4. **Project termination.** (p.21) At any point in time, the project manager knows the total cumulative expenditure for the project, $E$. In addition, the manager knows what fraction $f_i$ of task $i$ remains to be completed, for each task $i = 1, \ldots, N$. Denote $f = (f_1, \ldots, f_N)^T$.

   The estimated time remaining before completion of the project is:

$$t = \widetilde{g}^T f \tag{11}$$

   where $\widetilde{g}$ is highly uncertain and its error is represented by an info-gap model of uncertainty:

$$\mathcal{U}_g(h, \widetilde{g}) = \left\{ g : \ (g - \widetilde{g})^T W (g - \widetilde{g}) \le h^2 \right\}, \quad h \ge 0 \tag{12}$$

   $W$ is a known, real, symmetric, positive definite matrix.

   Given an estimate of the time remaining before completion of the project, the projected remaining cost of the project is:

$$\widetilde{c}(t) = c_0 \sqrt{t} \tag{13}$$

   This projected remaining cost is highly uncertain and its error is estimated by an info-gap model of uncertainty:

$$\mathcal{U}_c(h, \widetilde{c}) = \left\{ c(t) : \ |c(t) - \widetilde{c}(t)| \le h \widetilde{c}(t) \right\}, \quad h \ge 0 \tag{14}$$

   The project fails if the total expenditure at the end of the project exceeds the budget $B$, which is specified.

   (a) Given the manager's knowledge of $E$ and $f$ at a particular point in time during the implementation of the project, how confident is the manager that the project will remain within the budget? Should the manager recommend that the project be terminated? How much additional budget would be needed to make in-budget completion fairly certain? How much budget could be removed from the project without jeopardizing successful in-budget completion?

   (b) The manager is in fact responsible for two projects, the project described above and an additional project with similar structure. For project $k = 1$ or 2, we denote the quantities $E_k$, $f^k$, $g^k$, $\widetilde{g}^k$, $W_k$, $c_{0,k}$ and $B_k$. Each project has info-gap models with the same structure. At a particular point in time during the implementation of both projects, the manager is informed of the status of the two projects by learning the values $E_k$ and $f^k$ for $k = 1$ and 2. The manager can move funds between the projects. How much money should be moved between the projects, if any? Should the manager recommend termination of one project and transfer of all its funds to the other project?

   (c) Repeat part 4a with the following modification. Given an estimate of the time remaining before completion of the project, based on eq.(11), the remaining cost $c(t)$ is a random variable. The estimate of the pdf of $c(t)$ is:

$$\widetilde{p}(c) = \frac{1}{\widetilde{c}} \mathrm{e}^{-c/\widetilde{c}}, \quad c \ge 0 \tag{15}$$

   where $\widetilde{c}$ is given in eq.(13). This pdf is highly uncertain and its error is represented by an info-gap model:

$$\mathcal{U}_p(h, \widetilde{p}) = \left\{ p(c) : \ p(c) \in \mathcal{P}, \ |p(c) - \widetilde{p}(c)| \le h \widetilde{p}(c) \right\}, \quad h \ge 0 \tag{16}$$

   where $\mathcal{P}$ is the set of non-negative pdfs normalized on $[0, \infty)$.

5. **Budgeting the manager's time** (p.22). The project manager must allocate his or her time among $N$ tasks. The planned allocation is $\widetilde{t} = (\widetilde{t}_1, \ldots, \widetilde{t}_N)^T$. In practice, the actual time which the manager devotes to the $N$ tasks will be $t = (t_1, \ldots, t_N)^T$. The uncertainty in the manager's attention times is represented by the following info-gap model:

$$\mathcal{U}(h, \widetilde{t}) = \left\{ t : \; (t - \widetilde{t})^T V (t - \widetilde{t}) \le h^2 \right\}, \quad h \ge 0 \tag{17}$$

where $V$ is real, symmetric, positive definite and known. The utility from time-allocation $t$ is $y^T t$ where $y$ is a known vector. It is required that the utility be no less than $u_{\rm c}$. Develop an explicit analytical expression for the robustness to uncertainty in $\widetilde{t}$.

6. **Uncertain budget** (p.22). The budget for each of the $N$ tasks in a project is $b = (b_1, \ldots, b_N)^T$, where each element of this vector can be any real number. $b$ is unknown and its uncertainty is represented by:

$$\mathcal{U}(h) = \left\{ b : \; b^T b \le h^2 \right\}, \quad h \ge 0 \tag{18}$$

The overall dis-utility to the firm of budget $b$ is expressed by $b^T F b$ where $F$ is a real, symmetric and positive definite matrix. It is required that the overall dis-utility be no greater than $u_{\rm c}$. The firm's management will determine the matrix $F$. Derive an explicit algebraic expression for the robustness to uncertainty in the budget, given the matrix $F$.

**Solution to Problem 4** (p.5).

**(4a)** We require:

$$E + c \le B \iff c \le B - E \tag{92}$$

where $c(t)$ depends on $t = g^T f$. Thus the robustness is defined as:

$$\widehat{h} = \max \left\{ h : \left( \max_{g \in \mathcal{U}_g(h, \widetilde{g})} \max_{c \in \mathcal{U}_c(h, \widetilde{c})} [E + c(g^T f)] \right) \le B \right\} \tag{93}$$

From the info-gap model in eq.(14) we know that, at horizon of uncertainty $h$:

$$c \le (1+h)\widetilde{c} \tag{94}$$

Thus we require:

$$(1+h)\widetilde{c} \le B - E \iff (1+h)c_0\sqrt{g^T f} \le B - E \tag{95}$$

We seek $\max_h g^T f$ by Lagrange optimization. Define:

$$H = g^T f + \lambda \left( h^2 - (g - \widetilde{g})^T W (g - \widetilde{g}) \right) \tag{96}$$

To find extreme values:

$$0 = \frac{\partial H}{\partial g} = f - 2\lambda W (g - \widetilde{g}) \iff g - \widetilde{g} = \frac{1}{2\lambda} W^{-1} f \tag{97}$$

Hence, from the constraint we find:

$$\frac{1}{2\lambda} = \frac{\pm h}{\sqrt{f^T W^{-1} f}} \tag{98}$$

Thus the maximizing $g$ is:

$$g = \widetilde{g} + \frac{h}{\sqrt{f^T W^{-1} f}} W^{-1} f \tag{99}$$

Hence:

$$\max_h g^T f = \widetilde{g}^T f + h\sqrt{f^T W^{-1} f} \tag{100}$$

Combining this with eq.(95) we find that the robustness is the lowest positive $h$ satisfying:

$$(1+h)c_0\sqrt{\widetilde{g}^T f + h\sqrt{f^T W^{-1} f}} = B - E \tag{101}$$

Or equivalently, the lowest positive $h$ satisfying:

$$(1+h)^2 c_0^2 \left( \widetilde{g}^T f + h\sqrt{f^T W^{-1} f} \right) = (B - E)^2 \tag{102}$$

This is a cubic expression in $h$. The least positive root is the robustness, $\widehat{h}(B)$.

Cancel the project if $\widehat{h}(B)$ is small.

Alternatively, add new budget, that is, change $B$, so that $\widehat{h}(B)$ is not small.

**Solution to Problem 5** (p.6).

The robustness is:

$$\widehat{h}(\widetilde{t}, u_{\rm c}) = \max \left\{ h : \min_{t \in \mathcal{U}(h, \widetilde{t})} t^T y \ge u_{\rm c} \right\} \tag{103}$$

Let $\mu(h)$ denote the inner minimum in eq.(103). We use Lagrange optimization to evaluate this minimum. Define:

$$H = t^T y + \lambda [h^2 - (t - \widetilde{t})^T V (t - \widetilde{t})] \tag{104}$$

Differentiating wrt $t$ and equating to 0 yields:

$$0 = \frac{{\rm d}H}{{\rm d}t} = y - 2\lambda V(t - \widetilde{t}) \quad \Longrightarrow \quad t - \widetilde{t} = \frac{1}{2\lambda} V^{-1} y \tag{105}$$

Employing the constraint yields:

$$h^2 = \frac{1}{4\lambda^2} y^T V^{-1} V V^{-1} y \quad \Longrightarrow \quad \frac{1}{2\lambda} = \frac{\pm h}{\sqrt{y^T V^{-1} y}} \tag{106}$$

Hence:

$$t = \widetilde{t} \pm \frac{h}{\sqrt{y^T V^{-1} y}} V^{-1} y \tag{107}$$

Thus:

$$\mu(h) = y^T \widetilde{t} - h \sqrt{y^T V^{-1} y} \tag{108}$$

which must be no less than $u_{\rm c}$. Hence the robustness is:

$$\widehat{h}(\widetilde{t}, u_{\rm c}) = \begin{cases} \dfrac{y^T \widetilde{t} - u_{\rm c}}{\sqrt{y^T V^{-1} y}} & \text{if } y^T \widetilde{t} \ge u_{\rm c} \\ 0 & \text{else} \end{cases} \tag{109}$$

**Solution to Problem 6** (p.6).

The robustness is:

$$\widehat{h}(F, u_{\rm c}) = \max \left\{ h : \max_{b \in \mathcal{U}(h)} b^T F b \le u_{\rm c} \right\} \tag{110}$$

Let $M(h)$ denote the inner maximum in eq.(110). We use Lagrange optimization to evaluate this maximum. Define:

$$H = b^T F b + \lambda (h^2 - b^T b) \tag{111}$$

Differentiating wrt $b$ and equating to 0 yields:

$$0 = \frac{{\rm d}H}{{\rm d}b} = 2Fb - 2\lambda b \tag{112}$$

Hence the optimizing choice of $b$ is an eigenvector of $F$. Define the ortho-normal eigenvectors and corresponding eigenvalues of $F$:

$$F v_i = \mu_i v_i, \quad i = 1, \ldots, N, \quad 0 < \mu_1 \le \mu_2 \le \cdots \le \mu_N, \quad v_i^T v_j = \delta_{ij} \tag{113}$$

Thus the $b$ which we seek, satisfying the constraint, is:

$$b = h v_k \tag{114}$$

for some $k$. Thus:

$$b^T F b = h^2 v_k^T F v_k = \mu_k h^2 \le u_{\rm c} \tag{115}$$

The greatest $h$ for all realizations of $b$ requires that we select $k = N$. Thus the robustness is:

$$\widehat{h}(F, u_{\rm c}) = \sqrt{\frac{u_{\rm c}}{\mu_N}} \tag{116}$$