# 15 Strategic Asset Allocation

§ This section based on section 4.4 of Yakov Ben-Haim, 2010, *Info-Gap Economics: An Operational Introduction,* Palgrave.

§ **Generic idea of an asset:**
- Energy supply to different actuators: motion on complex terrain; robotics.
- Duration and force at load points for deflection, especially in non-linear system.
- Duration at search locations (looking for treasure or enemies).
- People developing innovative ideas or projects.
- Stocks or bonds in finance: monetary return.

§ **Generic idea of strategic allocation:**
- Dynamic setting: multiple time steps.
- Allocation at each time step.
- Budget limitation.
- "Returns" or "outcomes" at each step determine resources for next step.

§ **Basic idea of asset allocation ("financial" model):**
- Choose an allocation of resources (e.g. budget) between different assets.
- The future returns are random and the pdf is uncertain.
- You require high probability that the future balance is acceptable.
  That is, the future **capital reserve** (or profit) must be adequate with high probability.

## 15.1 Budget Constraint

**Basic variables:**

$x_{it}$ is the **quantity of the $i$th asset which is purchased** at time $t$. $x_{it}$ can be either positive or negative. The allocation vector is $x_t = (x_{1t}, \ldots, x_{Nt})^T$. This is **chosen at time $t$.**

$p_{it}$ is the **ex-dividend price**[3] **of the $i$th asset** for purchase at time $t$. The vector of prices is $p_t = (p_{1t}, \ldots, p_{Nt})^T$. **Known at time $t$.**

$y_{it}$ is the **payoff of the $i$th asset** at time $t+1$. The vector of payoffs is $y_t = (y_{1t}, \ldots, y_{Nt})^T$. **Not known at time $t$.**

$c_t$ is the **capital reserve** of the financial institution[4] at time $t+1$. **Not known at time $t$.**

The **budget constraint:**

$$c_t + p_t^T x_t = y_t^T x_{t-1} \tag{235}$$

[3] Ex-dividend price of a stock is the price without the value of the next dividend payment.
[4] For an individual investor $c_t$ could be thought of as consumption.

## 15.2 Uncertainty

§ **Moderate uncertainty:**

- $y_t$ is random and known to be normally distributed.
- Moments are estimated but uncertain:
  - ∘ Estimated mean of the payoff vector is $\mu_{yt}$.
  - ∘ Estimated covariance matrix of the payoff is $\Sigma_{yt}$.

§ Thus, from the budget constraint in eq.(235), the capital reserve is a normal random variable with estimated mean and variance:

$$\widetilde{\mu}_{ct} = -p_t^T x_t + \mu_{yt}^T x_{t-1} \tag{236}$$

$$\widetilde{\sigma}_{ct}^2 = x_{t-1}^T \Sigma_{yt} x_{t-1} \tag{237}$$

§ Error values of the estimated mean and standard deviation, $\widetilde{\mu}_{ct}$ and $\widetilde{\sigma}_{ct}$, are $\varepsilon_\mu$ and $\varepsilon_\sigma$.

§ Info-gap model for uncertainty in the distribution of the capital reserve, $c_t$:

$$\mathcal{U}(h) = \left\{ f(c_t) \sim N(\mu_{ct}, \sigma_{ct}^2) : \ \left| \frac{\mu_{ct} - \widetilde{\mu}_{ct}}{\varepsilon_\mu} \right| \le h, \ \left| \frac{\sigma_{ct} - \widetilde{\sigma}_{ct}}{\varepsilon_\sigma} \right| \le h, \ \sigma_{ct} \ge 0 \right\}, \quad h \ge 0 \tag{238}$$

## 15.3 Performance and Robustness

**Performance requirement.**
The $\alpha$ **quantile** of the distribution $f(c_t)$, denoted $q(\alpha, f)$, is the value of $c_t$ for which the probability of being less than this value equals $\alpha$. This quantile is defined in:

$$\alpha = \int_{-\infty}^{q(\alpha,f)} f(c_t)\, \mathrm{d}c_t \tag{239}$$

$\alpha$ is typically small so $q(\alpha, f)$ may be negative.

§ The **performance requirement** is:

$$q(\alpha, f) \geq r_{\mathrm{c}} \tag{240}$$

We will use the robustness function to evaluate the confidence in satisfying this requirement for chosen investment, $x_t$.

**Robustness function:**

$$\widehat{h}(x_t, r_{\mathrm{c}}) = \max \left\{ h : \left( \min_{f \in \mathcal{U}(h)} q(\alpha, f) \right) \geq r_{\mathrm{c}} \right\} \tag{241}$$

§ $z_\alpha$ is the $\alpha$ quantile of the standard normal distribution.
- Assume: $\alpha < 1/2$ so that $z_\alpha < 0$.
- Typically $\alpha$ around 0.01.

§ One can show:

$$\widehat{h}(x_t, r_{\mathrm{c}}) = \frac{r_{\mathrm{c}} - q(\alpha, \widetilde{f})}{\varepsilon_\sigma z_\alpha - \varepsilon_\mu} \tag{242}$$

or zero if this is negative.
- The numerator and denominator are both negative, so the robustness decreases as $r_{\mathrm{c}}$ increases towards $q(\alpha, \widetilde{f})$.

## 15.4 Opportuneness Function

§ **Windfall aspiration is:**

$$q(\alpha, f) \geq r_{\mathrm{w}} > r_{\mathrm{c}} \tag{243}$$

§ **Opportuneness:**

$$\widehat{\beta}(x_t, r_{\mathrm{w}}) = \min \left\{ h : \left( \max_{f \in \mathcal{U}(h)} q(\alpha, f) \right) \geq r_{\mathrm{w}} \right\} \tag{244}$$

§ Inverse of opportuneness:

- $M(h)$ denotes the **inner maximum** in eq.(244).
- $M(h)$ is the **inverse of the opportuneness.**
- That is, a plot of $M(h)$ vs. $h$ is the same as a plot of $r_{\mathrm{w}}$ vs. $\widehat{\beta}(x_t, r_{\mathrm{w}})$.
- We will derive an explicit expression from which to evaluate $M(h)$.

§ **Ramp function:** $r(x) = 0$ if $x < 0$ and $r(x) = x$ if $x \geq 0$.

§ **One assumption:**

- $z_\alpha$ is the $\alpha$ quantile of the standard normal distribution.
- We assume that $\alpha < 1/2$, so that $z_\alpha < 0$.

§ **One can show:**

$$q(\alpha, f) = \sigma_{ct} z_\alpha + \mu_{ct} \tag{245}$$

Proof:

$$\alpha = \mathrm{Prob}\left(x \leq q(\alpha, f)\right) \tag{246}$$

$$= \mathrm{Prob}\left( \frac{x - \mu_{ct}}{\sigma_{ct}} \leq \frac{q(\alpha, f) - \mu_{ct}}{\sigma_{ct}} \right) \tag{247}$$

Note that:

$$z = \frac{x - \mu_{ct}}{\sigma_{ct}} \sim \mathcal{N}(\mu_{ct}, \sigma_{ct}) \tag{248}$$

$$z_\alpha = \frac{q(\alpha, f) - \mu_{ct}}{\sigma_{ct}} \tag{249}$$

Re-arranging eq.(249) leads to eq.(245).

§ **Inverse of opportuneness function:**

$$M(h) = r(\widetilde{\sigma}_{ct} - \varepsilon_\sigma h) z_\alpha + \widetilde{\mu}_{ct} + \varepsilon_\mu h \tag{250}$$

## 15.5 Policy Exploration

§ **Example:**

- One risk-free asset, $i = 1$, and a one uncorrelated risky asset, $i = 2$.
- Select the allocation.
- Price vector is $p_t = (7, 10)$.
- The level of confidence of the quantile is $\alpha = 0.01$.
- The standard deviation of the payoff of the risky asset is 5% of its estimated mean unless indicated otherwise.
- Thus $(\Sigma_{yt})_{22} = (0.05\mu_{yt,2})^2$. The other elements of the $2 \times 2$ covariance matrix $\Sigma_{yt}$ are zero.

§ **Trade-offs and zeroing** (fig. 20):

- Robustness vs critical reserve.
- Opportuneness vs windfall reserve.

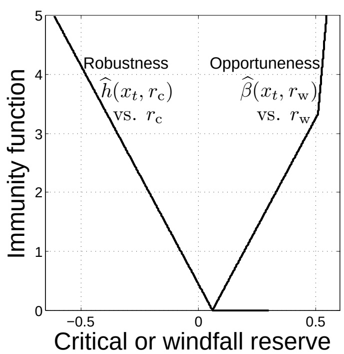


Figure 20: Robustness and opportuneness curves. $x_{t-1} = x_t = (0.7, 0.3)^T$. $\mu_{yt} = (1.04p_{1t}, 1.08p_{2t})^T$. $\varepsilon_\mu = 0.05\widetilde{\mu}_{ct}$. $\varepsilon_\sigma = 0.3\widetilde{\mu}_{ct}$.

| Portfolio | $\mu_{yt,1}/p_{1t}$ | $\mu_{yt,2}/p_{2t}$ | $\widetilde{\mu}_{ct}$ | $\widetilde{\sigma}_{ct}$ | $\varepsilon_\mu/\widetilde{\mu}_{ct}$ | $\varepsilon_\sigma/\widetilde{\sigma}_{ct}$ |
|---|---|---|---|---|---|---|
| 1 | 0.04 | 0.08 | 0.436 | 0.162 | 0.05 | 0.1 |
| 2 | 0.036 | 0.076 | 0.404 | 0.161 | 0.035 | 0.075 |

Table 1: Parameters of two portfolios. Robustness curves in fig. 21.

**Choose between two portfolios,** table 1.

- First portfolio has higher estimated mean payoffs and higher errors.
- Classical dilemma: portfolio 1 is better on average, but more uncertain.

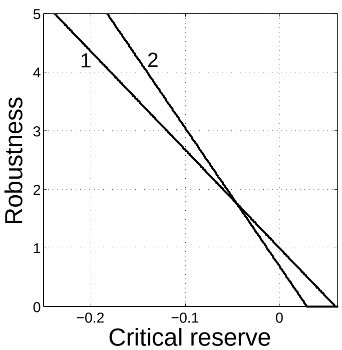


Figure 21: Robustness curves. $x_{t-1} = x_t = (0.7, 0.3)^T$. See table 1.

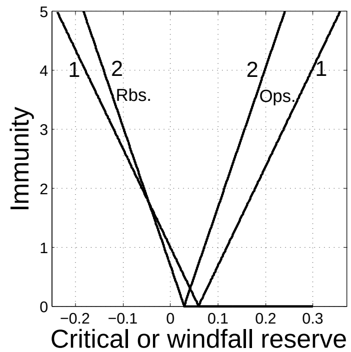


Figure 22: Robustness and opportuneness curves for portfolios in fig. 21.

§ **Preference reversal,** fig. 21.

§ **Robustness and opportuneness,** fig. 22.

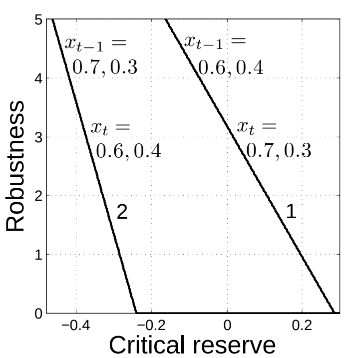


Figure 23: Robustness curves for two sequences of investments.

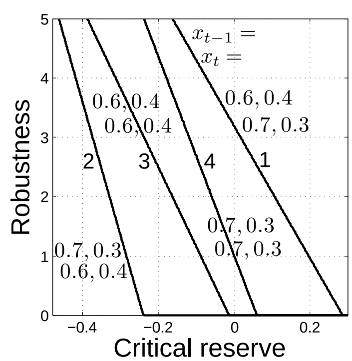


Figure 24: Robustness curves for 4 sequences of investments. Curves 1 and 2 reproduced from fig. 23.

§ **Sequence matters,** fig. 23.

- Sequence of investment vectors are reversed between the two portfolios.
- Two differences between outcomes:
  - ○ Portfolio 1 has much higher nominal $\alpha$ quantile (horizontal intercept).
  - ○ Portfolio 2 has steeper slope, which implies lower cost of robustness.

§ **Sequence matters,** fig. 24.

- Portfolios 1 and 2 same as fig. 23.
- Portfolio 3 and 4 are similar, and without investment change over time.