Lecture Notes on

Portfolio Management

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Source material:[1]

• Yakov Ben-Haim, 2010, *Info-Gap Economics: An Operational Introduction,* Chapter 4: Financial Stability, Palgrave-Macmillan.

• Yakov Ben-Haim, 2006, *Info-Gap Decision Theory: Decisions Under Severe Uncertainty,* 2nd edition, Academic Press. Section 3.2.7.

A Note to the Student: These lecture notes are not a substitute for the thorough study of books. These notes are no more than an aid in following the lectures.

Contents

[0]\lectures\Econ-Dec-Mak\portfolio-mgt001.tex 19.6.2022

[1]Additional material in the file: Yakov Ben-Haim, Lecture notes on Robustness and Opportuneness, section 12: Portfolio Investment, \lectures\risk\lectures\ro02.pdf.

1 Probabilistic Mean-Variance Analysis

§ **Random asset returns:**

- u_i is the future return on \$1 investment in asset i.
- u_i is a random variable.
- $u \in \Re^N$ is a vector of random returns on N different assets.
- q is a vector of investments: \$$q_i$ invested in asset i.
  - $q_i > 0$ means purchase of asset i.
  - $q_i < 0$ means sale of asset i.
- The future return on this investment is:

$$r(q,u) = \sum_{i=1}^{N} u_i q_i = q^T u \tag{1}$$

§ **Budget constraint:**

$$B = \sum_{i=1}^{N} q_i = q^T \mathbf{1} \tag{2}$$

where B is the fixed budget and **1** is an N-vector whose elements are all ones.

§ **Mean and variance of return:**

- We assume that we know the mean, $\overline{u}$, and covariance matrix , C, of u (**what's that?**)[2].
- Thus we can calculate the mean, $\overline{r}$, and variance, σ_r^2, of r:

$$\begin{aligned}
\overline{r} &= \mathrm{E}(q^T u) = q^T \overline{u} \quad \textbf{Why?} && (3)\\
\sigma_r^2 &= \mathrm{E}\left[(r-\overline{r})^2\right] && (4)\\
&= \mathrm{E}\left[\left(q^T u - q^T \overline{u}\right)^2\right] && (5)\\
&= \mathrm{E}\left[\left(q^T (u - \overline{u})\right)^2\right] && (6)\\
&= q^T \underbrace{\mathrm{E}\left[(u-\overline{u})(u-\overline{u})^T\right]}_{C} q && (7)\\
&= q^T C q && (8)
\end{aligned}$$

§ **Risk-adjusted mean return:**

- We must choose q subject to the budget constraint, eq.(2).
- We would like **large average return:** large $q^T\overline{u}$.
- We would like **small variation** on this average return: small σ_r.
- **One approach** is to choose q to maximize:

$$\begin{aligned}
F &= \left(q^T \overline{u}\right)^2 - \lambda \sigma_r^2 && (9)\\
&= q^T \overline{u}\,\overline{u}^T q - \lambda q^T C q && (10)
\end{aligned}$$

where $\lambda > 0$, chosen to weight the "risk" term against the "return" term.

- **Another approach:** explore trade off between mean, $q^T\overline{u}\,\overline{u}^T q$, and variance, $q^T C q$.

[2] The covariance matrix is defined as $C = \mathrm{E}[(u-\overline{u})(u-\overline{u})^T]$ where u and $\overline{u}$ are column vectors.

§ **Optimal allocation:**

- Write eq.(10) as:

$$F = q^T\left(\overline{u}\,\overline{u}^T - \lambda C\right)q \quad (11)$$
$$= q^T Z q \quad (12)$$

which defines the $N \times N$ real symmetric matrix Z, whose eigenvalues and eigenvectors are:

$$Z\zeta_i = \gamma_i\zeta_i, \quad i = 1, \ldots, N, \quad \gamma_1 \le \cdots \le \gamma_N, \quad \zeta_i^T\zeta_j = \delta_{ij} \quad (13)$$

δ_{ij} is the Kronecker delta function.

- The ortho-normal eigenvectors span $\Re^N$ so q can be expanded as:

$$q = \sum_{i=1}^{N}\alpha_i\zeta_i \quad (14)$$

- Thus eq.(12) becomes:

$$F = \left(\sum_{i=1}^{N}\alpha_i\zeta_i^T\right)Z\left(\sum_{j=1}^{N}\alpha_j\zeta_j\right) \quad (15)$$
$$= \left(\sum_{i=1}^{N}\alpha_i\zeta_i^T\right)\left(\sum_{j=1}^{N}\alpha_j\gamma_j\zeta_j\right) \quad (16)$$
$$= \sum_{i=1}^{N}\alpha_i^2\gamma_i \quad (17)$$

- γ_N is greatest eigenvalue, so F is maximized, subject to the budget constraint in eq.(2), if:

$$\alpha_1 = \cdots = \alpha_{N-1} = 0, \quad \alpha_N = \frac{B}{\zeta_N^T\mathbf{1}} \quad (18)$$

That is, from eqs.(14) and (18), the optimizing budget-constrained allocation is:

$$q^\star = \frac{B}{\zeta_N^T\mathbf{1}}\zeta_N \quad (19)$$

- From eq.(11) we see that $q^\star$ usually mixes all N assets.

§ **What have we assumed:**

- Knowledge of past mean and covariance, $\overline{u}$ and C.
- Knowledge that future moments will be (at least nearly) the same as past moments.
- When are these reasonable assumptions? When not?
  (E.g. innovative start-ups vs conventional enterprises.)

2 Info-Gap Extension of Mean-Variance Analysis

§ **Source material:** *Info-Gap Decision Theory,* section 3.2.7.[3]

§ **In section 1 we assumed:**
- u, the returns vector, is a random variable.
- Mean and covariance of u are known.

§ **We now consider weaker information:**
- Nominal or typical $\widetilde{u}$ is known, perhaps calculated as historical mean or expert's best guess.
- Shape of clusters of u-vectors is roughly known or "guesstimated".
  E.g., we may have the historical covariance of u-vectors, or expert's judgment.
- We will adopt an ellipsoid-bound **info-gap model of uncertainty**:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u = \widetilde{u} + v : \ v^T W v \le h^2 \right\}, \quad h \ge 0 \tag{20}$$

W is a known, real, symmetric, positive definite matrix. E.g. inverse of covariance matrix.
- Explain intuitively why W (ellipsoidal shape matrix) could be the inverse covariance matrix:
  Shape of the uncertain cluster expresses variance and covariance.

§ **System model:** returns from investment:

$$r(q, u) = q^T u \tag{21}$$

§ **Performance requirement:**
- Adequately large return (satisficing, not optimizing):

$$r(q, u) \ge r_{\rm c} \tag{22}$$

§ **Maximize robustness** against uncertain returns.

2.1 Robustness Function

§ Robustness depends on 3 components:
system model, eq.(21), performance requirement, eq.(22), and uncertainty model, eq.(20).

§ $\widehat{h}(q, r_{\rm c}) =$ greatest uncertainty at which reward is no less than $r_{\rm c}$ for investment portfolio q:

$$\widehat{h}(q, r_{\rm c}) = \max \left\{ h : \ \left(\min_{u \in \mathcal{U}(h, \widetilde{u})} r(q, u) \right) \ge r_{\rm c} \right\} \tag{23}$$

To evaluate $\widehat{h}(q, r_{\rm c})$ we must determine the inner minimum in eq.(23):

$$\min_{u \in \mathcal{U}(h, \widetilde{u})} r(q, u) = q^T \widetilde{u} + \min_{v^T W v \le h^2} q^T v \tag{24}$$

§ To evaluate this optimum we use **Lagrange optimization.** Define:

$$H = q^T v + \lambda \left(h^2 - v^T W v \right) \tag{25}$$

[3] See also section 12 of Lecture Notes on Robustness and Opportuneness, \lectures\risk\lectures\ro02.tex.

where λ will be chosen to satisfy the constraint:

$$v^T W v = h^2 \tag{26}$$

Question: Why must eq.(26) hold?

The condition for an extremum of H where $H = q^T v + \lambda \left(h^2 - v^T W v \right)$:

$$0 = \frac{\partial H}{\partial v} = q - 2\lambda W v \tag{27}$$

$$\Longrightarrow \qquad v = \frac{1}{2\lambda} W^{-1} q \tag{28}$$

Using the constraint from eq.(26):

$$h^2 = v^T W v = \frac{1}{4\lambda^2} q^T W^{-1} W W^{-1} q \tag{29}$$

which leads to:

$$\frac{1}{2\lambda} = \frac{\pm h}{\sqrt{q^T W^{-1} q}} \tag{30}$$

Hence:

$$v = \frac{\pm h}{\sqrt{q^T W^{-1} q}} W^{-1} q \tag{31}$$

So the minimum is:

$$\min_{v^T W v \leq h^2} q^T v = -h \sqrt{q^T W^{-1} q} \tag{32}$$

Consequently:

$$\min_{u \in \mathcal{U}(h, \widetilde{u})} r(q, u) = q^T \widetilde{u} - h \sqrt{q^T W^{-1} q} \tag{33}$$

§ To find $\widehat{h}$: Equate this minimum to $r_{\rm c}$ and solve for h:

$$\widehat{h}(q, r_{\rm c}) = \frac{q^T \widetilde{u} - r_{\rm c}}{\sqrt{q^T W^{-1} q}} \tag{34}$$

unless this is negative, in which case:

$$\widehat{h}(q, r_{\rm c}) = 0 \tag{35}$$

Why?

§ Two properties of robustness function in eq.(34):

- **Zeroing:** No robustness at nominal return, $q^T \widetilde{u}$.
- **Trade-off** between robustness, $\widehat{h}(q, r_{\rm c})$, and satisficed return, $r_{\rm c}$.

2.2 Robust Optimal Investment

§ **Question: how to choose the investment vector q?**

Strategy:

- $\widehat{h}(q, r_{\rm c})$ depends on the decision vector q.
- For $\widehat{h}$ we know that: "bigger is better".
- So, choose q to maximize $\widehat{h}(q, r_{\rm c})$ subject to budget constraint:

$$\sum_{i=1}^{N} q_i = q^T \mathbf{1} = B = \text{total available budget} \tag{36}$$

$\mathbf{1}$ is an N-vector whose elements each equal one.
$q_i > 0 \implies$ buy commodity i.
$q_i < 0 \implies$ sell commodity i.

- We **optimize** robustness to uncertainty and **satisfice** financial return: **robust satisficing.**
- We can try to optimize the return. How? Choose large r_c. What is resulting robustness?

§ **Consider a special (simple) case:**

$$\widetilde{u}_i = u_o \quad \text{for all} \quad i \tag{37}$$

That is: all commodities have the same nominal value.
Of course, the uncertainties may differ between commodities.
Eq.(37) can be expressed:

$$\widetilde{u} = u_o \mathbf{1} \tag{38}$$

§ **The robustness,** eq.(34), becomes:

$$\widehat{h}(q, r_c) = \frac{u_o q^T \mathbf{1} - r_c}{\sqrt{q^T W^{-1} q}} \tag{39}$$

$$= \frac{u_o B - r_c}{\sqrt{q^T W^{-1} q}} \tag{40}$$

§ So, how to choose the investment vector q?
From eq.(40) we maximize $\widehat{h}$
by choosing q to minimize $q^T W^{-1} q$
subject to the constraint $q^T \mathbf{1} = B$.

§ We again use Lagrange optimization. Define:

$$H = q^T W^{-1} q + \lambda \left(B - q^T \mathbf{1} \right) \tag{41}$$

Extrema are obtained from:

$$0 = \frac{\partial H}{\partial q} = 2W^{-1} q - \lambda \mathbf{1} \quad \implies \quad q = \frac{\lambda}{2} W \mathbf{1} \tag{42}$$

Choose λ to satisfy the constraint (**Why** must this constraint hold?):

$$B - q^T \mathbf{1} = 0 \tag{43}$$

Combining eqs.(42) and (43):

$$B = \frac{\lambda}{2} \mathbf{1}^T W \mathbf{1} \quad \implies \quad \frac{\lambda}{2} = \frac{B}{\mathbf{1}^T W \mathbf{1}} \tag{44}$$

Combining eqs.(42) and (44), the **robust optimal** q is found to be:

$$\widehat{q}_c = \frac{B}{\mathbf{1}^T W \mathbf{1}} W \mathbf{1} \tag{45}$$

Note: $\widehat{q}_c$ depends on the budget, B, and on the uncertainty via W.

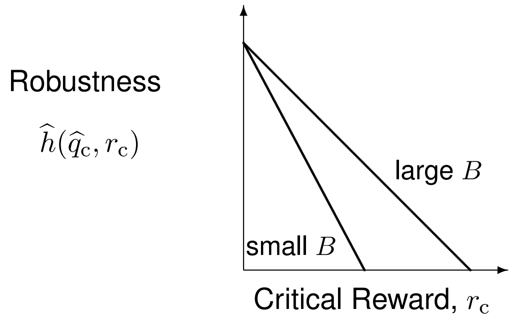

Figure 1: Robustness function vs critical reward, eq.(46), p.7.

Substitute eq.(45) into eq.(40) and the **optimal robustness** becomes:

$$\widehat{h}(\widehat{q}_{\rm c}, r_{\rm c}) = \frac{(u_o B - r_{\rm c})\sqrt{\mathbf{1}^T W \mathbf{1}}}{B} \tag{46}$$

This shows the usual trade-off between robustness vs. critical reward, as in fig.1:
Slope $\propto -\frac{1}{B}$, where $B =$ total investment.
Question: Are things better or worse with large investment B?
Answers:

- ◦ Greater robustness at fixed aspiration $r_{\rm c}$, for larger B. (Big B is good.)
- ◦ Greater cost of robustness at larger B:
  larger reduction in requirement, $r_{\rm c}$, to achieve a given increase in robustness.
  (Big B is not so good.)

• Note $\widehat{h} \propto \sqrt{\mathbf{1}^T W \mathbf{1}}$. **Question:** Does this make sense?
(W is inverse covariance matrix or something like that.)

2.3 Comparing Portfolios

§ Consider 2 sets of investment options, each with:

- Constant nominal return, $\widetilde{u}_i = u_{o,i}\mathbf{1}$, $i = 1, 2$.
- Ellipsoid-bound info-gap models of uncertainty as in eq.(20), p.4:

$$\mathcal{U}_i(h, \widetilde{u}_i) = \left\{ u = \widetilde{u}_i + v : \ v^T W_i v \le h^2 \right\}, \quad h \ge 0, \ i = 1, 2 \tag{47}$$

Consider the following special case:

$$u_{o,1} < u_{o,2} \tag{48}$$
$$\mathbf{1}^T W_1 \mathbf{1} > \mathbf{1}^T W_2 \mathbf{1} \tag{49}$$

- Eq.(48) implies that option 1 is nominally worse than option 2.
- Eq.(49) implies that option 1 is nominally less uncertain than option 2.
  (Recall: W is **inverse** covariance matrix or something similar).

The optimum robustness function for investment option i is, from eq.(46), p.7:

$$\widehat{h}_i(\widehat{q}_{{\rm c}i}, r_{\rm c}) = \frac{(u_{o,i} B - r_{\rm c})\mathbf{1}^T W_i \mathbf{1}}{B} \tag{50}$$

§ These two optimal robustness functions appear schematically in fig. 2, p.8:

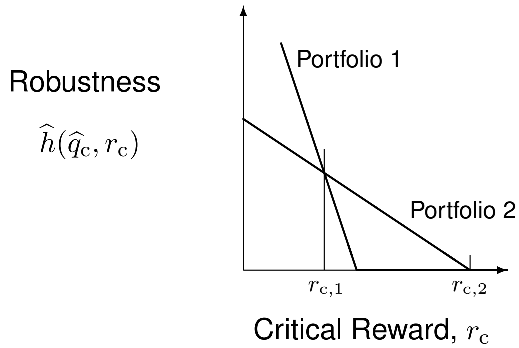

Figure 2: Robustness functions for two different portfolio investment alternatives, eq.(50), p.7.

Clearly:

- We prefer portfolio 1 for rewards $r_{\rm c} < r_{\rm c,1}$.
  Portfolio 2 is more risky than portfolio 1 in this range.
- We prefer portfolio 2 for rewards $r_{\rm c,1} < r_{\rm c} < r_{\rm c,2}$.
  Portfolio 1 is more risky than portfolio 2 in this range.
- Neither portfolio is acceptable for rewards $r_{\rm c,2} < r_{\rm c}$.
  Both portfolios very risky.
- Neither portfolio is **robust dominant.**
- Crossing robustness curves entails potential for **reversal of preference.**

Further questions that we did not address:

- Is the available robustness adequate?
- How much robustness is needed?

2.4 Opportuneness Function

§ **Two sides** of the “uncertainty coin”:

- Uncertainty can be **pernicious:** Robustness function protects against this.
- Uncertainty can be **propitious:** Opportuneness function exploits this.

§ We now develop the opportuneness function, $\widehat{\beta}(q, r_{\rm w})$.
$\widehat{\beta}(q, r_{\rm w}) =$ least uncertainty needed to enable (not guarantee) wonderfully large reward $r_{\rm w}$:

$$\widehat{\beta}(q, r_{\rm w}) = \min \left\{ h : \left(\max_{u \in \mathcal{U}(h, \widetilde{u})} r(q, u) \right) \geq r_{\rm w} \right\} \tag{51}$$

where:

$$r_{\rm w} \gg r_{\rm c} \tag{52}$$

Compare this to the robustness function, eq.(23) on p.4:
$\widehat{h}(q, r_{\rm c}) =$ maximum uncertainty tolerable to guarantee adequately large (critical) reward $r_{\rm c}$:

$$\widehat{h}(q, r_{\rm c}) = \max \left\{ h : \left(\min_{u \in \mathcal{U}(h, \widetilde{u})} r(q, u) \right) \geq r_{\rm c} \right\} \tag{53}$$

§ $\widehat{\beta}(q, r_{\rm w})$ and $\widehat{h}(q, r_{\rm c})$ are **dual functions,** responding to **dual** nature of uncertainty.

§ Distinct decision strategies:
$\widehat{\beta}(q, r_{\rm w})$: **windfalling** at $r_{\rm w}$. Aspiring to a wildly wonderful windfall.
$\widehat{h}(q, r_{\rm c})$: **satisficing** at $r_{\rm c}$. Requiring an essential critical return.

§ Proceeding as before with Lagrange optimization we find:

$$\max_{u \in \mathcal{U}(h, \widetilde{u})} q^T u = q^T \widetilde{u} + h\sqrt{q^T W^{-1} q} \tag{54}$$

Equate this to $r_{\rm w}$ and solve for h to find the opportuneness function:

$$\widehat{\beta}(q, r_{\rm w}) = \frac{r_{\rm w} - q^T \widetilde{u}}{\sqrt{q^T W^{-1} q}} \tag{55}$$

Note trade-off of certainty vs. windfall reward. Compare with robustness in eq.(34), p.5:

$$\widehat{h}(q, r_{\rm c}) = \frac{q^T \widetilde{u} - r_{\rm c}}{\sqrt{q^T W^{-1} q}} \tag{56}$$

- $\widehat{\beta}$ and $\widehat{h}$ both have trade offs in opposite directions: what do they mean?
- $\widehat{\beta}$ and $\widehat{h}$ both have zeroing at same value, but with different meanings.

§ Impose the same budget constraint:

$$q^T \mathbf{1} = B \tag{57}$$

Also, assume as before:

$$\widetilde{u} = u_o \mathbf{1} \tag{58}$$

The opportuneness function becomes:

$$\widehat{\beta}(q, r_{\rm w}) = \frac{r_{\rm w} - u_o B}{\sqrt{q^T W^{-1} q}} \tag{59}$$

Recall the robustness function, eq.(46) on p. 7:

$$\widehat{h}(q, r_{\rm c}) = \frac{u_o B - r_{\rm c}}{\sqrt{q^T W^{-1} q}} \tag{60}$$

§ Recall "Bigger is better" for $\widehat{h}$
$\Longrightarrow$ choose q to maximize $\widehat{h}$.
"Big is bad" for $\widehat{\beta}$
$\Longrightarrow$ choose q to minimize $\widehat{\beta}$.

§ Can we optimize $\widehat{h}$ and $\widehat{\beta}$ with the same q?
- $\max \widehat{h}$ requires minimum $q^T W^{-1} q$.
- $\min \widehat{\beta}$ requires maximum $q^T W^{-1} q$.

So we cannot simultaneously optimize $\widehat{h}$ and $\widehat{\beta}$:
Any change in q which increases $\widehat{h}$ also increases $\widehat{\beta}$.
Any change in q which decreases $\widehat{h}$ also decreases $\widehat{\beta}$.
Thus $\widehat{h}$ and $\widehat{\beta}$ are **antagonistic.** Each one trades off against the other.

§ Trade-off between robustness and opportuneness. From eqs.(59) and (60):

$$\frac{\mathrm{d}\widehat{h}(q,r_\mathrm{c})}{\mathrm{d}q} = -\frac{u_o B - r_\mathrm{c}}{q^T W^{-1} q} \underbrace{\frac{\mathrm{d}\sqrt{q^T W^{-1} q}}{\mathrm{d}q}}_{v} = -\frac{u_o B - r_\mathrm{c}}{q^T W^{-1} q} v \tag{61}$$

$$\frac{\mathrm{d}\widehat{\beta}(q,r_\mathrm{w})}{\mathrm{d}q} = -\frac{r_\mathrm{w} - u_o B}{q^T W^{-1} q} \underbrace{\frac{\mathrm{d}\sqrt{q^T W^{-1} q}}{\mathrm{d}q}}_{v} = -\frac{r_\mathrm{w} - u_o B}{q^T W^{-1} q} v \tag{62}$$

Hence:

$$\frac{\mathrm{d}\widehat{h}}{\mathrm{d}\widehat{\beta}} = \frac{u_o B - r_\mathrm{c}}{r_\mathrm{w} - u_o B} > 0 \tag{63}$$

- Eq.(63) is positive. **Question:** Good news or bad news?
- The trade-off between robustness and opportuneness is shown schematically in fig. 3.

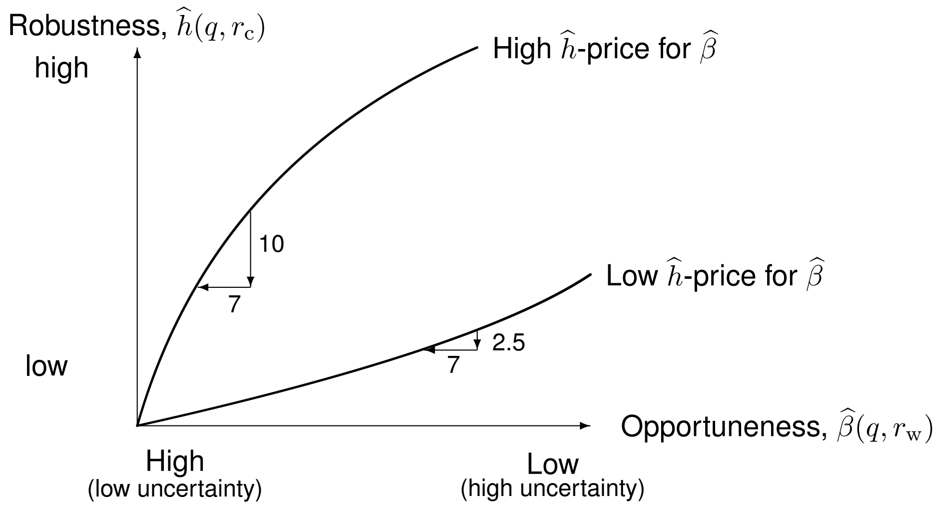

Figure 3: Trade-off between robustness and opportuneness.

§ Does $\widehat{\beta}$ have an optimum?
Can we maximize $q^T W^{-1} q$ subject to $q^T \mathbf{1} = B$?
No. See fig. 4.
For any constant = $q^T W^{-1} q$
There is a q that also satisfies the constraint.
However, as q moves far from the origin,
other constraints may become active.

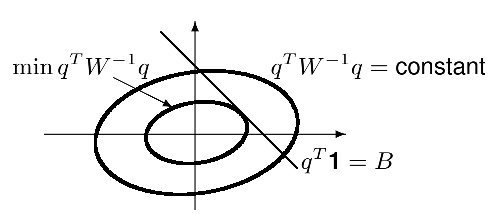

Figure 4: Schematic illustration of constrained optimization of $q^T W^{-1} q$.

3 Value at Risk in Financial Economics with Info-Gaps

§ **Three problems:**

- **Joint probabilities:** Uncertain common-mode defaults or economy-wide challenges.
  Consider two firms in different sectors (e.g. IT and manufacturing):
  They are connected by operating in the same economy.
  They both require workers and thus both may face deficiency if employment is high.
  They both require capital and thus both face high interest if inflation is high.
- **Fat tails** of true (but uncertain) distribution of returns:
  - Extreme outcomes more frequent than anticipated. **Question:** Good or bad news?
  - High percentiles under-estimated by putative distribution.
- **Past vs future:** Statistics vary in time. The future may be hard to predict.

§ **Two foci of uncertainty:**

- **Statistical** fluctuations:
  - Randomness, "noise".
  - Estimation uncertainty.
- **Knightian** uncertainty:
  - Surprises, e.g. innovations.
  - Structural changes to the economy, e.g. social change, environmental regulation, etc.
  - Historical data used to predict future.

§ **Outline:**

- Structured securities: Junior and senior tranches. Section 3.1.
- Origin of fat tails: Parameter uncertainty. Section 3.2.
- Value at risk (section 3.3):
  - Fat tails.
  - Thin tails.

3.1 Structured Securities

§ **Source:** Ben-Haim, *Info-Gap Economics: An Operational Introduction,* section 4.1.

§ **Two securities:**

- Jane takes a mortgage (or sells shares in a start-up) in Montana.
- John takes a mortgage (or sells shares in a start-up) in Mississippi.

§ **Tranche:** An issue of bonds derived from a pooling of similar obligations
(such as securitized mortgage debt of Jane and John).

§ **Two types of tranches:**

- Junior tranche: pays off if **neither** security fails. Both securities essential.
- Senior tranche: pays off if **at most one** security fails. Either security sufficient.

§ **Default of a security** (the security fails): The security does not pay.
E.g. John doesn't pay his mortgage or his start-up fails.

§ **Probabilities of default:**
- F_i = marginal default probability of security $i, \ i = 1, 2$.
- F_{12} = probability of joint default.

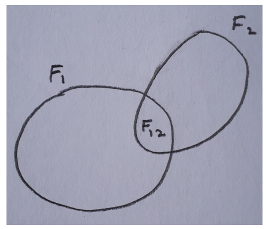

Figure 5: Venn diagram of marginal default probabilities, F_1 and F_2, and joint default probability F_{12}.

§ **Probabilities of tranche default:**
- Junior tranche default probability, see fig. 5, p.12. (**Why** "$-F_{12}$"):

$$F_{\rm J} = F_1 + F_2 - F_{12} \tag{64}$$

- Senior tranche default probability:

$$F_{\rm S} = F_{12} \tag{65}$$

§ **Example.**
- Estimates: $\widetilde{F}_i = 0.01. \quad \widetilde{F}_{12} = 0.0001$
- Junior tranche default probability: $F_{\rm J} = 0.0199$. Not too good.
- Senior tranche default probability: $F_{\rm S} = 0.0001$. Not too bad.

§ **Problem:**
Estimates of F_i and F_{12} uncertain, especially concerning innovative ventures.

$$\left| F_i - \widetilde{F}_i \right| \le h, \ i = 1, \, 2 \tag{66}$$

$$\left| F_{12} - \widetilde{F}_{12} \right| \le h \tag{67}$$

Horizon of uncertainty, h, is unknown. No known worst case.

§ **Info-gap model** (additional constraints assure mathematically legitimate probabilities):[4]

$$\mathcal{U}(h) = \Big\{ F_1, F_2, F_{12} : \quad \begin{array}{l} F_1 \ge 0, \quad F_2 \ge 0, \quad F_{12} \ge 0. \\ F_{12} \le \min[F_1, \ F_2]. \\ F_1 + F_2 - F_{12} \le 1. \\ \left| F_i - \widetilde{F}_i \right| \le h, \ i = 1, \, 2. \\ \left| F_{12} - \widetilde{F}_{12} \right| \le h \Big\}, \qquad h \ge 0 \end{array} \tag{68}$$

- Contraction: $\mathcal{U}(0) = \{\widetilde{F}_1, \widetilde{F}_2, \widetilde{F}_{12}\}$.
- Nesting: $h < h' \implies \mathcal{U}(h) \subseteq \mathcal{U}(h')$.

[4]Derivation in appendix 4.5 in *Info-Gap Economics,* pp.132–133.

- Family of nested sets.
- No known worst case.
- Non-probabilistic uncertainty about probabilities of default.

§ **Performance requirements:** Acceptable default probabilities:

- Junior:

$$F_{\mathrm{J}} \le F_{\mathrm{cJ}} \tag{69}$$

- Senior:

$$F_{\mathrm{S}} \le F_{\mathrm{cS}} \tag{70}$$

§ **Question:** Where do F_{cJ} and F_{cS} come from?

- Regulator requirements.
- Investor requirements or judgments.
- Competition between products and sectors.

§ **Robustness functions: definitions.**

- Maximum tolerable uncertainty.
- Max horizon of uncertainty at which default probability is acceptable.
- Junior and senior robustness functions:

$$\widehat{h}_{\mathrm{j}}(F_{\mathrm{cJ}}) = \max\left\{h : \left(\max_{F_i, F_{12} \in \mathcal{U}(h)} F_{\mathrm{J}}\right) \le F_{\mathrm{cJ}}\right\} \tag{71}$$

$$\widehat{h}_{\mathrm{s}}(F_{\mathrm{cS}}) = \max\left\{h : \left(\max_{F_i, F_{12} \in \mathcal{U}(h)} F_{\mathrm{S}}\right) \le F_{\mathrm{cS}}\right\} \tag{72}$$

§ **Assume estimated marginal default probabilities are equal:**

$$\widetilde{F}_1 = \widetilde{F}_2 = \widetilde{F} \tag{73}$$

§ **Robustness functions: expressions,** figs. 6, 7, p.14:[5]

- Junior tranche:

$$\widehat{h}_{\mathrm{j}}(F_{\mathrm{cJ}}) = \begin{cases} 0, & \text{if } F_{\mathrm{cJ}} < 2\widetilde{F} - \widetilde{F}_{12} \\ \dfrac{F_{\mathrm{cJ}} - (2\widetilde{F} - \widetilde{F}_{12})}{3}, & \text{if } 2\widetilde{F} - \widetilde{F}_{12} \le F_{\mathrm{cJ}} \le 2(\widetilde{F} + \widetilde{F}_{12}) \\ \dfrac{F_{\mathrm{cJ}}}{2} - \widetilde{F}, & \text{else} \end{cases} \tag{74}$$

- Senior tranche:

$$\widehat{h}_{\mathrm{s}}(F_{\mathrm{cS}}) = F_{\mathrm{cS}} - \widetilde{F}_{12} \tag{75}$$

  or zero if this is negative.
- **Trade off:** Robustness goes up (good) as critical default probability goes up (bad).
- **Zeroing:** No robustness of estimated default probability.

[5] Derivations in *Info-Gap Economics,* p.92.

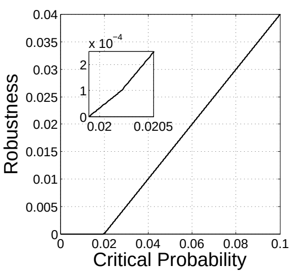

Figure 6: Junior robustness. $\widetilde{F}_i = 0.01$, $\widetilde{F}_{12} = 0.0001$.

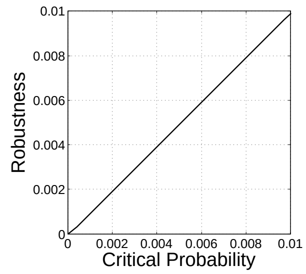

Figure 7: Senior robustness. $\widetilde{F}_i = 0.01$, $\widetilde{F}_{12} = 0.0001$.

§ **Numerical Results (figs. 6, 7):**

- **Trade off:** Robustness vs. critical probability.
- **Zeroing:** No robustness of estimated default probability.
- **What does $\widehat{h} = 0.02$ mean?** Is this low robustness? See info-gap model in eq.(68), p.12:

$$\mathcal{U}(h) = \Big\{ F_1, F_2, F_{12} : \quad F_1 \ge 0, \quad F_2 \ge 0, \quad F_{12} \ge 0. \\ F_{12} \le \min[F_1,\ F_2]. \\ F_1 + F_2 - F_{12} \le 1. \\ \left| F_i - \widetilde{F}_i \right| \le h,\ i = 1,\ 2. \\ \left| F_{12} - \widetilde{F}_{12} \right| \le h \Big\}, \qquad h \ge 0 \tag{76}$$

Recall: $\widetilde{F}_i = 0.01$, $\widetilde{F}_{12} = 0.0001$.
Thus $\widehat{h} = 0.02$ is **moderate** for junior tranche, and **large** for senior tranche.

- **High cost of rbs:** small slope.
  - ◦ Junior: $\Delta\widehat{h}/\Delta F_{\rm cj} =$ 0.5. **Why** is this “high” cost of robustness?
  - ◦ Senior: $\Delta\widehat{h}/\Delta F_{\rm cs} =$ 1. **Question:** Is this also high cost of robustness?

§ **Implication:**

Ignoring uncertainties in default probabilities causes unrealistic estimation of credit risk.

3.2 Parameter Uncertainty and FAT Tails

§ **Exponential tails** (thin tails) of pdf's are common (normal, exponential, Gamma) but not universal.

§ **Basic idea** of fat tails:

- Fat tails of uncertain true distribution:
  - Tail decays slower than exponential.
  - Not all moments exist: you don't know the mean and/or variance. **Why** does that matter?
- Estimated thin-tail pdf may have uncertain parameters.
- Total pdf may be fat tailed even if marginal pdf's are thin-tailed.

§ We now develop an **example** of a fat-tailed distribution.

§ **Exponential distribution of t.**

- t is a random variable:

$$f(t|\lambda) = \lambda \mathrm{e}^{-\lambda t}, \quad t \geq 0 \tag{77}$$

- All moments of $t|\lambda$ exist.
- $f(t|\lambda)$ is **not** fat-tailed.
- Common distribution for random time between events.

§ **Gamma distribution of λ: mixture of populations.**

- λ is a random variable.

$$\pi(\lambda) = \frac{\alpha}{\Gamma(k)}(\alpha\lambda)^{k-1}\mathrm{e}^{-\alpha\lambda}, \quad \lambda \geq 0, \quad \underbrace{\alpha > 0, \quad k > 0}_{\text{parameters}} \tag{78}$$

- All moments of λ exist.
- $\pi(\lambda)$ is **not** fat-tailed.
- Widely used for skewed positive-valued variables, e.g. disease rates, insurance claims, rainfall.

§ **What is marginal distribution of t?**

- $t|\lambda$ is exponential (thin).
- λ is Gamma (thin).
- t is Pareto (fat). (We won't prove this.)

§ We will now discuss the **Pareto distribution.**

- Define a dimensionless random variable where T is from eq.(77), α is from eq.(78):

$$Y = 1 + \frac{T}{\alpha} \tag{79}$$

- Y has a Pareto distribution:

$$f(y) = k\left(\frac{1}{y}\right)^{k+1}, \quad y \geq 1 \tag{80}$$

  Note the non-exponential fat tail.
- Mean and variance:

$$\mathrm{E}(y) = \frac{k}{k-1} \quad \text{if} \quad k > 1 \tag{81}$$

$$\mathrm{var}(y) = \frac{k}{k-2} - \left(\frac{k}{k-1}\right)^2 \quad \text{if} \quad k > 2 \tag{82}$$

- Not all moments exist (fat tails) if $k = 1$ or 2.
- Parameter uncertainty may imply fat tails (**Why?**) even if estimated moments are bounded.

§ **In summary:**
Exponential distribution (thin tail) combined with Gamma distribution (thin tail) produced Pareto distribution that may be fat tailed.

3.3 Value at Risk

§ **Source:** Ben-Haim, *Info-Gap Economics: An Operational Introduction,* section 4.2.

§ **Question:** why are tails (especially fat tails) of distribution of returns important?

3.3.1 Formulation: Quantile Risk

§ **Outcome, return, r:**
- Scalar random variable.
- Large is better than small.
- Too small is unacceptable.

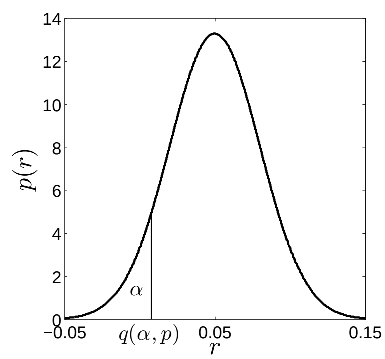

Figure 8: α quantile, $q(\alpha, p)$.

§ **Two questions:**
- What is the probability, α, of losing more the \$$q$?
- What is the quantity, \$$q$, for which α is the probability of losing more?

The quantile function provides answers.

§ α **Quantile,** $q(\alpha, p)$**:** The value of r whose probability of non-exceedence is α:

$$\alpha = \int_{-\infty}^{q(\alpha,p)} p(r)\,\mathrm{d}r \tag{83}$$

- α is probability of failure; usually small.
- $q(\alpha, p)$ is usually negative.
- $q(\alpha, p)$ is **value at risk:** probability of α to lose more than $|q(\alpha, p)|$.

§ At fixed α (e.g $\alpha = 0.02$):
- **small quantile (far left, very negative) $\Longleftrightarrow$ high risk.**
- Prefer slightly negative quantile over very negative quantile. **Why?**

E.g.: Prefer 2% probability of losing $1,000 over 2% probability of losing $10,000.

§ **Performance requirement** (reserve requirement) is r_c:

$$q(\alpha, p) \geq r_c \tag{84}$$

This is called a reserve requirement because you must (or should) keep \$$r_c$ in reserve.

§ **Problem:**
- pdf of r highly uncertain, especially on the tails.
- May have fat tails.
- Hence $q(\alpha, p)$ highly uncertain.

3.3.2 Example: Fat Tails

§ **Info-gap model of uncertainty:**
- $\widetilde{p}(r)$ is estimated pdf; e.g. normal. No fat tails.
- Envelope-bound uncertainty:

$$|p(r) - \widetilde{p}(r)| \leq g(r)h \tag{85}$$

- Some $p(r)$ distributions may have fat tails. Hence the uncertainty very important.

§ **How to choose envelope?**
- Economic judgment. E.g. one might expect Pareto distribution as discussed earlier.
- Analogical reasoning: Similarity to historical cases.
- Assess "equivalent risk": Comparing thin and fat tailed distributions.

§ **Example: $g(r)$ is "$1/r^2$" on tails: Fat tails.**

$$g(r) = \begin{cases} \dfrac{(\mu - r_s)^2 \, \widetilde{p}(\mu - r_s)}{r^2} & \text{if } r < \mu - r_s \\[2ex] \widetilde{p}(r) & \text{if } |r - \mu| \leq r_s \\[2ex] \dfrac{(\mu + r_s)^2 \, \widetilde{p}(\mu + r_s)}{r^2} & \text{if } r > \mu + r_s \end{cases} \tag{86}$$

§ **A pdf with $1/r^2$ tails:**
- Normalized.
- Finite mean.
- Unbounded variance.

§ **Info-gap model:**

$$\mathcal{U}(h) = \left\{ p(r) : \ p(r) \geq 0, \ \int_{-\infty}^{\infty} p(r) \, \mathrm{d}r = 1, \ |p(r) - \widetilde{p}(r)| \leq g(r)h \right\}, \quad h \geq 0 \tag{87}$$

§ **Robustness:**

Max horizon of uncertainty at which loss is acceptable (satisfies reserve requirement):

$$\widehat{h}(\alpha, r_c) = \max \left\{ h : \ \left(\min_{p \in \mathcal{U}(h)} q(\alpha, p) \right) \geq r_c \right\} \tag{88}$$

§ **Robustness depends on:**

- Underlying securities.
- Designated failure probability, α.
- Critical loss (reserve requirement), $r_{\rm c}$.

§ **Robustness is a decision function:**

- Satisfy requirement: Quantile is large enough.
- Maximize robustness.
- Don't try to minimize risk, α, or to maximize the quantile, $q(\alpha, p)$.

§ **Example: Uncertain fat tails.**

- $\widetilde{p}(r)$ normal with mean μ and variance σ^2.
- Info-gap model of eq.(87) with $r_{\rm s} = 2\sigma$. **Why** 2σ?[6]
- 1% risk so $\alpha = 0.01$ unless otherwise stated.
- 3 Portfolios: lo, med, hi expected return.

[6] We are fairly confident in $\widetilde{p}(r)$, based on evidence, within 2σ of the estimated mean.

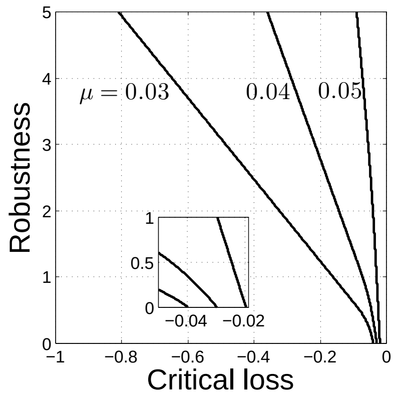

Figure 9: Robustness, $\widehat{h}$, vs. critical loss, r_c, for 3 values of μ. $\sigma = 0.03$. Uncertain fat tails.

§ **Results in fig. 9:**

- Zeroing: No robustness of estimated critical loss.
- Trade off: robustness improves (goes up) as critical loss gets worse (goes down).
- Effects of increasing the mean return:
  - ○ Shift to higher robustness (good).
  - ○ Increase slope: reduce cost of robustness (good).

§ **Calibrating the robustness in fig. 9:**

- Is robustness of 3 or 5 “large”?
- $\widehat{h} = 3$ means 3 $\times$ the fat tail is tolerable.
- Standard normal at -2σ: 0.0228.
- Fat tail at -2σ: 0.1080. Depends on $g(r)$ from eq.(86), p.17.
- Robustness of 3 $\times$ the fat tail seems substantial.
- This is a judgment based on contextual understanding, experience, and risk preference.

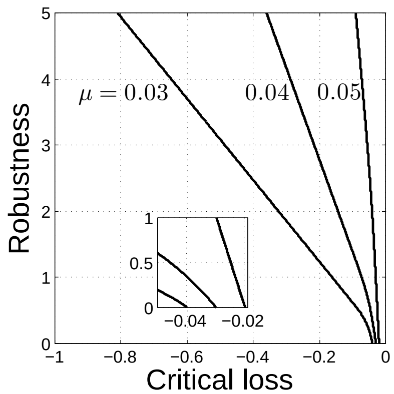

Figure 10: Robustness, $\widehat{h}$, vs. critical loss, r_c, for 3 values of μ. $\sigma = 0.03$. Same as fig. 9. Uncertain fat tails.

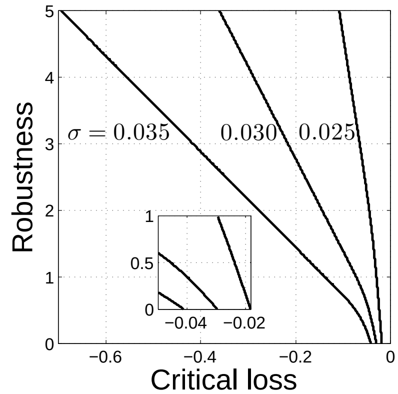

Figure 11: Robustness, $\widehat{h}$, vs. critical loss, r_c, for 3 values of σ. $\mu = 0.04$. Uncertain fat tails.

§ **Mean-variance trade-off: comparing figs. 10 and 11.**

- Fig. 10: robustness curves for various μ at fixed σ.
- Fig. 11: robustness curves for various σ at fixed μ.
- Figs. 10 and 11 quite similar.
- Increasing mean by 0.01 roughly equivalent to decreasing standard deviation by 0.005.
- Recall previous discussion of balancing mean and variance, eq.(9), p.2, by maximizing:

$$F = \left(q^T \overline{u}\right)^2 - \lambda \sigma_r^2 \tag{89}$$

This required knowledge of moments and of the tails.

- Current approach manages uncertainty in the pdf.

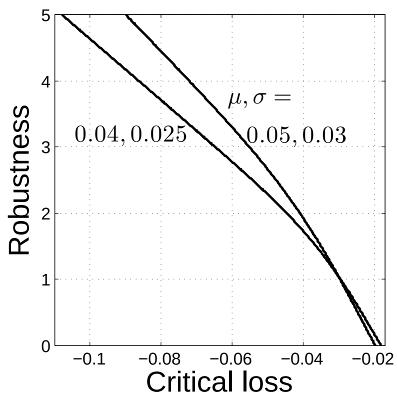

Figure 12: Robustness, $\widehat{h}$, vs. critical loss, r_c, for two different combinations of μ and σ. Uncertain fat tails.

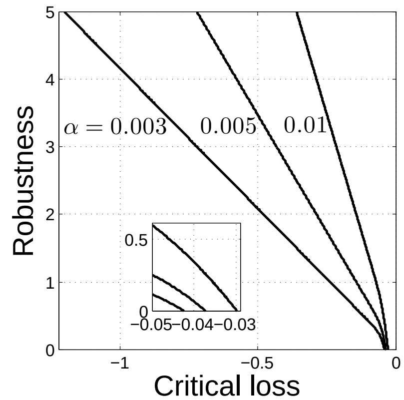

Figure 13: Robustness, $\widehat{h}$, vs. critical loss, r_c, for three different probabilities of failure, α. $\mu = 0.04$ and $\sigma = 0.03$. Uncertain fat tails.

§ **Preference reversal: comparing two portfolios in fig. 12.**

- $(\mu, \sigma) = (0.04, 0.025) \succ_{\mathbf{est.}} (0.05, 0.03)$. Putative, zero-robustness comparison.
  Question: Is $\succ_{\mathbf{est.}}$ a good basis for decision?
- $(\mu, \sigma) = (0.05, 0.03) \ \succ_{\mathbf{rbs.}} (0.04, 0.025)$ at $\widehat{h} > 1$.
  Question: Is the robust advantage of $(\mu, \sigma) = (0.05, 0.03)$ significant at critical loss -0.06?

§ **Effect of greater probability of failure (greater acceptable risk), α, fig. 13:**

- Greater robustness (curves shift right) as α increases.
- Lower cost of robustness (steeper curves) as α increases.
- Trade offs: μ, σ and α.

3.3.3 Example: Thin Tails

§ **Less severe uncertainty:**

- Thin tails: Less uncertainty about extreme events.
- Known pdf family: Normal distributions.
- Uncertain moments of normal pdf:

$$\mathcal{U}(h) \;=\; \left\{ p(r) \sim N(\mu, \sigma^2) : \; \left| \frac{\mu - \widetilde{\mu}}{\varepsilon_\mu} \right| \le h, \; \left| \frac{\sigma - \widetilde{\sigma}}{\varepsilon_\sigma} \right| \le h, \; \sigma \ge 0 \right\}, \quad h \ge 0 \tag{90}$$

Note **non-probabilistic** uncertainty about moments of a **probability** distribution.

- Robustness as in eq.(88), p.17:
  Max horizon of uncertainty at which loss is acceptable (satisfies reserve requirement):

$$\widehat{h}(\alpha, r_c) = \max \left\{ h : \; \left(\min_{p \in \mathcal{U}(h)} q(\alpha, p) \right) \ge r_c \right\} \tag{91}$$

§ **Results, fig. 14:**

- Trade off: Robustness vs. critical loss.
- Zeroing: No rbs. of est. critical loss.
- Robustness: Greater than for fat tails (but not enormously greater).

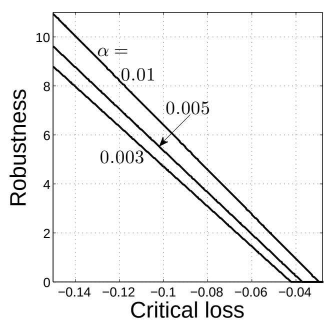

Figure 14: Robustness, $\widehat{h}$, vs. critical loss, r_c, for three different probabilities of failure, α. $\widetilde{\mu} = 0.04$, $\widetilde{\sigma} = 0.03$, $\varepsilon_\mu = 0.004$, $\varepsilon_\sigma = 0.003$. Uncertain moments.

§ **Preference reversal, fig. 15:**

- $(\mu, \sigma) = (0.04, 0.03) \;\succ_{\mathbf{est.}}\; (0.02, 0.025)$. Low-robustness.
- $(\mu, \sigma) = (0.02, 0.025) \succ_{\mathbf{rbs.}} (0.04, 0.03)$. $\widehat{h} > 2$.

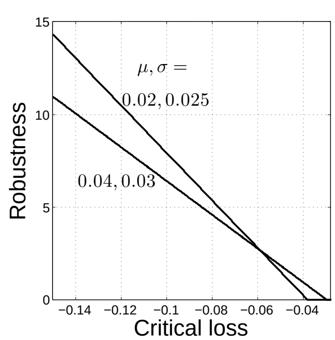

Figure 15: Robustness, $\widehat{h}$, vs. critical loss, r_c, for two different combinations of μ and σ. $\alpha = 0.01$. Uncertain moments.