and variance σ_i^2, and that the revenues of different tasks are uncorrelated statistically:

$$\widetilde{p}_i(u_i) \sim \mathcal{N}(\widetilde{\mu}_i, \widetilde{\sigma}_i^2), \quad i = 1, \ldots, N \tag{7}$$

$$\widetilde{p}(u) = \prod_{i=1}^{N} \widetilde{p}_i(u_i) \tag{8}$$

The moments $\widetilde{\mu}_i$ and $\widetilde{\sigma}_i^2$ are highly uncertain, and the uncertainty is represented by an info-gap model:

$$\mathcal{U}(h, \widetilde{p}_i) = \left\{ \mu_i, \sigma_i : \ \left| \frac{\mu_i - \widetilde{\mu}_i}{\widetilde{\mu}_i} \right| \le h, \ \max[0, \ (1-h)\widetilde{\sigma}_i] \le \sigma_i \le (1+h)\widetilde{\sigma}_i \right\}, \quad h \ge 0 \tag{9}$$

For any pdf $p(u)$, the probability of success of allocation q is:

$$P_s(q, p) = \mathrm{Prob}[R(q, u) \ge R_c] \tag{10}$$

The project owners require that the probability of success be no less than P_c. Formulate an expression for the robustness, to uncertainty in the pdf, of the probability of success of allocation q. Develop this expression as far as you can, and use it to choose the allocation q.

4. **Project termination.** (p.21) At any point in time, the project manager knows the total cumulative expenditure for the project, E. In addition, the manager knows what fraction f_i of task i remains to be completed, for each task $i = 1, \ldots, N$. Denote $f = (f_1, \ldots, f_N)^T$.

   The estimated time remaining before completion of the project is:

$$t = \widetilde{g}^T f \tag{11}$$

   where $\widetilde{g}$ is highly uncertain and its error is represented by an info-gap model of uncertainty:

$$\mathcal{U}_g(h, \widetilde{g}) = \left\{ g : \ (g - \widetilde{g})^T W (g - \widetilde{g}) \le h^2 \right\}, \quad h \ge 0 \tag{12}$$

   W is a known, real, symmetric, positive definite matrix.

   Given an estimate of the time remaining before completion of the project, the projected remaining cost of the project is:

$$\widetilde{c}(t) = c_0 \sqrt{t} \tag{13}$$

   This projected remaining cost is highly uncertain and its error is estimated by an info-gap model of uncertainty:

$$\mathcal{U}_c(h, \widetilde{c}) = \left\{ c(t) : \ |c(t) - \widetilde{c}(t)| \le h \widetilde{c}(t) \right\}, \quad h \ge 0 \tag{14}$$

   The project fails if the total expenditure at the end of the project exceeds the budget B, which is specified.

   (a) Given the manager's knowledge of E and f at a particular point in time during the implementation of the project, how confident is the manager that the project will remain within the budget? Should the manager recommend that the project be terminated? How much additional budget would be needed to make in-budget completion fairly certain? How much budget could be removed from the project without jeopardizing successful in-budget completion?

   (b) The manager is in fact responsible for two projects, the project described above and an additional project with similar structure. For project $k = 1$ or 2, we denote the quantities E_k, f^k, g^k, $\widetilde{g}^k$, W_k, $c_{0,k}$ and B_k. Each project has info-gap models with the same structure. At a particular point in time during the implementation of both projects, the manager is informed of the status of the two projects by learning the values E_k and f^k for $k = 1$ and 2. The manager can move funds between the projects. How much money should be moved between the projects, if any? Should the manager recommend termination of one project and transfer of all its funds to the other project?

   (c) Repeat part 4a with the following modification. Given an estimate of the time remaining before completion of the project, based on eq.(11), the remaining cost $c(t)$ is a random variable. The estimate of the pdf of $c(t)$ is:

$$\widetilde{p}(c) = \frac{1}{\widetilde{c}} \mathrm{e}^{-c/\widetilde{c}}, \quad c \ge 0 \tag{15}$$

   where $\widetilde{c}$ is given in eq.(13). This pdf is highly uncertain and its error is represented by an info-gap model:

$$\mathcal{U}_p(h, \widetilde{p}) = \left\{ p(c) : \ p(c) \in \mathcal{P}, \ |p(c) - \widetilde{p}(c)| \le h \widetilde{p}(c) \right\}, \quad h \ge 0 \tag{16}$$

   where $\mathcal{P}$ is the set of non-negative pdfs normalized on $[0, \infty)$.

5. **Budgeting the manager's time** (p.22). The project manager must allocate his or her time among N tasks. The planned allocation is $\widetilde{t} = (\widetilde{t}_1, \ldots, \widetilde{t}_N)^T$. In practice, the actual time which the manager devotes to the N tasks will be $t = (t_1, \ldots, t_N)^T$. The uncertainty in the manager's attention times is represented by the following info-gap model:

$$\mathcal{U}(h, \widetilde{t}) = \left\{ t : \ (t - \widetilde{t})^T V (t - \widetilde{t}) \le h^2 \right\}, \quad h \ge 0 \tag{17}$$

where V is real, symmetric, positive definite and known. The utility from time-allocation t is $y^T t$ where y is a known vector. It is required that the utility be no less than u_c. Develop an explicit analytical expression for the robustness to uncertainty in $\widetilde{t}$.

6. **Uncertain budget** (p.22). The budget for each of the N tasks in a project is $b = (b_1, \ldots, b_N)^T$, where each element of this vector can be any real number. b is unknown and its uncertainty is represented by:

$$\mathcal{U}(h) = \left\{ b : \ b^T b \le h^2 \right\}, \quad h \ge 0 \tag{18}$$

The overall dis-utility to the firm of budget b is expressed by $b^T F b$ where F is a real, symmetric and positive definite matrix. It is required that the overall dis-utility be no greater than u_c. The firm's management will determine the matrix F. Derive an explicit algebraic expression for the robustness to uncertainty in the budget, given the matrix F.

7. **Hybrid uncertainty: uncertain probabilistic profit.** The profit from a new project is a random variable x which is distributed exponentially:

$$P(x|\lambda) = 1 - \mathrm{e}^{-\lambda x}, \quad x \geq 0 \tag{19}$$

The estimated value of λ is $\widetilde{\lambda}$, but this estimate is highly uncertain:

$$\mathcal{U}(h, \widetilde{\lambda}) = \left\{ \lambda : \ \lambda > 0, \ |\lambda - \widetilde{\lambda}| \leq h\widetilde{\lambda} \right\}, \quad h \geq 0 \tag{20}$$

It is required that profit at least x_{c} occur with probability no less than P_{c}:

$$\mathrm{Prob}(x \geq x_{\mathrm{c}}|\lambda) \geq P_{\mathrm{c}} \tag{21}$$

Derive an explicit algebraic expression for the robustness to uncertainty in λ.

8. **Project profitability.** The profit from a project is described by:

$$R(q, u) = q_1 u + q_2 \tag{22}$$

where $q = (q_1, q_2)$ are positive numbers chosen by the manager. The parameter u is uncertain:

$$\mathcal{U}(h, \widetilde{u}) = \{u : \ |u - \widetilde{u}| \le h\}, \quad h \ge 0 \tag{23}$$

The profit must be at least $R_{\rm c}$. Profit as large as $R_{\rm w}$ would be wonderful. Derive expressions for the robustness and opportuneness functions.

9. **Customer satisfaction.** The satisfaction of the customer is measured as:

$$S = qA + q^2 B \tag{24}$$

where q is controlled by the manager and A and B are uncertain:

$$\mathcal{U}(h, \widetilde{A}, \widetilde{B}) = \left\{ A, B : \ (A - \widetilde{A})^2 + (B - \widetilde{B})^2 \le h^2 \right\}, \quad h \ge 0 \tag{25}$$

The requirement is that the satisfaction be at least $S_{\rm c}$. Derive the robustness of decision q.

10. **Hybrid uncertainty: uncertain probabilistic task time.** (p.26) The probability that the project will be completed within the critical duration t_c is:

$$P = \frac{t_c}{t_c + uq} \tag{26}$$

where q is the time required to set up the project before actual work begins, and controlled by the manager, and u is uncertain:

$$\mathcal{U}(h, \widetilde{u}) = \{u : \ |u - \widetilde{u}| \le h\widetilde{u}\}, \quad h \ge 0 \tag{27}$$

The customer demands that the task complete within duration t_c with probability no less than P_c. Derive an explicit algebraic expression for the robustness of q.

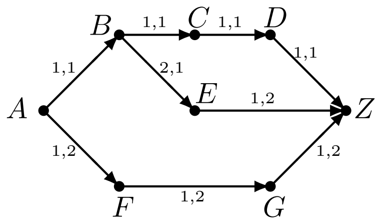


Figure 1: Transportation network for problem 11.

11. **Transportation network.** (p.27) A transportation network is specified by the directed graph in fig. 1. You must travel from node A to node Z along one of the three paths:

$$\begin{array}{ll} \text{Path 1:} & A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow Z \\ \text{Path 2:} & A \longrightarrow B \longrightarrow E \longrightarrow Z \\ \text{Path 3:} & A \longrightarrow F \longrightarrow G \longrightarrow Z \end{array}$$

The estimated transit time between nodes i and j is $\widetilde{t}_{ij}$, while the unknown true transit time is t_{ij}. The info-gap model for transit-time uncertainty is:

$$\mathcal{U}(h, \widetilde{t}) = \left\{ t : \ |t_{ij} - \widetilde{t}_{ij}| \le h w_{ij} \widetilde{t}_{ij}, \text{ for all } \ i, j \right\}, \quad h \ge 0 \tag{28}$$

where w_{ij} is a non-negative known uncertainty weight. The values of $\widetilde{t}_{ij}$, w_{ij} appear along the edges of the graph in fig. 1. For instance, $\widetilde{t}_{EZ}, w_{EZ} = 1, 2$.

(a) It is required to get from A to Z in a duration no longer than $t_{\rm c}$. Derive expressions for the robustness to transit-duration uncertainty for each of the three paths.

(b) For what values of $t_{\rm c}$ will you prefer each of the three paths, based on the criterion of maximizing the robustness and satisficing the total transit time?

Budget item	In-House			Out-Source		
	$\widetilde{c}_i$	w_i	d_i	$\widetilde{c}_i$	w_i	d_i
1	1.5	1.0	1.0	1.3	1.3	1.0
2	2.7	0.7	0.7	2.2	1.0	0.9
3	1.4	1.2	0.8	1.5	1.3	0.8

Table 1: Data for problem 12.

12. **In-house or out-source?** (p.29) Our firm is planning a project which can either be implemented in-house, or purchased by out-sourcing. The project involves three main budget items, whose expenses are incurred at different stages of the project, and thus entail different financing costs. The total expense of the project is:

$$E(c) = \sum_{i=1}^{3} d_i c_i \tag{29}$$

where c_i is the cost of stage i and d_i is the discount factor for that stage. We require that the expense not exceed the budget, B.

The estimated costs $\widetilde{c}_i$, cost-uncertainties w_i, and discount factors d_i for the three budget items, for in-house and out-source options, are listed in table 1. Use a fractional-error info-gap model:

$$\mathcal{U}(h, \widetilde{c}) = \{c: \ |c_i - \widetilde{c}_i| \le w_i \widetilde{c}_i h, \ i = 1, 2, 3\}, \quad h \ge 0 \tag{30}$$

Study the choice between in-house and out-source options, as a function of the total budget B. Which is preferable, as a function of the total budget constraint?

13. **Failure probability and financial loss.** We are designing a production system which is subject to failures (cracks, leaks, down times, etc.). The severity of failure is the random variable x, where large x means large failure. We must choose between two alternative systems. The less expensive system, $i = 1$, is more prone to failure, but can be implemented with greater redundancy so the financial loss due to failure is lower. The more expensive system, $i = 2$, is less failure-prone but failures are more disruptive and hence more expensive.

The best estimate of the probability density function for failure with system i is exponential:

$$\widetilde{p}_i(x) = \lambda_i \mathrm{e}^{-\lambda_i x}, \quad x \ge 0 \tag{31}$$

$0 < \lambda_1 < \lambda_2$, expressing the fact that system 1 is more prone to failure. The estimated pdf, $\widetilde{p}_i(x)$, is highly uncertain, especially far out on the tail (large failure), and the true pdf, $p_i(x)$, is unknown. An info-gap model for uncertainty in the pdf is:

$$\mathcal{U}(h, \widetilde{p}_i) = \left\{ p(x): \ p(x) \ge 0, \ \int_0^\infty p(x)\, \mathrm{d}x = 1, \ |p(x) - \widetilde{p}_i(x)| \le h \widetilde{p}_i(x) \right\}, \quad h \ge 0 \tag{32}$$

The financial loss resulting from failure of severity x, with option i, is:

$$L_i(x) = c_i x^2 \tag{33}$$

$0 < c_1 < c_2$ expresses the fact that failures in system 1 are less costly than in system 2.

We require that the probability of loss exceeding L_c be less than P_c. It would be wonderful if the probability of loss exceeding L_w (which is less than L_c) were less than P_w (which is less than P_c). Derive robustness and opportuneness functions for the two systems, and discuss the implications for choosing between the systems.

Simplification: assume that $\sqrt{L_\mathrm{c}/c_i}$ is greater than the median of $\widetilde{p}_i(x)$.

14. **Value of managerial attention.** (p.30) The value of managerial attention usually increases in time as the project approaches completion at time T. However, the value function is highly uncertain. The estimated value of attention at time t is:

$$\widetilde{v}(t) = v_1 t, \quad 0 \le t \le T \tag{34}$$

where v_1 is known and positive. However, the uncertainty in the true value, which may in fact be negative and need not always be positive or monotonic, is represented by the following info-gap model:

$$\mathcal{U}(h) = \left\{ v(t) : \left| \frac{v(t) - \widetilde{v}(t)}{v_1 T} \right| \le h \right\}, \quad h \ge 0 \tag{35}$$

The manager will plan his degree of attention to the project in question, as specified by a "focus function" $f(t)$ which expresses a degree of managerial attention devoted to the project. The utility of focus $f(t)$ at time t depends on the value of attention at that time. The overall value of the manager's focus function is:

$$R(f, v) = \int_0^T f(t) v(t) \, \mathrm{d}t \tag{36}$$

We will consider a simple focus function of the form:

$$f(t) = f_0 t \tag{37}$$

where the manager must choose the value of f_0.

Evaluate the robustness of focus f_0 given critical value R_c.

15. **Product development effort.** A firm is developing a new product, whose anticipated reward is $\widetilde{r}_1$ per unit of effort, E, which is invested. Thus the anticipated reward from developing product 1 is:

$$g(E) = \widetilde{r}_1 E \tag{38}$$

This same effort could be invested in an alternative product whose anticipated reward is $\widetilde{r}_2$ per unit of effort, E, which is invested, where $\widetilde{r}_2 > \widetilde{r}_1$. However, a sunk cost of c would be lost in abandoning the first product and moving to the second product. The sunk cost is small, so $\widetilde{r}_2 - c > \widetilde{r}_1$. The anticipated reward from developing product 2 is:

$$g(E) = \widetilde{r}_2 E - c \tag{39}$$

The fractional errors of the estimates, $\widetilde{r}_1$ and $\widetilde{r}_2$, are unknown. Also, the project fails if the reward is less g_c.

(a) Derive the robustness function for choosing to develop each product with effort E.

(b) The firm will go out of business if it earns less than g_c. What is the greatest value of g_c at which the firm should prefer staying with product 1?

(c) Now consider product 1 again, but with a different info-gap model for uncertainty in the reward:

$$g(E) = \widetilde{r}_1 E + \sum_{m=2}^{N} r_m E^m \tag{40}$$

$\widetilde{r}_1$ is known but the coefficients r_m are uncertain. Let $\rho = (r_2, \ldots, r_N)^T$ and $\epsilon = (E^2, \ldots, E^N)^T$. Use an ellipsoidal info-gap model:

$$\mathcal{U}(h) = \left\{ \rho : \ \rho^T W \rho \le h^2 \right\}, \quad h \ge 0 \tag{41}$$

where W is a known, real, symmetric, positive definite matrix. Derive the robustness function.

16. **A new product.** The quantity demanded, price per unit and total production cost, as a function of time, are $q(t)$, $p(t)$ and $c(t)$. The time horizon is $t \in [0, 1]$. The net income is:

$$R = \int_0^1 [q(t)p(t) - c(t)]\,\mathrm{d}t \tag{42}$$

We have estimates of these functions, which are constant in time: $\widetilde{q}$, $\widetilde{p}$ and $\widetilde{c}$. The actual time-varying values are uncertain, but we have the following understanding:

- Quantity demanded is substantially higher in mid-season, maybe by about 50%, but this is a new product so we don't know very well.
- Sale price is proportional to quantity demanded, $p(t) = \kappa q(t)$. The value of κ is around 2.5 plus or minus about 0.5, but it can fluctuate greatly due to fads and fashions or competition.
- Production costs are fairly stable compared to the other factors.

(a) Construct an info-gap model for uncertainty in the functions of this model.

(b) Use this info-gap model to construct the robustness function, given the requirement that the income exceed R_{c}.

(c) Use this info-gap model to construct the opportuneness function, given the aspiration that the income exceed R_{w}.

17. **Budgeting uncertain costs** (p.32). Two budget items have estimated costs $\widetilde{c}_1 = 10 \pm 4$ and $\widetilde{c}_2 = 6 \pm 2$. The errors are estimates based on judgment and past experience. In addition, experience has shown that when expenditure on one of these items increases by \$1, the expenditure on the other tends to decrease by about \$0.2. Thus the covariance is roughly estimated as $-\$2$.

(a) How would you quantify the uncertainty in the expenses?

(b) What budget do you need?

18. **Managerial attention** (p.33). A manager must allocate time between two tasks. The time allocated to the ith task is t_i, and the total time available is T. The reward from task i resulting from this allocation is:

$$r_i(t_i) = \frac{1}{\lambda_i t_i} \tag{43}$$

The total reward, $r(t)$, is the sum of the two single-task rewards. We require that the total reward exceed r_{c}.

(a) The coefficients λ_i are uncertain, as expressed by the following info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda_i : \ \lambda_i > 0, \ \left| \frac{\lambda_i - \widetilde{\lambda}_i}{\widetilde{\lambda}_i} \right| \le h \right\}, \quad h \ge 0 \tag{44}$$

Derive an expression for the robustness.

(b) Continuing part (a), find the time-allocation which maximizes the robustness, given the constraint on the total time available and assuming that each task must receive at least a duration $0 < \tau \ll T/2$.

(c) Now change the info-gap model of eq.(44) to include information which distinguishes between the fractional errors of $\widetilde{\lambda}_1$ and $\widetilde{\lambda}_2$:

$$\mathcal{U}(h) = \left\{ \lambda_i : \ \lambda_i > 0, \ \left| \frac{\lambda_i - \widetilde{\lambda}_i}{s_i} \right| \le h \right\}, \quad h \ge 0 \tag{45}$$

Derive an expression for the inverse of the robustness function and compare it with the inverse of the robustness function from part (a).

(d) Now change the info-gap model to the following envelope-bound model:

$$\mathcal{U}(h) = \left\{ r_i(t_i) : \ \left| \frac{r_i(t_i) - \widetilde{r}_i(t_i)}{\widetilde{r}_i(t_i)} \right| \le h \right\}, \quad h \ge 0 \tag{46}$$

where $\widetilde{r}_i(t_i)$ is defined by eq.(43) with $\widetilde{\lambda}_i$. Derive the robustness function.

(e) Now change the info-gap model of eq.(46) to the following envelope-bound model:

$$\mathcal{U}(h) = \left\{ r_i(t_i) : \ \left| \frac{r_i(t_i) - \widetilde{r}_i(t_i)}{s_i(t_i)} \right| \le h \right\}, \quad h \ge 0 \tag{47}$$

where $\widetilde{r}_i(t_i)$ is defined by eq.(43) with $\widetilde{\lambda}_i$. We allow negative reward (penalty). Derive the robustness function.

(f) Now consider a probabilistic version of this problem. Let $t_{\max}$ denote the larger of the two time allocations:

$$t_{\max} = \max[t_1, t_2] \tag{48}$$

Let us suppose that the total reward depends only on $t_{\max}$. Furthermore, our estimate of the pdf of the total reward is the following gamma distribution:

$$\widetilde{p}(r) = t_{\max}^2 r \mathrm{e}^{-r t_{\max}}, \quad r \ge 0 \tag{49}$$

Let $\widetilde{P}(r)$ denote the estimated cumulative distribution function for total reward.

The actual distribution of reward is uncertain, with pdf and cdf $p(r)$ and $P(r)$, respectively. In particular, we suspect that there is non-zero probability of negative reward (penalty) though we have no idea what the pdf for negative r is, though we believe that the shape of the gamma distribution describes the distribution of positive rewards. Thus we use the following info-gap model:

$$\mathcal{U}(h) = \{ p(r) : \ P(r < 0) = h, \ p(r) = (1-h)\widetilde{p}(r) \text{ for } r \ge 0 \}, \quad h \ge 0 \tag{50}$$

We require that the probability of reward at least as large as r_{c} must be no less than P_{c}:

$$P(r \ge r_{\mathrm{c}}) \ge P_{\mathrm{c}} \tag{51}$$

Derive the robustness function for $r_{\mathrm{c}} > 0$.

(g) Let $t = (t_1, t_2)^T$ denote the column vector of task times and suppose that the total reward is:

$$r(t) = \rho^T t \tag{52}$$

where ρ is uncertain with info-gap model:

$$\mathcal{U}(h) = \left\{ \rho : \ (\rho - \widetilde{\rho})^T W (\rho - \widetilde{\rho}) \le h^2 \right\}, \quad h \ge 0 \tag{53}$$

We require total reward no less than r_{c}. Derive the robustness function.

Solutions for

Problem Set on Info-Gap Risks in Project Management

List of problem solutions:

Solution to Problem 1.

1a Define the vector y whose elements are:

$$y_i = \mathrm{e}^{r(T-t_i)} \tag{54}$$

Thus the total project cost, eq.(1), is:

$$C(c,t) = y^T c \tag{55}$$

The robustness of task-initiation times t, with failure-cost C_f, is:

$$\widehat{h}(t, C_\mathrm{f}) = \max \left\{ h : \max_{c \in \mathcal{U}(h, \widetilde{c})} y^T c \le C_\mathrm{f} \right\} \tag{56}$$

To evaluate the inner maximum in the robustness function use Lagrange optimization and define:

$$H = y^T c + \lambda \left[h^2 - (c - \widetilde{c})^T V^{-1} (c - \widetilde{c}) \right] \tag{57}$$

Differentiating:

$$0 = \frac{\partial H}{\partial c} = y - 2\lambda V^{-1}(c - \widetilde{c}) \quad \Longrightarrow \quad c - \widetilde{c} = \frac{1}{2\lambda} V y \tag{58}$$

Use the constraint to determine λ

$$h^2 = \frac{1}{4\lambda^2} y^T V V^{-1} V y \quad \Longrightarrow \quad \frac{1}{2\lambda} = \frac{\pm h}{\sqrt{y^T V y}} \tag{59}$$

Hence the optimizing cost vector is:

$$c = \widetilde{c} \pm \frac{h}{\sqrt{y^T V y}} V y \tag{60}$$

The inner maximum in eq.(56) is:

$$\max_{c \in \mathcal{U}(h, \widetilde{c})} y^T c = y^T \widetilde{c} + h \sqrt{y^T V y} \tag{61}$$

Equating this to C_f and solving for h yields the robustness:

$$\widehat{h}(t, C_\mathrm{f}) = \begin{cases} \dfrac{C_\mathrm{f} - y^T \widetilde{c}}{\sqrt{y^T V y}} & \text{if } C_\mathrm{f} \ge y^T \widetilde{c} \\ 0 & \text{else} \end{cases} \tag{62}$$

1b If the covariance matrix is diagonal, then the positive part of the robustness function in eq.(62) is:

$$\widehat{h}(t, C_\mathrm{f}) = \frac{C_\mathrm{f} - y^T \widetilde{c}}{\sqrt{\displaystyle\sum_{i=1}^{N} v_i \mathrm{e}^{2r(T-t_i)}}} \tag{63}$$

Tasks with large variance, v_i, should be delayed so that $T - t_i$ is small.

Consider the possibility of choosing t_i so that:

$$v_i \mathrm{e}^{2r(T-t_i)} = \gamma \tag{64}$$

where γ is a constant greater than the greatest variance. (Does this maximize the robustness?) Thus:

$$t_i = T - \frac{1}{2r} \ln \frac{\gamma}{v_i} = T - \frac{1}{2r} \ln \gamma + \frac{1}{2r} \ln v_i \tag{65}$$

Suppose we have labelled the tasks so that $v_{i+1} > v_i$. Then the tasks are spaced as:

$$t_{i+1} - t_i = \frac{1}{2r}\left(\ln v_{i+1} - \ln v_i\right) = \frac{1}{2r}\ln\frac{v_{i+1}}{v_i} \tag{66}$$

1c The maximum of $y^T c$, at horizon of uncertainty h, occurs when $c_i = (\widetilde{c}_i + h\sqrt{v_i})$. Thus:

$$\max_{c\in\mathcal{U}(h,\widetilde{c})} y^T c = y^T\widetilde{c} + h\sum_i y_i\sqrt{v_i} \tag{67}$$

$$= y^T\widetilde{c} + h y^T V^{1/2}\mathbf{1} \tag{68}$$

where $\mathbf{1}$ is a vector of ones. Requiring that this maximum cost not exceed C_f, and solving for h, yields the robustness:

$$\widehat{h}(t, C_\mathrm{f}) = \begin{cases} \dfrac{C_\mathrm{f} - y^T\widetilde{c}}{y^T V^{1/2}\mathbf{1}} & \text{if } C_\mathrm{f} \geq y^T\widetilde{c} \\ 0 & \text{else} \end{cases} \tag{69}$$

Note the similarity to the robustness in eqs.(62) and (63).

1d We would like to choose task-initiation times, subject to operational and substantive constraints, so that the robustness is large. Considering the robustness in eq.(69), this means that we would like both $y^T\widetilde{c}$ and $y^T V^{1/2}\mathbf{1}$ to be small. That is:

$$y^T\widetilde{c} = \sum_i \widetilde{c}_i \mathrm{e}^{r(T-t_i)} \qquad \text{so} \qquad \text{large } \widetilde{c}_i \text{ implies large } t_i \tag{70}$$

$$y^T V^{1/2}\mathbf{1} = \sum_i \sqrt{v_i}\mathrm{e}^{r(T-t_i)} \qquad \text{so} \qquad \text{large } v_i \text{ implies large } t_i \tag{71}$$

In other words, costly tasks should be delayed, eq.(70), but also tasks with large cost uncertainty should be delayed, eq.(71). Operational constraints probably prevent actual optimization of the robustness. However, costly or uncertain tasks should be delayed where possible.

Another approach: The similarity between eqs.(69) and (63) suggests that the solution here can be quite similar to the solution of problem 1b. For instance, consider choosing t_i so that:

$$y_i\sqrt{v_i} = \gamma \tag{72}$$

where γ is a constant. Thus:

$$\gamma = \sqrt{v_i}\mathrm{e}^{r(T-t_i)} \quad \Longrightarrow \quad t_i = T - \frac{1}{r}\ln\frac{\gamma}{\sqrt{v_i}} \tag{73}$$

Hence the time increments between tasks would be:

$$t_{i+1} - t_i = \frac{1}{r}\ln\sqrt{\frac{v_{i+1}}{v_i}} \tag{74}$$

Does this maximize $\widehat{h}$? Probably not.

Solution to Problem 3.

3a. The robustness of allocation q, with required additional revenue $R_{\rm c}$, is:

$$\widehat{h}(q,R_{\rm c}) = \max\left\{h: \ \min_{u\in\mathcal{U}(h,\widetilde{u})} R(q,u) \ge R_{\rm c}\right\} \tag{75}$$

The added revenue is $R(q,u)=q^T u$. We assume the all allocations are non-negative: $q_i \ge 0$. Thus, the inner minimum in eq.(75) occurs when $u_i = \widetilde{u}_i - hw_i$. Thus:

$$\min_{u\in\mathcal{U}(h,\widetilde{u})} q^T u = \sum_{i=1}^{N} q_i(\widetilde{u}_i - hw_i) = q^T\widetilde{u} - hq^T w \tag{76}$$

Equating this to $R_{\rm c}$ and solving for h yields the robustness:

$$\widehat{h}(q,R_{\rm c}) = \left\{\begin{array}{cl} 0 & \text{if } R_{\rm c} > q^T\widetilde{u} \\ \dfrac{q^T\widetilde{u} - R_{\rm c}}{q^T w} & \text{else} \end{array}\right. \tag{77}$$

We choose the allocation to maximize the robustness, at specified required revenue $R_{\rm c}$. That is, choose q as the solution of:

$$\max \widehat{h}(q,R_{\rm c}) \quad \text{subject to} \quad \sum_{i=1}^{N} q_i = Q \tag{78}$$

In other words, considering the robustness function in eq.(77), we allocate the budget Q among the tasks to balance between (1) increasing the anticipated revenue $q^T\widetilde{u}$ and (2) decreasing the impact of uncertainty $q^T w$. This is a linear optimization problem.

3b. The opportuneness of allocation q, with windfall additional revenue $R_{\rm w}$, is:

$$\widehat{\beta}(q,R_{\rm w}) = \min\left\{h: \ \max_{u\in\mathcal{U}(h,\widetilde{u})} R(q,u) \ge R_{\rm w}\right\} \tag{79}$$

The added revenue is $R(q,u)=q^T u$. We assume the all allocations are non-negative: $q_i \ge 0$. Thus, the inner maximum in eq.(79) occurs when $u_i = \widetilde{u}_i + hw_i$. Thus:

$$\max_{u\in\mathcal{U}(h,\widetilde{u})} q^T u = \sum_{i=1}^{N} q_i(\widetilde{u}_i + hw_i) = q^T\widetilde{u} + hq^T w \tag{80}$$

Equating this to $R_{\rm w}$ and solving for h yields the opportuneness:

$$\widehat{\beta}(q,R_{\rm w}) = \left\{\begin{array}{cl} 0 & \text{if } R_{\rm w} < q^T\widetilde{u} \\ \dfrac{R_{\rm w} - q^T\widetilde{u}}{q^T w} & \text{else} \end{array}\right. \tag{81}$$

We choose the allocation to minimize the opportuneness, at specified windfall revenue $R_{\rm w}$. That is, choose q as the solution of:

$$\min \widehat{\beta}(q,R_{\rm w}) \quad \text{subject to} \quad \sum_{i=1}^{N} q_i = Q \tag{82}$$

In other words, considering the opportuneness function in eq.(81), we allocate the budget Q among the tasks to balance between (1) increasing the anticipated revenue $q^T\widetilde{u}$ and (2) increasing the opportuneness of uncertainty $q^T w$. This is a linear optimization problem.

3c. To evaluate the inner minimum in the robustness function use Lagrange optimization and define:

$$H = q^T u + \lambda \left[h^2 - (u - \widetilde{u})^T V^{-1} (u - \widetilde{u}) \right] \tag{83}$$

Differentiating:

$$0 = \frac{\partial H}{\partial u} = q - 2\lambda V^{-1}(u - \widetilde{u}) \quad \Longrightarrow \quad u - \widetilde{u} = \frac{1}{2\lambda} V q \tag{84}$$

Use the constraint to determine λ

$$h^2 = \frac{1}{4\lambda^2} q^T V V^{-1} V q \quad \Longrightarrow \quad \frac{1}{2\lambda} = \frac{\pm h}{\sqrt{q^T V q}} \tag{85}$$

Hence the optimizing revenue vector is:

$$u = \widetilde{u} \pm \frac{h}{\sqrt{q^T V q}} V q \tag{86}$$

The inner minimum in eq.(75) is:

$$\min_{u \in \mathcal{U}(h, \widetilde{u})} q^T u = q^T \widetilde{u} - h \sqrt{q^T V q} \tag{87}$$

Equating this to R_c and solving for h yields the robustness:

$$\widehat{h}(q, R_c) = \begin{cases} \dfrac{q^T \widetilde{u} - R_c}{\sqrt{q^T V q}} & \text{if } R_c \le q^T \widetilde{u} \\ 0 & \text{else} \end{cases} \tag{88}$$

The choice of q to maximize $\widehat{h}(q, R_c)$, subject to the budget constraint, is a bit complicated. Nonetheless, like the solution of part 3a, we allocate the budget Q among the tasks to balance between (1) increasing the anticipated revenue $q^T \widetilde{u}$ and (2) decreasing the impact of uncertainty $q^T V q$. (A special case is discussed in the lecture on portfolio selection.)

3d. The robustness is:

$$\widehat{h}(q, P_c) = \max \left\{ h : \min_{p \in \mathcal{U}(h, \widetilde{p})} P_s(q, p) \ge P_c \right\} \tag{89}$$

The probability of success, given pdf $p(u)$, is:

$$P_s(q, p) = \text{Prob}[R(q, u) \ge R_c] \tag{90}$$

$$= \int_{q^T u \ge R_c} p(u) \, du \tag{91}$$

What we need is the pdf of $q^T u$, based on the pdf of u. This pdf is the convolution of normal distributions, which itself is normal. Given the moments of the underlying normal distributions, we known also the moments of the convolution.

Solution to Problem 4 (p.5).

(4a) We require:

$$E + c \le B \iff c \le B - E \tag{92}$$

where $c(t)$ depends on $t = g^T f$. Thus the robustness is defined as:

$$\widehat{h} = \max\left\{ h : \left(\max_{g \in \mathcal{U}_g(h,\widetilde{g})} \max_{c \in \mathcal{U}_c(h,\widetilde{c})} [E + c(g^T f)] \right) \le B \right\} \tag{93}$$

From the info-gap model in eq.(14) we know that, at horizon of uncertainty h:

$$c \le (1+h)\widetilde{c} \tag{94}$$

Thus we require:

$$(1+h)\widetilde{c} \le B - E \iff (1+h)c_0\sqrt{g^T f} \le B - E \tag{95}$$

We seek $\max_h g^T f$ by Lagrange optimization. Define:

$$H = g^T f + \lambda\left(h^2 - (g - \widetilde{g})^T W (g - \widetilde{g})\right) \tag{96}$$

To find extreme values:

$$0 = \frac{\partial H}{\partial g} = f - 2\lambda W(g - \widetilde{g}) \iff g - \widetilde{g} = \frac{1}{2\lambda} W^{-1} f \tag{97}$$

Hence, from the constraint we find:

$$\frac{1}{2\lambda} = \frac{\pm h}{\sqrt{f^T W^{-1} f}} \tag{98}$$

Thus the maximizing g is:

$$g = \widetilde{g} + \frac{h}{\sqrt{f^T W^{-1} f}} W^{-1} f \tag{99}$$

Hence:

$$\max_h g^T f = \widetilde{g}^T f + h\sqrt{f^T W^{-1} f} \tag{100}$$

Combining this with eq.(95) we find that the robustness is the lowest positive h satisfying:

$$(1+h)c_0\sqrt{\widetilde{g}^T f + h\sqrt{f^T W^{-1} f}} = B - E \tag{101}$$

Or equivalently, the lowest positive h satisfying:

$$(1+h)^2 c_0^2 \left(\widetilde{g}^T f + h\sqrt{f^T W^{-1} f}\right) = (B - E)^2 \tag{102}$$

This is a cubic expression in h. The least positive root is the robustness, $\widehat{h}(B)$.

Cancel the project if $\widehat{h}(B)$ is small.

Alternatively, add new budget, that is, change B, so that $\widehat{h}(B)$ is not small.

Solution to Problem 5 (p.6).

The robustness is:

$$\widehat{h}(\widetilde{t}, u_{\rm c}) = \max\left\{ h : \min_{t \in \mathcal{U}(h, \widetilde{t})} t^T y \geq u_{\rm c} \right\} \tag{103}$$

Let $\mu(h)$ denote the inner minimum in eq.(103). We use Lagrange optimization to evaluate this minimum. Define:

$$H = t^T y + \lambda[h^2 - (t - \widetilde{t})^T V (t - \widetilde{t})] \tag{104}$$

Differentiating wrt t and equating to 0 yields:

$$0 = \frac{{\rm d}H}{{\rm d}t} = y - 2\lambda V(t - \widetilde{t}) \quad \Longrightarrow \quad t - \widetilde{t} = \frac{1}{2\lambda} V^{-1} y \tag{105}$$

Employing the constraint yields:

$$h^2 = \frac{1}{4\lambda^2} y^T V^{-1} V V^{-1} y \quad \Longrightarrow \quad \frac{1}{2\lambda} = \frac{\pm h}{\sqrt{y^T V^{-1} y}} \tag{106}$$

Hence:

$$t = \widetilde{t} \pm \frac{h}{\sqrt{y^T V^{-1} y}} V^{-1} y \tag{107}$$

Thus:

$$\mu(h) = y^T \widetilde{t} - h\sqrt{y^T V^{-1} y} \tag{108}$$

which must be no less than $u_{\rm c}$. Hence the robustness is:

$$\widehat{h}(\widetilde{t}, u_{\rm c}) = \begin{cases} \dfrac{y^T \widetilde{t} - u_{\rm c}}{\sqrt{y^T V^{-1} y}} & \text{if } y^T \widetilde{t} \geq u_{\rm c} \\ 0 & \text{else} \end{cases} \tag{109}$$

Solution to Problem 6 (p.6).

The robustness is:

$$\widehat{h}(F, u_{\rm c}) = \max\left\{ h : \max_{b \in \mathcal{U}(h)} b^T F b \leq u_{\rm c} \right\} \tag{110}$$

Let $M(h)$ denote the inner maximum in eq.(110). We use Lagrange optimization to evaluate this maximum. Define:

$$H = b^T F b + \lambda(h^2 - b^T b) \tag{111}$$

Differentiating wrt b and equating to 0 yields:

$$0 = \frac{{\rm d}H}{{\rm d}b} = 2Fb - 2\lambda b \tag{112}$$

Hence the optimizing choice of b is an eigenvector of F. Define the ortho-normal eigenvectors and corresponding eigenvalues of F:

$$F v_i = \mu_i v_i, \quad i = 1, \ldots, N, \quad 0 < \mu_1 \leq \mu_2 \leq \cdots \leq \mu_N, \quad v_i^T v_j = \delta_{ij} \tag{113}$$

Thus the b which we seek, satisfying the constraint, is:

$$b = h v_k \tag{114}$$

for some k. Thus:

$$b^T F b = h^2 v_k^T F v_k = \mu_k h^2 \leq u_{\rm c} \tag{115}$$

The greatest h for all realizations of b requires that we select $k = N$. Thus the robustness is:

$$\widehat{h}(F, u_{\rm c}) = \sqrt{\frac{u_{\rm c}}{\mu_N}} \tag{116}$$

Solution to Problem 7.

The robustness is:

$$\widehat{h}(P_c) = \max\left\{h : \min_{\lambda\in\mathcal{U}(h,\widetilde{\lambda})} \text{Prob}(x \ge x_c|\lambda) \ge P_c\right\} \tag{117}$$

where $\text{Prob}(x \ge x_c|\lambda) = \mathrm{e}^{-\lambda x_c}$.

Let $\mu(h)$ denote the inner minimum in eq.(117), which occurs when:

$$\lambda = (1+h)\widetilde{\lambda} \tag{118}$$

Thus:

$$\mu(h) = \mathrm{e}^{-(1+h)\widetilde{\lambda}} \tag{119}$$

which must be no less than P_c. Hence the robustness is:

$$\widehat{h}(P_c) = \begin{cases} -1 - \frac{1}{\widetilde{\lambda}x_c}\ln P_c & \text{if } 1 \le -\frac{1}{\widetilde{\lambda}x_c}\ln P_c \\ 0 & \text{else} \end{cases} \tag{120}$$

Solution to Problem 8.

The robustness and opportuneness are:

$$\widehat{h}(q, R_\mathrm{c}) = \max \left\{ h : \left(\min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \right) \geq R_\mathrm{c} \right\} \tag{121}$$

$$\widehat{\beta}(q, R_\mathrm{w}) = \min \left\{ h : \left(\max_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \right) \geq R_\mathrm{w} \right\} \tag{122}$$

First consider the robustness. The inner minimum occurs at $u = \widetilde{u} - h$, so the minimum is:

$$\min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) = (\widetilde{u} - h) q_1 + q_2 \tag{123}$$

Equating this to R_c and solving for h yields the robustness:

$$\widehat{h}(q, R_\mathrm{c}) = \widetilde{u} - \frac{R_\mathrm{c} - q_2}{q_1} \tag{124}$$

or zero if this is negative.

Now consider the opportuneness. The inner maximum occurs at $u = \widetilde{u} + h$, so the maximum is:

$$\max_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) = (\widetilde{u} + h) q_1 + q_2 \tag{125}$$

Equating this to R_w and solving for h yields the opportuneness:

$$\widehat{\beta}(q, R_\mathrm{w}) = \frac{R_\mathrm{w} - q_2}{q_1} - \widetilde{u} \tag{126}$$

Solution to Problem 9.

The robustness is:

$$\widehat{h}(q, R_{\rm c}) = \max \left\{ h : \left(\min_{A,B \in \mathcal{U}(h,\widetilde{A},\widetilde{B})} (qA + q^2 B) \right) \geq S_{\rm c} \right\} \tag{127}$$

Define the objective function:

$$H = qA + q^2 B + \lambda[h^2 - (A - \widetilde{A})^2 - (B - \widetilde{B})^2] \tag{128}$$

Conditions for an extremum are:

$$0 = \frac{\partial H}{\partial A} = q - 2\lambda(A - \widetilde{A}) \implies A = \frac{1}{2\lambda} q + \widetilde{A} \tag{129}$$

$$0 = \frac{\partial H}{\partial B} = q^2 - 2\lambda(B - \widetilde{B}) \implies B = \frac{1}{2\lambda} q^2 + \widetilde{B} \tag{130}$$

From the constraint we find:

$$h^2 = \frac{1}{4\lambda^2}(q^2 + q^4) \implies \frac{1}{2\lambda} = \frac{\pm h}{\sqrt{q^2 + q^4}} \tag{131}$$

Thus:

$$\min_{A,B \in \mathcal{U}(h,\widetilde{A},\widetilde{B})} S(q, A, B) = q\widetilde{A} + q^2 \widetilde{B} - \frac{h}{\sqrt{q^2 + q^4}}(q^2 + q^4) \tag{132}$$

Equating this to $S_{\rm c}$ and solving for h yields the robustness:

$$\widehat{h}(q, S_{\rm c}) = \frac{q\widetilde{A} + q^2 \widetilde{B} - S_{\rm c}}{\sqrt{q^2 + q^4}} \tag{133}$$

or zero if this is negative.

Solution to Problem 10. (p.26)

The robustness is:

$$\widehat{h}(q, P_{\rm c}) = \max \left\{ h : \ \left(\min_{u \in \mathcal{U}(h,\widetilde{u})} \frac{t_{\rm c}}{t_{\rm c} + uq} \right) \geq P_{\rm c} \right\} \tag{134}$$

This minimum occurs at $u = (1+h)\widetilde{u}$ so:

$$\min_{u \in \mathcal{U}(h,\widetilde{u})} \frac{t_{\rm c}}{t_{\rm c} + uq} = \frac{t_{\rm c}}{t_{\rm c} + (1+h)\widetilde{u}} \tag{135}$$

Equating this to $P_{\rm c}$ and solving for h yields the robustness:

$$\widehat{h}(q, P_{\rm c}) = \frac{(1 - P_{\rm c})t_{\rm c}}{q\widetilde{u}P_{\rm c}} - 1 \tag{136}$$

or zero if this is negative.

Solution to Problem 11, transportation network. (p.11)

The duration of the m path is c_m, which is the sum of the transit times in that path. Specifically:

$$c_1(t) = t_{AB} + t_{BC} + t_{CD} + t_{DZ} \quad (137)$$
$$c_2(t) = t_{AB} + t_{BE} + t_{EZ} \quad (138)$$
$$c_3(t) = t_{AF} + t_{FG} + t_{GZ} \quad (139)$$

The robustness to transit-time uncertainty for the mth path is:

$$\widehat{h}(m, t_{\mathrm{c}}) = \max \left\{ h : \left(\max_{u \in \mathcal{U}(h, \widetilde{t})} c_m(t) \right) \le t_{\mathrm{c}} \right\} \quad (140)$$

Let $\mu_m(h)$ denote the inner maximum. For instance:

$$\mu_1(h) = (1 + hw_{AB})\widetilde{t}_{AB} + (1 + hw_{BC})\widetilde{t}_{BC} + (1 + hw_{CD})\widetilde{t}_{CD} + (1 + hw_{DZ})\widetilde{t}_{DZ} \quad (141)$$
$$= c_1(\widetilde{t}) + h \sum_{(1)} w_{ij} t_{ij} \quad (142)$$

where $\sum_{(1)}$ denotes summation over the elements in path 1. Equating eq.(142) to t_{c} and solving for h yields the robustness of path 1:

$$\widehat{h}(1, t_{\mathrm{c}}) = \frac{t_{\mathrm{c}} - c_1(\widetilde{t})}{\sum_{(1)} w_{ij} t_{ij}} \quad (143)$$

or zero if this is negative. Likewise we find the other path robustnesses:

$$\widehat{h}(2, t_{\mathrm{c}}) = \frac{t_{\mathrm{c}} - c_2(\widetilde{t})}{\sum_{(2)} w_{ij} t_{ij}} \quad (144)$$
$$\widehat{h}(3, t_{\mathrm{c}}) = \frac{t_{\mathrm{c}} - c_3(\widetilde{t})}{\sum_{(3)} w_{ij} t_{ij}} \quad (145)$$

The coefficients in eqs.(143)–(145) are:

$$c_1(\widetilde{t}) = 4, \qquad \sum_{(1)} w_{ij} t_{ij} = 4 \quad (146)$$
$$c_2(\widetilde{t}) = 4, \qquad \sum_{(2)} w_{ij} t_{ij} = 5 \quad (147)$$
$$c_3(\widetilde{t}) = 3, \qquad \sum_{(3)} w_{ij} t_{ij} = 6 \quad (148)$$

Thus the path-robustness, shown schematically in fig. 2, functions are:

$$\widehat{h}(1, t_{\mathrm{c}}) = \frac{t_{\mathrm{c}} - 4}{4} \quad (149)$$
$$\widehat{h}(2, t_{\mathrm{c}}) = \frac{t_{\mathrm{c}} - 4}{5} \quad (150)$$
$$\widehat{h}(3, t_{\mathrm{c}}) = \frac{t_{\mathrm{c}} - 3}{6} \quad (151)$$

From these relations we see that path 2 is never preferred. The robustness curves for paths 1 and 3 cross at $t_{\times} = 6$. Thus path 3 is preferred over path 1 if $t_{\mathrm{c}} < 6$, and path 1 is preferred otherwise.

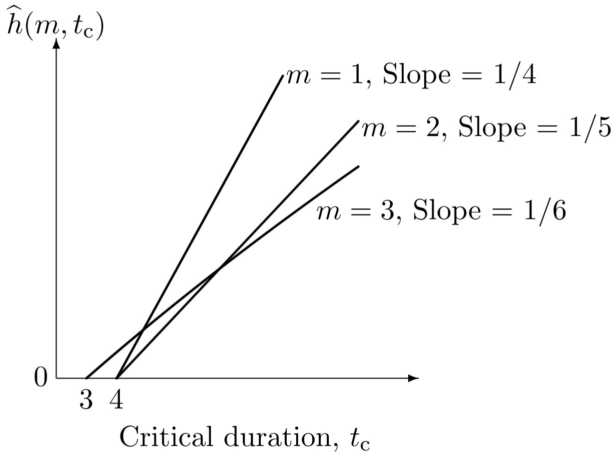


Figure 2: Robustness function in eqs.(149)–(151).

Solution to Problem 12. (p.12)

The options 'in-house' and 'out-source' are denoted $q = 1$ and $q = 2$, respectively. The total expenses are $E(c)$, and we require that $E(c) \le B$. Thus, whichever option we choose (and using the corresponding estimated values), the robustness is defined as:

$$\widehat{h} = \max\left\{h : \left(\max_{c\in\mathcal{U}(h)} E(c)\right) \le B\right\} \tag{152}$$

Denote the inner maximum by $\mu(h)$, which is:

$$\mu(h) = \sum_{i=1}^{3}(\widetilde{c}_i + hw_i\widetilde{c}_i)d_i = \sum_{i=1}^{3}\widetilde{c}_i d_i + h\sum_{i=1}^{3} w_i\widetilde{c}_i d_i \tag{153}$$

Equating to B and solving for h yields the robustness:

$$\widehat{h} = \frac{B - \sum_i d_i\widetilde{c}_i}{\sum_i d_i w_i \widetilde{c}_i} \tag{154}$$

Given two financing structures, their robustness curves cross if and only if the following conditions hold:

$$\sum_i d_i\widetilde{c}_i < \sum_i d_i'\widetilde{c}_i' \tag{155}$$

$$\sum_i d_i w_i\widetilde{c}_i > \sum_i d_i' w_i'\widetilde{c}_i' \tag{156}$$

We find (outsrc.m):

$$\text{In-house:} \quad \textstyle\sum_i d_i\widetilde{c}_i = 4.51 \quad \sum_i d_i w_i\widetilde{c}_i = 4.167 \tag{157}$$

$$\text{Out-source:} \quad \textstyle\sum_i d_i\widetilde{c}_i = 4.48 \quad \sum_i d_i w_i\widetilde{c}_i = 5.23 \tag{158}$$

So, in-house is nominally more expensive, but its robustness curve is steeper. The two robustness curves cross at $(B_\times,\ h_\times) = (4.63,\ 0.02822)$. So, we prefer the in-house option (which is nominally more expensive) if we can obtain a budget in excess of 4.63, and/or if we need robustness in excess of 0.028.

Solution to Problem 13.

The probability of unacceptable financial loss with system i, for pdf $p(x)$, is:

$$P_i(L_c, p) = \int_{L_i(x)\ge L_c}^{\infty} p(x)\,\mathrm{d}x \tag{159}$$

In light of the definition of $L_i(x)$ in eq.(33) this can be written:

$$P_i(L_c, p) = \int_{\sqrt{L_c/c_i}}^{\infty} p(x)\,\mathrm{d}x \tag{160}$$

The robustness and opportuneness functions for system i are:

$$\widehat{h}(i) = \max\left\{h : \left(\max_{p\in\mathcal{U}(h,\widetilde{p}_i)} P_i(L_c, p)\right) \le P_c\right\} \tag{161}$$

$$\widehat{\beta}(i) = \min\left\{h : \left(\min_{p\in\mathcal{U}(h,\widetilde{p}_i)} P_i(L_w, p)\right) \le P_w\right\} \tag{162}$$

We first derive the **robustness function.** Let $\mu(h)$ denote the inner maximum in eq.(161). The robustness is the greatest value of h at which $\mu(h) = P_c$. This maximum occurs when $p(x)$ is as large as possible, at horizon of uncertainty h, for $x \geq \sqrt{L_c/c_i}$. Since $\sqrt{L_c/c_i}$ exceeds the median of $\widetilde{p}_i(x)$ we can choose the upper tail of $p(x)$ as $(1+h)\widetilde{p}_i(x)$, and still be able to normalize the density by choosing $p(x)$ less that $\widetilde{p}_i(x)$ for x below $\sqrt{L_c/c_i}$. So:

$$\mu(h) = (1+h)\int_{\sqrt{L_c/c_i}}^{\infty} \widetilde{p}_i(x)\,\mathrm{d}x = (1+h)\mathrm{e}^{-\lambda_i\sqrt{L_c/c_i}} \tag{163}$$

Equating this to P_c and solving for h yields the robustness for system i:

$$\widehat{h}(i) = P_c \mathrm{e}^{\lambda_i\sqrt{L_c/c_i}} - 1 \tag{164}$$

or zero if this is negative.

The robustness functions for the two systems are illustrated schematically in fig. 3 and 4, showing that one system is always more robust than the other. Which it is depends on the values of the parameters.

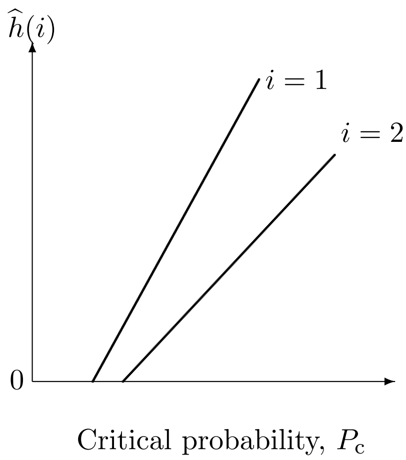


Figure 3: Robustness function in eq.(164), if $\lambda_1/\sqrt{c_1} > \lambda_2/\sqrt{c_2}$.

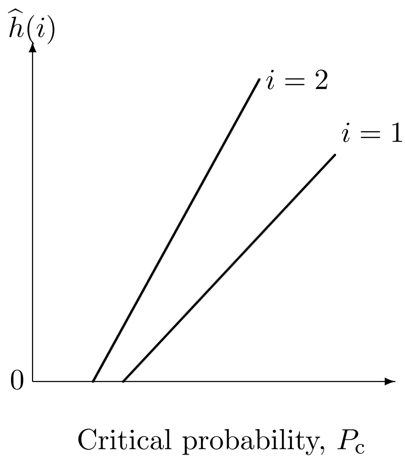


Figure 4: Robustness function in eq.(164), if $\lambda_1/\sqrt{c_1} < \lambda_2/\sqrt{c_2}$.

We now derive the **opportuneness function.** Let $M(h)$ denote the inner minimum in eq.(162). The opportuneness is the smallest value of h at which $M(h) = P_w$. Arguing as before, we see that $M(h)$ is obtained by choosing[1] $p(x) = (1-h)\widetilde{p}_i(x)$ for $x \geq \sqrt{L_c/c_i}$. This results in:

$$M(h) = (1-h)\int_{\sqrt{L_c/c_i}}^{\infty} \widetilde{p}_i(x)\,\mathrm{d}x = (1-h)\mathrm{e}^{-\lambda_i\sqrt{L_c/c_i}} \tag{165}$$

Equating this to P_w and solving for h yields the opportuneness for system i:

$$\widehat{\beta}(i) = 1 - P_w \mathrm{e}^{\lambda_i\sqrt{L_c/c_i}} \tag{166}$$

or zero if this is negative.

The opportuneness functions are shown in figs. 3 and 4, showing that one system is always more opportune than the other. Which it is depends on the values of the parameters. Note that the two systems are sympathetic: the more robust is also the more opportune.

Solution to Problem 14: Value of managerial attention.

The robustness of focus-function coefficient f_0 given critical value R_c is:

$$\widehat{h}(f_0, R_c) = \max\left\{h : \left(\min_{v\in\mathcal{U}(h)} R(f_0, v)\right) \geq R_c\right\} \tag{167}$$

[1]This choice is only value for $h \leq 1$, but we will see that the opportuneness cannot be greater than unity in any case.

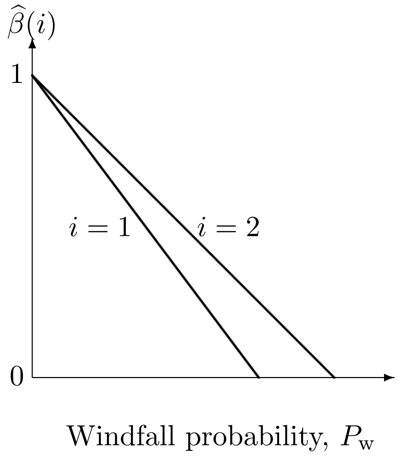


Figure 5: Opportuneness function in eq.(166), if $\lambda_1/\sqrt{c_1} > \lambda_2/\sqrt{c_2}$.

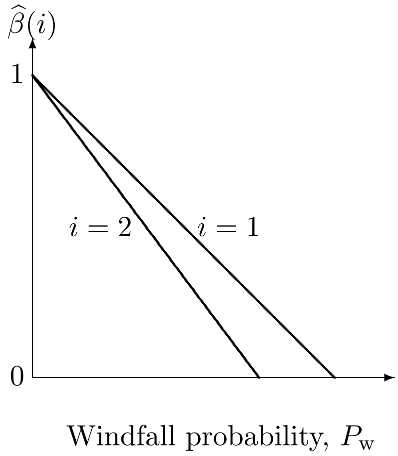


Figure 6: Opportuneness function in eq.(166), if $\lambda_1/\sqrt{c_1} < \lambda_2/\sqrt{c_2}$.

Denote the inner minimum in eq.(167) by $\mu(h)$, which occurs when $v(t) = \widetilde{v}(t) - hv_1T$. Thus:

$$\mu(h) = f_0 \int_0^T t(v_1 t - hv_1 T)\, {\rm d}t = \frac{f_0 v_1 T^3}{3} - \frac{f_0 h v_1 T^3}{2} \tag{168}$$

Thus the robustness is:

$$\widehat{h}(f_0, R_{\rm c}) = \frac{2}{3} - \frac{2R_{\rm c}}{f_0 v_1 T^3} \tag{169}$$

or zero is this expression is negative.

Solution to Problem 15.

(a) Let $q = 1$ denote staying with product 1, and $q = -1$ moving to product 2. The unknown actual reward from product i with effort E is:

$$g(q=1, E, r_1) = r_1 E \tag{170}$$

$$g(q=2, E, r_2) = r_2 E - c \tag{171}$$

The gain can be written more compactly as:

$$g(q, E, r) = \frac{1+q}{2} r_1 E + \frac{1-q}{2}(r_2 E - c) \tag{172}$$

Define the info-gap model as:

$$\mathcal{U}(h, \widetilde{r}) = \left\{ r = (r_1, r_2) : \ \left| \frac{r_i - \widetilde{r}_i}{\widetilde{r}_i} \right| \le h, \ \ i = 1,\ 2 \right\}, \quad h \ge 0 \tag{173}$$

The robustness function is:

$$\widehat{h}(E, q, g_{\rm c}) = \max \left\{ h : \ \left(\min_{r \in \mathcal{U}(h, \widetilde{r})} g(q, E, r) \right) \ge g_{\rm c} \right\} \tag{174}$$

Let $\mu(h)$ denote the inner minimum in eq.(174).

For $q = 1$ we find:

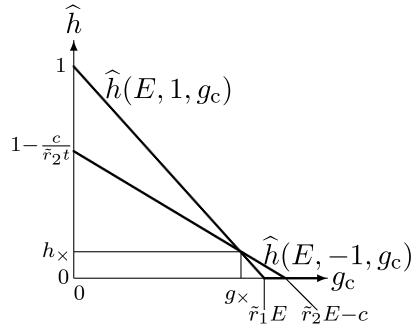
$$\mu(h) = (1-h)\widetilde{r}_1 E \ge g_{\rm c} \quad \text{implies} \quad \widehat{h}(E, 1, g_{\rm c}) = 1 - \frac{g_{\rm c}}{\widetilde{r}_1 E} \tag{175}$$

For $q = -1$ we find:

$$\mu(h) = (1-h)\widetilde{r}_2 E - c \ge g_{\rm c} \quad \text{implies} \quad \widehat{h}(E, -1, g_{\rm c}) = 1 - \frac{g_{\rm c} + c}{\widetilde{r}_2 E} \tag{176}$$

(b) These robustness curves are shown schematically in fig. 7 on the assumption that $\tilde{r}_2 E - c > \tilde{r}_1 E$. The robustness curves are shown in fig. 7 using the assumption that, nominally, $q = -1$ is preferred: $\tilde{r}_2 E - c > \tilde{r}_1 E$. We see crossing of the robustness curves, which implies reversal of preference either if robustness greater than $h_\times$ is needed, or if performance less than $g_\times$ is adequate. Specifically, the firm should prefer staying with product 1 if gain equal or less than $g_\times$ is adequate. By equating eqs.(175) and (176) we find:

$$g_\times = \frac{c\tilde{r}_1}{\tilde{r}_2 - \tilde{r}_1} \tag{177}$$



two

Figure 7: Comparison of two options: reversal of preferences for problem 15.

(c) The robustness is defined as in eq.(174). Let $\mu(h)$ denote the inner minimum, which we evaluate using Lagrange optimization:

$$H = \tilde{r}_1 E + \rho^T \epsilon + \lambda(h^2 - \rho^T W \rho) \tag{178}$$

$$0 = \frac{\partial H}{\partial \rho} = \epsilon - 2\lambda W \rho \Longrightarrow \rho = \frac{1}{2\lambda} W^{-1} \epsilon \Longrightarrow h^2 = \frac{1}{4\lambda^2} \epsilon^T W^{-1} \epsilon \Longrightarrow \frac{1}{2\lambda} = \frac{\pm h}{\sqrt{\epsilon^T W^{-1} \epsilon}} \tag{179}$$

Thus:

$$\mu(h) = \tilde{r}_1 E - h\sqrt{\epsilon^T W^{-1} \epsilon} \tag{180}$$

Hence:

$$\hat{h} = \frac{\tilde{r}_1 E - g_c}{\sqrt{\epsilon^T W^{-1} \epsilon}} \tag{181}$$

or zero if this is negative.

Solution to Problem 17.

(a) An info-gap model of uncertainty is:

$$\mathcal{U}(h) = \left\{ c: \ (c - \tilde{c})^T W^{-1} (c - \tilde{c}) \le h^2 \right\}, \quad h \ge 1 \tag{182}$$

where W is the covariance matrix:

$$W = \begin{pmatrix} 16 & -2 \\ -2 & 4 \end{pmatrix} \tag{183}$$

(b) The robustness of budget B is:

$$\hat{h} = \max \left\{ h: \ \left(\max_{c \in \mathcal{U}(h)} c^T \mathbf{1} \right) \le B \right\} \tag{184}$$

where $\mathbf{1}$ is a vector of ones.

Use Lagrange optimization to find the inner minimum in the definition of the robustness. The result is the following robustness function:

$$\widehat{h}(B) = \frac{B - \mathbf{1}^T \widetilde{c}}{\sqrt{\mathbf{1}^T W \mathbf{1}}} = \frac{B - 16}{4} \tag{185}$$

Suppose we judge that robustness against 3 units of error in the "covariance matrix" W is adequate. Thus we want $\widehat{h} \approx 3$. This implies a budget of $B = 28$. The nominal budget is $\mathbf{1}^T \widetilde{c} = 16$.

Solution to Problem 18.

(a) For a given time allocation, t, the robustness is defined as:

$$\widehat{h}(t, r_{\mathrm{c}}) = \max \left\{ h : \left(\min_{\lambda_1, \lambda_2 \in \mathcal{U}(h)} r(t, \lambda_1, \lambda_2) \right) \geq r_{\mathrm{c}} \right\} \tag{186}$$

The reward from task i decreases as λ_i increases:

$$\frac{\partial r_i(t_i, \lambda_i)}{\partial \lambda_i} = \frac{-1}{\lambda_i^2 t_i} \tag{187}$$

Thus the inner minimum in eq.(186), which we denote $\mu(h)$, occurs for $\lambda_i = (1+h)\widetilde{\lambda}_i$. The inner minimum is:

$$\begin{aligned} \mu(h) &= \frac{1}{(1+h)\widetilde{\lambda}_1 t_1} + \frac{1}{(1+h)\widetilde{\lambda}_2 t_2} && (188) \\ &= \frac{1}{1+h} \underbrace{\frac{t_1 \widetilde{\lambda}_1 + t_2 \widetilde{\lambda}_2}{t_1 t_2 \widetilde{\lambda}_1 \widetilde{\lambda}_2}}_{\widetilde{r}} && (189) \end{aligned}$$

which defines $\widetilde{r}$. Note that $\widetilde{r}$ is the estimated total reward (see eq.(191)). Equating $\mu(h)$ to r_{c} and solving for h yields the robustness:

$$\widehat{h}(t, r_{\mathrm{c}}) = \frac{\widetilde{r}}{r_{\mathrm{c}}} - 1 \tag{190}$$

or zero if this is negative.

(b) The robustness is maximized by maximizing $\widetilde{r}$. Note that, from the constraint $t_2 = T - t_1$:

$$\widetilde{r} = \frac{1}{t_1 \widetilde{\lambda}_1} + \frac{1}{t_2 \widetilde{\lambda}_2} = \frac{1}{t_1 \widetilde{\lambda}_1} + \frac{1}{(T - t_1)\widetilde{\lambda}_2} \tag{191}$$

Also:

$$\frac{\partial \widetilde{r}}{\partial t_1} = \frac{\widetilde{\lambda}_1 t_1^2 - \widetilde{\lambda}_2 (T - t_1)^2}{\widetilde{\lambda}_1 \widetilde{\lambda}_2 t_1^2 (T - t_1)^2} \tag{192}$$

Thus $\widetilde{r}$ has a single *minimum* at an intermediate value of t_1. Thus the *maximum* $\widetilde{r}$ occurs either for $t_1 = \tau$ or $t_1 = T - \tau$. Consider these two cases:

$$\begin{aligned} \widetilde{r}(t_1 = \tau) &= \frac{1}{\tau \widetilde{\lambda}_1} + \frac{1}{(T-\tau)\widetilde{\lambda}_2} = \frac{\tau \widetilde{\lambda}_1 + (T-\tau)\widetilde{\lambda}_2}{\tau (T-\tau) \widetilde{\lambda}_1 \widetilde{\lambda}_2} && (193) \\ \widetilde{r}(t_2 = \tau) &= \frac{1}{\tau \widetilde{\lambda}_2} + \frac{1}{(T-\tau)\widetilde{\lambda}_1} = \frac{\tau \widetilde{\lambda}_2 + (T-\tau)\widetilde{\lambda}_1}{\tau (T-\tau) \widetilde{\lambda}_1 \widetilde{\lambda}_2} && (194) \end{aligned}$$

We see that:

$$\begin{aligned} \widetilde{r}(t_1 = \tau) > \widetilde{r}(t_2 = \tau) \quad &\text{iff} \quad \tau \widetilde{\lambda}_1 + (T-\tau)\widetilde{\lambda}_2 > \tau \widetilde{\lambda}_2 + (T-\tau)\widetilde{\lambda}_1 && (195) \\ &\text{iff} \quad (T-\tau)(\widetilde{\lambda}_2 - \widetilde{\lambda}_1) > \tau (\widetilde{\lambda}_2 - \widetilde{\lambda}_1) && (196) \\ &\text{iff} \quad \widetilde{\lambda}_2 - \widetilde{\lambda}_1 > 0 && (197) \end{aligned}$$

where the last relation results from the fact that $T - \tau > \tau > 0$.

Thus the robustness is maximized by choosing:

$$\begin{array}{ll} t_1 = \tau & \text{if } \widetilde{\lambda}_2 > \widetilde{\lambda}_1 \\ t_1 = T - \tau & \text{else} \end{array} \tag{198}$$

This is the allocation which maximizes the estimated total reward.

(c) The inverse of the robustness is the inner minimum in eq.(186), which we denote $\mu(h)$. In the present case, as distinct from eq.(188), we find:

$$\mu(h) = \frac{1}{(\widetilde{\lambda}_1 + hs_1)t_1} + \frac{1}{(\widetilde{\lambda}_2 + hs_2)t_2} \tag{199}$$

Comparing this with eq.(188) at $h = 0$ we note that the robustness vanishes at the same value of r_c. However, the slopes of the robustness curves differ in the two cases.

(d) The inner minimum in the definition of the robustness is:

$$\mu(h) = (1 - h)[\widetilde{r}_1(t_1) + \widetilde{r}_2(t_2)] \tag{200}$$

or zero if $h > 1$. Thus the robustness is:

$$\widehat{h} = 1 - \frac{r_c}{\widetilde{r}_1(t_1) + \widetilde{r}_2(t_2)} \tag{201}$$

or zero if this is negative. Note that the robustness is maximized by maximizing the estimated total reward, $\widetilde{r}_1(t_1) + \widetilde{r}_2(t_2)$.

(e) The inner minimum in the definition of the robustness is:

$$\mu(h) = \widetilde{r}_1(t_1) - hs_1(t_1) + \widetilde{r}_2(t_2) - hs_2(t_2) \tag{202}$$

Thus the robustness is:

$$\widehat{h} = \frac{\widetilde{r}_1(t_1) + \widetilde{r}_2(t_2) - r_c}{s_1(t_1) + s_2(t_2)} \tag{203}$$

or zero if this is negative. Note that the robustness is maximized by maximizing the estimated total reward, $\widetilde{r}_1(t_1) + \widetilde{r}_2(t_2)$. However, the slope may differ from the previous cases.

(f) The mean and variance of the estimate pdf are:

$$\mathrm{E}(r) = \frac{2}{t_{\max}}, \quad \mathrm{var}(r) = \frac{2}{t_{\max}^2} \tag{204}$$

The robustness is defined as:

$$\widehat{h} = \max \left\{ h : \left(\min_{p \in \mathcal{U}(h)} P(r \geq r_c) \right) \geq P_c \right\} \tag{205}$$

The inner minimum, $\mu(h)$, occurs with the greatest possible "leakage" of probability into the negative region:

$$\mu(h) = (1 - h)[1 - \widetilde{P}(r_c)] \tag{206}$$

Equating this to P_c and solving for h yields:

$$\widehat{h} = 1 - \frac{r_c}{1 - \widetilde{P}(r_c)} \tag{207}$$

or zero if this is negative.

(g) The robustness is defined as:

$$\widehat{h} = \max \left\{ h : \left(\min_{\rho \in \mathcal{U}(h)} \rho^T t \right) \geq r_c \right\} \tag{208}$$

Use Lagrange optimization to find the inner minimum, which we denote $\mu(h)$:

$$H = \rho^T t + \lambda[h^2 - (\rho - \widetilde{\rho})^T W(\rho - \widetilde{\rho})] \tag{209}$$

$$0 = \frac{\partial H}{\partial \rho} = t - 2\lambda W(\rho - \widetilde{\rho}) \implies \rho - \widetilde{\rho} = \frac{1}{2\lambda} W^{-1} t \implies h^2 = \frac{1}{4\lambda^2} t^T W^{-1} t \tag{210}$$

Thus:

$$\rho = \widetilde{\rho} - \frac{h}{\sqrt{t^T W^{-1} t}} W^{-1} t \implies \mu(h) = \widetilde{\rho}^T t - h\sqrt{t^T W^{-1} t} \tag{211}$$

Equating this to r_c and solving for h yields the robustness:

$$\widehat{h} = \frac{\widetilde{\rho}^T t - r_\mathrm{c}}{\sqrt{t^T W^{-1} t}} \tag{212}$$

or zero if this is negative.