Problem Set on

Economic Decision Making for Engineers

List of problems:

[0] \lectures\Intro-Econ-DM\EconDecMakHW02.tex 29.6.2023

1. **Compound interest.** (p.52)[1] An amount, F, 6 years in the future, is equivalent to \$1,500 now, when the annual interest rate is 12%. What is the value of F?

2. **Compound interest.** (p.53)[2] How much interest is payable each year on a loan of \$2,000 if the interest rate is 10% per year when half of the loan principal will be repaid as a lump sum at the end of 4 years and the other half will be repaid as a lump sum at the end of 8 years? How much interest will be paid over the 8-year period?

3. **Compound interest.** (p.54)[3] In problem 2, if the interest had not been paid each year and if the principal was repaid entirely at the end, how much interest would be due at the end of the 8th year?

4. **Equivalent annual payment.** (p.55)[4] A present obligation of \$20,000 is to be repaid in equal uniform annual amounts, each of which includes repayment of the debt (principal) and interest on the debt, over a period of 5 years. If the interest rate is 10% per year, what is the amount of the annual payment?

5. **Compound interest.** (p.56)[5] Suppose that the \$20,000 in problem 4 is to be repaid at the rate of \$4,000 per year plus the interest that is owed based on the beginning-of-year unpaid principal. Compute the total amount of interest paid in this situation and compare it with problem 4.

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[1] DeGarmo, 3-10, p.123.
[2] DeGarmo, 3-4, p.122.
[3] DeGarmo, ∼3-5, p.122.
[4] DeGarmo, 3-11, p.123.
[5] DeGarmo, 3-12, p.123.

6. **Investment and return.** (p.57)[6] Do you recommend the following investment over 8 years in a new system? The initial cost is $640,000. The income is $180,000 at the end of each year. The annual cost of operating the new system is $44,000 at the end of each of the first two years, and decreases by $4,000 in each of the subsequent years. The company's minimal acceptable rate of return (MARR) is 15%. The system has no salvage value at the end of 8 years.

[6] DeGarmo, 4-2a, p.177.

7. **Size of a grant.** (p.58) You are considering an engineering project that has no financial return, but is needed in order to provide a public good. The operational life is 10 years. The operating costs are $15,000 at the end of each year. The re-sale (salvage) value of the equipment at the end of the project is $3,000. A grant of $S at the start of the project will fund the project. How large a grant should you take in order to cover the operating costs, accounting for the salvage value? Assume that un-used funds from the grant can be invested at 6% interest until they are needed.

8. **Present worth of a loan,** (p.58) Consider a loan P at annual interest i for N years. The principal is repaid in equal installments at the end of each year. The accrued interest is paid at the end of each year.

   (a) What is the PW of the loan payments for $N = 10$, $i = 0.06$ and $P =$\$100,000?

   (b) ‡[7] What is the PW for any N, i and P?

[7] Problems marked ‡ are challenging.

9. **Choose quality and lifetime.** (p.60) You must choose between two design concepts. Design 1 has higher quality, higher projected earnings R_1 at the end of each year, longer life, N_1 years, but higher initial cost S_1. Design 2 is the reverse, and has lower quality, lower projected earnings R_2 at the end of each year, shorter life, N_2 years, but lower initial cost S_2. Whichever design you choose, it must be implemented for $2N$ years. The operational lifetimes of designs 1 and 2 are $N_1 = 2N$ and $N_2 = N$, respectively. Thus, if you choose design 2, it must be re-purchased after N years. Which design do you prefer, as a function of the parameters? Suppose a MARR of 15%.

10. **Uncertain simple interest.** (p.63) A loan, P, with simple interest i for N periods incurs a future repayment $F = (1 + iN)P$. Given an estimate $\widetilde{i}$ with error s_i, use a fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ i : \left| \frac{i - \widetilde{i}}{s_i} \right| \le h \right\}, \quad h \ge 0 \tag{1}$$

(a) Derive an expression for the robustness of requiring that the future repayment will not exceed the critical value F_{c}.

(b) Given: $N = 5$, $\widetilde{i} = 0.06$, $s_i = 0.02$, $P = \$10,000$ and contextual information that the estimated interest of 6% can err by several percentage points or more. What is a reliable estimate of the future repayment?

11. **Uncertain compound interest.** (p.64) Invest $\$P$ at the start of N years with estimated yearly rate of return $\widetilde{i}$, where the estimate may err by tens of percent or more. While the actual rate of return, i, is unknown, assume that it is constant over time. Furthermore, assume that i cannot be less than -1 ($i = -1$ implies FW is zero, and any more negative value of i implies debt).
(a) Derive an expression for the robustness of requiring that the future worth of the investment will not be less than the critical value FW_{c}.
(b) Based on (a) and given that $N = 5$, $\widetilde{i} = 0.06$ and $P = \$10,000$, what is a reliable estimate of the future worth?
(c) Return to (a) and compare two alternatives with different estimated rates of return, $\widetilde{i}_1 < \widetilde{i}_2$. For what values of FW_{c} do you prefer each alternative, based on the robustness to uncertainty?
(d) Evaluate (c) for $\widetilde{i}_1 = 0.02$, $\widetilde{i}_2 = 0.03$, $N = 5$ and $P = \$10,000$.

12. **Investment with uncertain costs and returns.** (p.66) Consider the following project. The initial cost is S. The income is estimated to be $\widetilde{R}$ at the end of each year. The annual cost of operating the new system is estimated to be $\widetilde{C}$ at the end of each year. These estimates may err substantially. Use a fractional error info-gap model for costs and returns:

$$\mathcal{U}(h)\left\{R,C:\ \left|\frac{R_k-\widetilde{R}}{\varepsilon_R\widetilde{R}}\right|\le h,\ \left|\frac{C_k-\widetilde{C}}{\varepsilon_C\widetilde{C}}\right|\le h,\ k=1,\,\ldots,\,N\right\},\quad h\ge 0 \tag{2}$$

The company's minimal acceptable rate of return (MARR) is i. The system will operate for N years.

(a) Derive the robustness function of the PW.

(b) Compare two realizations of this system with the following characteristics:

$$\text{PW}(\widetilde{R}_1,\widetilde{C}_1) \quad < \quad \text{PW}(\widetilde{R}_2,\widetilde{C}_2) \tag{3}$$

$$\varepsilon_{1,R}\widetilde{R}_1+\varepsilon_{1,C}\widetilde{C}_1 \quad < \quad \varepsilon_{2,R}\widetilde{R}_2+\varepsilon_{2,C}\widetilde{C}_2 \tag{4}$$

For what values of the PW do you prefer option 1? Provide an intuitive explanation of the results.

13. **Investment with uncertain probabilistic returns.** (p.68) Consider the following project. The initial cost is S. The income is R at the end of each year. The annual cost of operating is C at the end of each year. The company's minimal acceptable rate of return (MARR) is i. The system will operate for N years.

(a) Suppose that R is the same each year but that its value is a random variable with an exponential distribution: $p(R) = \lambda e^{-\lambda R}$, for $R \geq 0$. Derive an expression for the probability of "failure": PW less than the critical value PW_c.

(b) Suppose that R is the same each year but that its value is a random variable with an exponential distribution as in part (a), and that λ is uncertain. The estimated value is $\widetilde{\lambda}$, but the fractional error of this estimate is unknown. We require that the probability of failure not exceed a critical value P_{fc}. Derive the robustness function.

(c)‡ Suppose that R is the same each year but that its value is a random variable whose distribution is thought to be exponential with coefficient $\widetilde{\lambda}$. However, the absolute error of this probability density is unknown. Consider the special case that the estimated probability of failure is much less than 1. Derive the robustness function for avoiding failure.

14. **Investment with uncertain costs and returns, revisited.** (p.70) Return to problem 6 and consider uncertainty in the operating costs and yearly returns. Here are the details of problem 6. The initial cost is $640,000. The income is estimated to be $180,000 at the end of each year, but this estimate may err by 30% or more. The annual cost of operating the new system is estimated to be $44,000 at the end of each of the first two years, and to decrease by $4,000 in each of the subsequent years. The operating costs may err by 10% or more. The company's minimal acceptable rate of return (MARR) is 15%. The system has no salvage value at the end of 8 years. Derive the robustness function of the PW.

15. **Benefit-cost ratio of two design concepts.** (p.71) Consider two design concepts. For both concepts the benefit and maintenance cost at the end of each year is $4,000 and $1,500, respectively. The interest rate is $i = 0.05$.

(a) Evaluate the benefit-cost ratios of the two design concepts, where the anticipated usable lifetimes of designs 1 and 2 are $N_1 = 3$ years and $N_2 = 5$ years. The initial investments in designs 1 and 2 are $S_1 = \$20,000$ $S_2 = \$33,333$. Which design has a better benefit-cost ratio (BCR)? Why? Note that $S_1/N_1 = S_2/N_2$. So why is the BCR result surprising?

(b) For what ratio of initial investment is the BCR the same for the two designs? What does this imply about the initial costs of the two designs?

16. **Present worth or benefit-cost ratio?** (p.73) Consider two design concepts for a system with $N = 5$ year expected life, financed at an interest rate of $i = 0.05$. For the jth system, the initial cost is S_j and the benefit and maintenance costs at the end of each year are B_j and C_j respectively. Consider specific values:

    $B_1 = \$1,270.35$, $C_1 = \$461.95$, $S_1 = \$1,000$.

    $B_2 = \$1,154.87$, $C_2 = \$415.75$, $S_2 = \$800$.

    (a) One team of analysts uses the present worth method to compare these concepts, and another team uses the benefit-cost ratio. What do they recommend? Do they agree? What does this imply?

    (b) Now consider fractional uncertainty in benefits and costs with the following info-gap model:

$$\mathcal{U}(h) = \left\{ B_j, C_j : \left| \frac{B_j - \widetilde{B}_j}{w_{B,j}} \right| \le h, \left| \frac{C_j - \widetilde{C}_j}{w_{C,j}} \right| \le h, \ j = 1, \ 2 \right\}, \quad h \ge 0 \tag{5}$$

The nominal values, $\widetilde{B}_j$ and $\widetilde{C}_j$, take the previous numerical values. The uncertainty weights, $w_{B,j}$ and $w_{C,j}$, are known positive values.

Derive separate robustness functions for satisficing the PW and satisficing the BCR, for each design. You will derive 4 robustness functions. Does the robust prioritization of the designs based on PW, necessarily agree with the robust prioritization based on BCR? Explain.

17. **Inspection system with uncertain diminishing inspection rate.** (p.75) Consider a new inspection system with initial cost S and end-of-year maintenance cost of C each year, financed at an interest rate of 4%. The system is expected to last N years. In the first year of operation the system performs B inspections, but thereafter the number of yearly inspections decreases by 6% each year. Treat this rate of decrease as a discount rate.
(a) Calculate the benefit-cost ratio. Evaluate it numerically for $S = \$20,000$, $C = \$1,500$, $B = 350,000$ and $N = 10$.
(b) The percentage yearly reduction of the inspection rate is estimated at $i_b = 0.06$. While i_b will be constant over time, its value may err by several tens of percent (plus or minus several times 0.006) or more. Evaluate the robustness of the BCR. Use the robustness to determine a reliable value of BCR, inspections/$.

18. **General inflation.** (p.77) (Based on DeGarmo, 9-2, p.395) If the average general inflation is expected to be 8% per year, how many years will it take to reduce the currency's purchasing power by one-half? Generalizing this, how long would it take to reduce the currency's purchasing power by a factor of $1/n$?

19. **Comparing alternatives under inflation.** (p.79) (DeGarmo, 9-4, p.396) The annual expenses for two programs are evaluated on different bases, as shown in table 1.

End of Year	Alternative A Nominal $	Alternative B Real $
1	−120,000	−100,000
2	−132,000	−110,000
3	−148,000	−120,000
4	−160,000	−130,000

Table 1: Problem 19.

If the average general price inflation is expected to be 6% per year, and the real rate of interest is 9% per year, evaluate the PW of the two alternatives. Which has the least negative PW?

20. **Salary erosion from inflation.** (p.80) (Based on DeGarmo, 9-6, p.396) An engineer received the nominal salaries shown in table 2 over the past 4 years, with inflation, f_k, in % indicated for each year.
    (a) If f_k is a measure of the general price inflation, evaluate the annual salaries in real year-0 dollars.
    (b) Now suppose that the inflation values in table 2 are estimates, where each estimate could err by $\pm$10% or more. You require that the real income in each year, $k = 1, \ldots, 4$, not be less than a specified value $R_{k,\text{c}}$. Derive an expression for the inverse of the robustness function for each year.

End of Year k	Nominal salary A_k ($)	f_k
1	34,000	7.1%
2	36,200	5.4%
3	38,800	8.9%
4	41,500	11.2%

Table 2: Data for problem 20.

21. **Exchange rate devaluation.** (p.82) (DeGarmo, 9-30, p.400) A US firm requires a 26% rate of return in US$ on an N-year investment in a foreign country. The real return in the foreign currency in year k is $R_{r,\text{for}}$. The year-0 exchange rate is $r_0 = 1$. The initial investment is S US$.

    (a) If the currency of the foreign country is expected to devalue at an average annual rate of 8% with respect to the US$, what rate of return in the foreign country would be required to meet the firm's requirement?

    (b) If the dollar is expected to devalue at an average annual rate of 6% with respect to the currency of the foreign country, what rate of return in the foreign country would be required to meet the firm's requirement?

22. ‡ **Real and nominal preferences with inflation and interest.** (p.83) Given real interest rate $i_{\rm r}$, inflation rate f, and two nominal series of expenses: $A_k^{(1)}$ and $A_k^{(2)}$, $k = 1, \ldots, N$.
   (a) Calculate present nominal worth and present real worth for each series, and show that they are equal.
   (b) Is it true that the preference between these two series is the same with the nominal and real PWs? Prove your assertion, stating any needed assumptions.
   (c) Extend part (a) for time-varying interest and inflation: $i_{{\rm r},k}$ and f_k.
   (d) Formulate a procedure for evaluating the robustness of the preference ranking in part 22a to uncertainty in $i_{\rm r}$ and f. The estimated values are constant and equal $\widetilde{i}$ and $\widetilde{f}$. Does your answer depend on the assumption that the PW is the same for the real and nominal series? This question can be interpreted in two ways.
      i. i_r and f are constant over time but their values are uncertain.
      ii. i_r and f are variable over time and their values are uncertain at each year.

   Explore the first interpretation.
   (e) Suppose that we confidently prefer series 1 over series 2 if:

$$\text{PW}_1 - \text{PW}_2 \geq \delta \qquad (6)$$

   for some given value of δ. If this holds with the estimated real interest and inflation, $\widetilde{i}_r$ and $\widetilde{f}$, how robust are we to uncertainty in $\widetilde{i}_r$ and $\widetilde{f}$?
   (f) Suppose that we confidently prefer series 1 over series 2 if eq.(6) holds. How robust is the preference ranking in part (a) to uncertainty in the nominal series $A_k^{(1)}$ and $A_k^{(2)}$?
   (g) Consider the nominal series of expenses in table 3 on p.20.
      i. Evaluate the robustnesses in part (e) with this data, given estimated real interest $\widetilde{i}_r = 0.05$ and estimated inflation $\widetilde{f} = 0.08$. Let $s_i = 0.01$ and $s_f = 0.02$. Use your judgment and experience to choose between these series, and to choose a realistic value of δ.
      ii. Consider the values in table 3 as estimates, with error estimates $s_k^{(i)} = 0.15 A_k^{(i)}$. Evaluate the robustnesses in part (f) with this data. Use your judgment and experience to choose between these series, and to choose a realistic value of δ.

Year	$A^{(1)}$	$A^{(2)}$
1	97	94
2	98	94
3	114	103
4	114	108
5	114	110
6	106	91
7	87	101
8	107	109
9	116	91
10	104	103
11	110	120
12	107	118
13	96	110
14	102	105
15	92	120

Table 3: Data for problem 22(g).

23. **Project financing.** (p.87)

(a) You are analyzing the financing of a project with 3 stages: (1) development, (2) implementation, and (3) operation. The 3 stages are performed in sequence: stage $n+1$ begins when stage n ends. The project starts at time $t=0$. The known duration (years) and known cost (in $t=0$ shekels) of stage n are t_n and c_n. The annual inflation rate is f. You will take a loan at the start of each stage to cover the cost (in current shekels) of that stage. The nominal interest rate of each loan is i_{nom}. You will pay back the entire cumulative loan t_4 years after completion of stage 3. Derive an explicit algebraic expression for the nominal shekel amount you will pay. What is the real shekel value of this payment (in the value of the shekel at the start of the project, $t=0$)?

(b) We now modify part (23a). At the end of each year during the 3rd stage—operation—you will receive income I_3 nominal shekels per year. This income is invested at interest i_{nom} until it is time to pay back the loans. There is no income during the first two stages. Everything else is the same as in part (23a). What is the nominal value of the cumulative income when you repay the loans? Assume all durations are integer years.

(c) We now return to part (23a). You could out-source the first stage—development—to a European country with currency in euros. The known duration (years) and cost (euros) of this first stage would be τ_1 and γ_1. The exchange rate is r shekels/euro. The inflation rates in euros and in shekels are f_{euro} and f_{nis}. Financing is done in shekels. You will take a loan in Israel, in current nominal shekels, at the start of each stage to cover the cost of that stage. The nominal shekel interest rate of each loan is i_{nom}. You will pay back the entire cumulative loan t_4 years after completion of stage 3. Derive an explicit algebraic expression for the nominal shekel amount you pay. What is the real shekel value of this payment (in the value of the shekel at the start of the project)?

(d) We now return to part (23a) and suppose that the costs, c_n, are uncertain. The estimated costs are $\widetilde{c}_n$ and the errors of these estimates are w_n. Use the following info-gap model:

$$\mathcal{U}(h) = \left\{ c : \ \left| \frac{c_n - \widetilde{c}_n}{w_n} \right| \le h, \ n = 1, 2, 3 \right\}, \quad h \ge 0 \tag{7}$$

We require that the real shekel value of the loan payment, in $t=0$ shekels, be no greater than the critical value R_{c}. Derive an explicit algebraic expression for the robustness to uncertainty in the project costs.

(e) We now return to part (23a) and suppose that the inflation rate, f, is constant but uncertain. The estimated inflation rate is $\widetilde{f}$ and the error of this estimate is s. Use the following info-gap model:

$$\mathcal{U}(h) = \left\{ f : \ \left| \frac{f - \widetilde{f}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{8}$$

We require that the real shekel value of the loan payment, in $t=0$ shekels, be no greater than the critical value R_{c}. The robustness to uncertainty in the inflation is a function of the critical value: $\widehat{h}(R_{\mathrm{c}})$. Derive an explicit algebraic expression for the inverse of the robustness function.

(f) We now return to part (23a) and suppose that the cost of stage 1, the development stage, is a normal random variable with mean $\widetilde{c}_1$ and variance σ_1^2. All other variables are known. The project owner will default on the loan if the real shekel value of the loan payment, in

$t = 0$ shekels, is greater than the critical value R_c. Derive an explicit analytical expression for the probability of default. Let $\Phi(z)$ denote the cumulative probability distribution function for a standard normal random variable z (zero-mean; unit variance).

24. **Compound interest.** (p.89) You will take a loan of \$6,000 for 3 years at 7% interest per year. You will pay the accumulated interest at the end of each year. How much interest will you pay at the end of each year if the loan principal will be repaid as \$1000k at the end of year k, for $k = 1$, 2, 3?

25. **Start up.** (p.89) Continuing problem 24, suppose that the loan is used in a start-up that produces revenue $R_k = \$800k^2$ at the end of year k, for $k = 1$, 2, 3. What is the numerical value of the present worth of the start-up, accounting for interest payment, principal payment and revenue for the 3-year loan period?

26. **Start up, continued.** (p.89) Let's generalize problem 25. Consider a loan for N years at interest rate i per year, with interest and principal payments I_k and P_k, respectively, at the end of each year. The loan will support a start-up whose return at the end of each year is roughly estimated at $\widetilde{R}_k$ (not necessarily the function in problem 25). These estimates may err by as much as s_k or more. $\widetilde{R}_k$ and s_k are known positive values. The uncertainty in the revenue is quantified with the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ R_k : \left| \frac{R_k - \widetilde{R}_k}{s_k} \right| \le h, \ k = 1, \ldots, N \right\}, \quad h \ge 0 \tag{9}$$

The investors require that the net present worth of the N-year start-up be no less than PW_c. Derive an explicit algebraic expression for the robustness to uncertainty.

27. **BCR of a measurement device.** (p.90) Your company is considering the purchase of an expensive measurement device. You will have to take a loan for N years at interest rate i_c per year to cover the cost, with interest and principal payments I_k and P_k, respectively, at the end of each year. The benefit of the device will be assessed by the number, B, of measurements each year with the device. B will be constant over time, but future benefit is discounted at the rate i_b because other competitive technologies will become available.

   (a) Derive an explicit algebraic expression for the benefit-cost ratio (BCR) of the device, recalling that B is constant over time.

   (b) B is constant over time, but its value is a random variable. Assume that B is exponentially distributed with probability density function $p(B) = \lambda \mathrm{e}^{-\lambda B}$, $B \ge 0$. You require that the BCR be no less than BCR_c. Derive an explicit algebraic expression for the probability of violating this requirement.

   (c) The value of the exponential coefficient is estimated as $\widetilde{\lambda}$, but the fractional error of this estimate is unknown (though λ must be positive). You require that the probability of violating the BCR requirement be less than P_c. Derive an explicit algebraic expression for the robustness to uncertainty in λ.

28. **Loan and investment.** (based on exam, 21.7.2014) (p.91) You will take a loan of \$60,000 at 7% yearly interest, at the start of year $k = 1$. We will consider different repayment schemes.

(a) You will repay \$20,000 of the principal at the end of each year $k = 1, 2, 3$. At the end of each year you will also repay the interest accrued up to that time. How much interest do you pay each year?

(b) Continue part 28a. At the beginning of year k you hold $\$60,000 - (k-1)\$20,000$, for $k = 1, 2, 3$. This sum will be invested with yearly rate of return $i_{\mathrm{inv}} = 0.15$ for the duration of the year. Evaluate the present worth of the returns on the investment (positive) and the interest payments (negative, with interest rate $i = 0.07$) over the 3 years of the loan, at discount rate i_{inv}.

(c) The initial loan, P, is \$60,000, but consider a different repayment scheme from part 28a. You will make 3 equal payments at the end of years $k = 1, 2, 3$. What is the yearly payment if these equal payments will entirely repay the loan and its interest at the end of 3 years? Explain the difference from part 28a.

(d) We now generalize the problem. Consider K yearly interest payments, I_k, $k = 1, \ldots, K$, where the I_k's are negative numbers (payments). The present worth of these payments, with annual discount rate i_{inv}, is denoted PW(i_{inv}). Note that the I_k's are themselves interest payments on a loan, and that the annual discount rate, i_{inv}, is not an interest rate on the loan. Rather, i_{inv} is a rate of return on investments made with the loan whose interest payments are I_k.

The discount rate is constant but uncertain with estimated value $\widetilde{i}_{\mathrm{inv}}$ and positive error estimate s:

$$\mathcal{U}(h) = \left\{ i_{\mathrm{inv}} : \ i_{\mathrm{inv}} \geq 0, \ \left| \frac{i_{\mathrm{inv}} - \widetilde{i}_{\mathrm{inv}}}{s} \right| \leq h \right\}, \quad h \geq 0 \qquad (10)$$

Derive an explicit algebraic expression for the minimum (most negative) present worth at horizon of uncertainty h. Denote this result $m(h)$.

(e) Continuing part 28d, we require that the PW be no more negative than the value PW$_{\mathrm{c}}$. The function $m(h)$ derived in part 28d is the inverse of the robustness function for this requirement, denoted $\widehat{h}$(PW$_{\mathrm{c}}$). Schematically (not numerically) sketch the robustness function, $\widehat{h}$(PW$_{\mathrm{c}}$) vs. PW$_{\mathrm{c}}$.

(f) Continuing part 28e, consider two alternative discount rates, and error estimates, for the interest payment scheme in part 28d:

$$\widetilde{i}_{1,\mathrm{inv}} < \widetilde{i}_{2,\mathrm{inv}} \qquad (11)$$

$$\frac{\widetilde{i}_{1,\mathrm{inv}}}{s_1} > \frac{\widetilde{i}_{2,\mathrm{inv}}}{s_2} \qquad (12)$$

Which discount scheme would you prefer if there were no uncertainty in the discount rate? Does this preference hold at all levels of uncertainty? Explain in terms of robustness against uncertainty of the two schemes.

29. **Future foreign earnings.** (based on exam, 21.7.2014) (p.93) Your employment contract states that at the end of k years from now you will receive a payment, in \$'s, of the fixed sum A_k. The general price inflation of \$'s will be f_j in year j for $j = 1, \ldots, k$. The exchange rate between \$'s and pesos at the end of year k will be r_k peso/\$. The general price inflation of pesos will be ϕ_j in year j for $j = 1, \ldots, k$.
   (a) What is the real value of A_k in \$'s at the start of year 1?
   (b) What is the real value of A_k in pesos at the start of year 1 if you transfer the payment when it is received, which is at the end of year k?
   (c) The value of the payment A_k is firmly fixed by contract. However, suppose the dollar inflation rates f_j are highly uncertain, while the peso inflation rates are very stable and well known. The exchange rate r_k is also highly uncertain. Consider the following info-gap model for f_j and r_k:

$$\mathcal{U}(h) = \left\{ r_k, f_j, j = 1, \ldots, k : \ r_k \ge 0, \ \left| \frac{r_k - \widetilde{r}_k}{\widetilde{r}_k} \right| \le h, \ \left| \frac{f_j - \widetilde{f}_j}{\widetilde{f}_j} \right| \le h \right\}, \quad h \ge 0 \qquad (13)$$

   You require that the real peso value, at the start of year 1, of the year-k earnings in \$'s, be no less than R_c. Derive an explicit algebraic expression for the robustness to uncertainty.

30. **Interest payments.** (based on exam, 1.10.2014) (p.93)
   (a) You will take a loan of \$40,000 for 4 years with annual interest rate of 0.05. You will repay \$11,000 at the end of years 1, 2 and 3. At the end of year 4 you will repay the remaining principal and all remaining accrued interest. How much interest accrues at the end of each of the 4 years, and what is your final payment?
   (b) Continue part 30a. The \$11,000 that is paid at the end of years 1, 2 and 3 could have been used instead for long-term investments beginning at the start of years 2, 3 and 4. These investments would have yielded 0.07 rates of return, compounded annually. What would have been the total value of each of these 3 investments at the end of year 4? Specify the values of each of the three investments separately.
   (c) Modification of part 30b. A sum I will be invested at the start of years $k = 1$, 2 and 3. The estimated rate of return, compounded annually, is $\widetilde{i}$. The actual rate of return may be less, but not negative, as expressed by an info-gap model:

$$\mathcal{U}(h) = \left\{ i : \ i \ge 0, \ \widetilde{i} - h \le i \le \widetilde{i} \right\}, \quad h \ge 0 \qquad (14)$$

   Derive an explicit algebraic expression for the least total combined value of all three investments at the end of year 3, at horizon of uncertainty h.

31. **Random returns.** (based on exam, 1.10.2014) (p.95)

(a) R dollars will be earned at the end of each of N years, $k = 1, \ldots, N$. The MARR is i. We require that the present worth of these earnings, at the start of year 1, be no less than PW_{c}. However, R is a uniformly distributed random variable on the interval $[R_1, R_2]$. What is the probability that the PW requirement will not be met? Denote this probability P_{f}.

(b) Continue part 31a and suppose that the value of P_{f} evaluated in part 31a is greater than 0 and less than 1. We require that P_{f} be no greater than the value P_{c}. However, the value of R_2 is uncertain:

$$\mathcal{U}(h) = \left\{ R_2 : \ R_2 \geq R_1, \ \left| \frac{R_2 - \widetilde{R}_2}{s_2} \right| \leq h \right\}, \quad h \geq 0 \tag{15}$$

where R_1 and $\widetilde{R}_2$ are known and $R_1 < \widetilde{R}_2$. s_2 is known and positive. Derive an explicit algebraic expression for the robustness to uncertainty in R_2.

(c) We modify the problem as follows. R_k dollars are earned at the end of year k. The dollar earnings are exchanged for pesos at the end of year k at exchange rate r_k peso/$. The peso inflation rate in year j is f_j, for $j = 1, \ldots, k$. What is the equivalent real value in pesos, at the start of year 1, of the dollar earnings for year k? Denote this real peso value by π_k.

(d) Continue part 31c. The exchange rate r_k and inflation rates, f_j, for $j = 1, \ldots, k$, are uncertain:

$$\mathcal{U}(h) = \left\{ r_k, f_j, j = 1, \ldots, k : \ r_k \geq 0, \ \left| \frac{r_k - \widetilde{r}_k}{\widetilde{r}_k} \right| \leq h, \ \left| \frac{f_j - \widetilde{f}_j}{\widetilde{f}_j} \right| \leq h \right\}, \quad h \geq 0 \tag{16}$$

where $\widetilde{r}_k$ and $\widetilde{f}_j$ are known and positive. Derive an explicit algebraic expression for the lowest value of π_k at horizon of uncertainty h.

32. **Loans, projects and investments.** (based on exam, 12.5.2015) (p.96)

(a) You will take a loan, L_n, at the start of year n, where $L_1 = \$5,000$, $L_2 = \$3,000$ and $L_3 = \$1,000$. The interest rate is $i = 0.06$ per year. You will repay the interest at the end of each year, and your will repay the principal at the end of year 3. What are the interest payments for years 1, 2 and 3?

(b) Like in part 32a, you will take a loan, L_n, at the start of year n, where $L_1 = \$5,000$, $L_2 = \$3,000$ and $L_3 = \$1,000$. The interest rate is $i = 0.06$ per year. However now you will pay all interest and principal at the end of year 3. What is the value of that payment?

(c) The start-up cost at time $t = 0$ of an N-year project is S. The income from the project is estimated to be $\widetilde{R}$ at the end of each year. Derive an explicit algebraic expression for the estimated present worth of the project if the MARR is i.

(d) Continue part 32c by including uncertainty in the annual income, R_k, at the end of year k, with this fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ R_1, \ldots, R_N : \; R_k \ge 0, \; \left| \frac{R_k - \widetilde{R}}{\widetilde{R}} \right| \le h, \; \forall \, k \right\}, \quad h \ge 0 \tag{17}$$

The project owner requires that the present worth be no less than the critical value $\mathrm{PW_c}$. Derive an explicit algebraic expression for the robustness.

(e) The return on an investment is a random variable R with an exponential probability density function: $p(R) = \lambda \mathrm{e}^{-\lambda R}$, $R \ge 0$. Derive an explicit algebraic expression for the probability that the return is less than a positive value R_c. Denote this probability $P_f(\lambda)$.

(f) Continue part 32e by considering uncertainty in the exponential coefficient with this info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda : \; \lambda \ge 0, \; \left| \frac{\lambda - \widetilde{\lambda}}{\widetilde{\lambda}} \right| \le h \right\}, \quad h \ge 0 \tag{18}$$

where $\widetilde{\lambda}$ is a known positive estimate. The owner requires that $P_f(\lambda)$ be no greater than P_c. Derive an explicit algebraic expression for the robustness.

33. **Earnings and investments.** (based on exam, 19.7.2015) (p.98)

(a) You will earn $150,000 at the end of 3 years. What is the present worth of this sum if the annual interest rate is 0.06? There is no inflation.

(b) You will earn $30,000 at the end of each year for 5 years. What is the sum of the present worth of this income stream if the annual interest rate is 0.06? There is no inflation.

(c) The US annual inflation is 4%. Your real earnings (in year-0 dollars) will be $30,000 at the end of each year for 5 years. At the end of each year you exchange your dollar earnings to pesos at the exchange rate of 20 peso/$. You hold the pesos in a zero-interest account. What is the present worth in pesos of this investment stream if the nominal annual interest rate on pesos is 0.06? The peso inflation rate is 8% per year.

(d) The US annual inflation is 4%. Your real earnings (in year-0 dollars) will be $30,000 at the end of each year for 5 years. At the end of each year you exchange your dollar earnings to pesos at the exchange rate of 20 peso/$. You invest the pesos and earn 8% nominal annual interest until the end of the 5 years.

- What is the total peso balance at the end of the 5 years?
- At the end of the 5 years you exchange the total peso sum to dollars. What is the nominal value of dollars at this time?
- What is the present worth (in year-0 dollars) of that dollar sum if the US annual real interest rate is 6%?

(e) We now generalize part 33b. You will earn $\$F$ at the end of each year for N years. There is no inflation, but F is uncertain according to this info-gap model:

$$\mathcal{U}(h) = \left\{ F : \left| \frac{F - \widetilde{F}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{19}$$

where $\widetilde{F}$ and s are known and positive. We require that the present worth of the sum of this income stream, at interest rate i, be no less than the critical value PW_{c}. Derive an explicit algebraic expression for the robustness.

(f) Continue from part 33e and compare two alternative income plans, where the estimated annual incomes differ, and the interest rates with which the present worth is calculated, also differ. Specifically:

$$\widetilde{F}_1 > \widetilde{F}_2 \quad \text{and} \quad \frac{s_2}{s_1} < \frac{\widetilde{F}_2}{\widetilde{F}_1} \tag{20}$$

For what values of PW_{c} do you robust-prefer plan 2? Explain the economic reason for this preference.

34. **Great idea for a start-up.** (based on exam, 19.7.2015) (p.101) You have a great idea for a start up. It will take N years to develop, but if it works you will then earn $\$R$. If it fails you will earn nothing. You will finance the project by taking a loan of $\$C$, at the beginning of year 1, at annual interest rate i. There is no inflation. The probability of success is p.
   (a) What is the present worth of the project if it succeeds and you earn $\$R$? What is the present worth of the project if it fails? What is the expected value of the present worth of the project?
   (b) We now extend part 34a. You anticipate that the return on the project will be $\widetilde{R}$, though this could err (for better or for worse) by as much as s or more. Thus you describe the uncertain return with an info-gap model:

$$\mathcal{U}(h) = \left\{ R : \left| \frac{R - \widetilde{R}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{21}$$

   You require that the expected value of the PW be no less than the critical value PW_{c}. Derive an explicit algebraic expression for the robustness.

35. **Artistic restoration.** (based on exam, 19.7.2015) (p.102) You have developed a process for restoring faded paintings. The process is slow and takes time. The annual cost is $\$A$ which is borrowed at annual interest i_c. There is no inflation. The curator is delighted, but would have preferred instantaneous restoration. The social benefit of the fully restored paintings will be R. The social benefit today is less if the painting is restored in the future. This can be approximated by discounting the social benefit at the rate i_b, compounded annually.

    (a) One version of the process acts simultaneously on all paintings that are fully restored after N years, and have no value until then. Derive expressions for:
    - The discounted present benefit of the paintings restored after N years.
    - The discounted present worth of the annual cost of the process over the N years.
    - The benefit cost ratio.

    (b) A different version of the process fully restores a fraction $1/N$ of the paintings each year, which obtain their full social value when restored. The annual cost of the process is $\$A$ as before. Derive expressions for:
    - The discounted benefit worth of the paintings restored yearly during N years.
    - The discounted present worth of the annual cost of the process over the N years.
    - The benefit cost ratio.

    (c) Compare the BCR's of the two options. Which is better, and what is the time-value explanation for the difference?

36. **Earnings and investments.** (based on exam, 15.10.2015) (p.103)

(a) You will earn F dollars at the end of 5 years. What must F equal if its present worth is \$100,000 and the annual interest rate is 0.07? There is no inflation.

(b) You will earn $[1 + 0.1(k - 1)]F$ dollars at the end of year k for $k = 1, 2, 3$. What must F equal if the sum of the present worth of this income stream is \$250,000? The annual interest rate is 0.03. There is no inflation.

(c) Your real earnings will be 50,000 peso at the end of each year for 3 years. You invest these earnings, at the end of each year, at 5% nominal annual interest. At the end of 3 years you withdraw the total peso balance and exchange it for dollars at 30 pesos/dollar. What is the dollar sum you obtain? The nominal annual interest rate for dollars is 0.06. The inflation rates for pesos and dollars are 0.07 and 0.03 respectively.

(d) We now generalize problem 36b as follows. You will earn $[1 + (k - 1)x]F$ (F is positive) at the end of year k, for $k = 1, \ldots, N$. The value of x is estimated at $\widetilde{x}$, but this could err by $\pm s$ or more. Represent the uncertainty in x with this info-gap model:

$$\mathcal{U}(h) = \left\{ x : \left| \frac{x - \widetilde{x}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{22}$$

We require that the present worth of the sum of the income stream must be no less than $\mathrm{PW_c}$. The annual interest rate is i. There is no inflation. Derive an explicit algebraic expression for the robustness function.

(e) We continue part 36d for the special case that the parameter values are those of part 36b: $N = 3$, $i = 0.03$ and $\widetilde{x} = 0.1$. Let F_{36b} denote the answer to part 36b, and choose $F = F_{36b}$. What is the numerical value of the robustness if $\mathrm{PW_c} = 250,000$?

37. **Investment and expected returns.** (based on exam, 15.10.2015) (p.105) You have a fixed budget of Q dollars, of which q_1 will be invested in project 1 and the remainder in project 2. Project j will take an integer number years, n_j, to complete. The return from project j, upon completion and if it succeeds, is $R_j = r_j q_j / Q$, where $r_j \geq 0$, for $j = 1, 2$. The return is zero if the project fails. The probability of success of project j is $p_j = q_j / Q$. The annual interest rate is i. There is no inflation.

(a) Derive an explicit algebraic expression for the expected value of the present worth of the investment.

(b) What choice of q_1 has maximal expected present worth?

(c) Continuing from part 37a, suppose that each r_j is uncertain, according to this info-gap model:

$$\mathcal{U}(h) = \left\{ r : \left| \frac{r_j - \widetilde{r}_j}{s_j} \right| \leq h, \ j = 1, 2 \right\}, \quad h \geq 0 \tag{23}$$

We require that the expected present worth be no less than the critical value, PW_c. Derive an explicit algebraic expression for the robustness.

(d) We now modify the problem and consider a single project whose return, at the end of n years, is R which is an exponential random variable: $p(R) = \lambda \text{e}^{-\lambda R}$. Derive an explicit algebraic expression for the probability that the present worth will be less than PW_c. The interest rate is i. Denote this probability P_f.

(e) Continuing part 37d, we consider λ to be uncertain according to this info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda : \ \lambda > 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{s} \right| \leq h \right\}, \quad h \geq 0 \tag{24}$$

We require that P_f not exceed the critical value P_fc. Derive an explicit algebraic expression for the robustness.

(f) Now consider a single project whose initial monetary cost, at time $t = 0$, is S. The monetary operating cost in year j is c_j. The non-monetary benefit in year j is b_j. The duration is N years. The monetary interest rate is i_c, and the non-monetary discount rate is i_b. Derive explicit algebraic expressions for the discounted total cost, the discounted total benefit, and the benefit-cost ratio. Also derive these results for the special case that costs and benefits are constant: $c_j = c$ and $b_j = b$.

38. **Income and uncertainty.** (based on exam, 23.5.2016) (p.107).

(a) You will take a loan of \$10,000 for 5 years with yearly compound interest of 5%. There is no inflation. You will repay \$2,500 at the end of each of the first 4 years. What is the payment at the end of the 5th year? What is the interest that accrues during the 5th year?

(b) You will earn NIS15,000 at the end of each year for 10 years. What is the present worth of this income stream if the interest is 4%. There is no inflation.

(c) At the end of each year, for N years, you will earn A and spend C. Both A and C are constant but uncertain with this info-gap model:

$$\mathcal{U}(h) = \left\{ (A, C) : \ \left| \frac{A - \widetilde{A}}{s_A} \right| \le h, \ \left| \frac{C - \widetilde{C}}{s_C} \right| \le h \right\}, \quad h \ge 0 \tag{25}$$

where $\widetilde{A}$, $\widetilde{C}$, s_A and s_c are all positive and known. You require that the present worth of this N-year program be no less than P_{c}. Derive an explicit algebraic expression for the robustness function.

(d) You invest A at time $t = 0$ with annual compound interest i. There is no inflation. You require that the future worth, after N years, be no less than F_{c}. The interest rate is constant over the N years but at the time of the investment all you know is that i is a random variable with an exponential probability density:

$$p(i) = \lambda \mathrm{e}^{-\lambda i}, \quad i \ge 0 \tag{26}$$

where the value of λ is known. What is the probability that your future-worth requirement will be satisfied? Call this probability P_{s}.

(e) Continuing from part 38d, now suppose that λ is uncertain according to this info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda : \ \lambda > 0, \ \left| \frac{\lambda - \widetilde{\lambda}}{\widetilde{\lambda}} \right| \le h \right\}, \quad h \ge 0 \tag{27}$$

You require that P_{s} be no less than the critical value P_{c}. Derive an explicit algebraic expression for the robustness for satisfying this requirement.

(f) You wish to choose an investment option, and you require that the future worth will be no less than F_{c}. You are offered two options between which you can choose. Option 1 guarantees a future worth of exactly F_1. The second option is uncertain, and its robustness function for future worth is:

$$\widehat{h}_2(F_{\mathrm{c}}) = 1 - \frac{F_{\mathrm{c}}}{2F_1} \tag{28}$$

or zero if this is negative. Which option would you choose, as a function of F_{c}, where your preference is for the more robust option?

39. **Present worth, interest, inflation and uncertainty.** (based on exam, 19.7.2016) (p.109).

(a) Consider a project for which the investment at the start of the first year is S, the revenue and cost at the end of each year are constant at R and C, the duration is N years, there is no inflation and the annual discount rate is i. Derive an explicit algebraic expression for the present worth.

(b) Continue part 39a with $S = \$10^4$, $R = \$2,000$, $C = \$1000$ and $i = 0.07$. Find the shortest project duration at which the present worth in non-negative.

(c) Continue part 39a with $R = \$2,000$, $C = \$1000$ and $i = 0.07$. What is the lowest initial investment for which the present worth is **negative** for all project durations?

(d) Continue part 39a and suppose that the discount rate, i, is uncertain with estimate $\widetilde{i}$ and uncertainty weight s_i, both known and positive. The info-gap model is:

$$\mathcal{U}(h) = \left\{ i : \ i \geq 0, \ \left| \frac{i - \widetilde{i}}{s_i} \right| \leq h \right\}, \quad h \geq 0 \tag{29}$$

We require that the present worth be no less than the critical value, PW_c. Derive an explicit algebraic expression for the inverse of the robustness function.

(e) Continue part 39a and suppose that R and C are uncertain as described by this info-gap model:

$$\mathcal{U}(h) = \left\{ R, C : \ \left| \frac{R - \widetilde{R}}{s_R} \right| \leq h, \ \left| \frac{C - \widetilde{C}}{s_C} \right| \leq h \right\}, \quad h \geq 0 \tag{30}$$

where $\widetilde{R}$, $\widetilde{C}$, s_R and s_C are known and positive. We require that the present worth be no less than the critical value, PW_c. Derive an explicit algebraic expression for the robustness function.

(f) Continue part 39a but now introduce constant annual inflation, f. The values R and C are in nominal dollars at the time of the initial investment and i is the nominal annual interest rate. Annual revenue and cost are constant in real dollars. Derive an explicit algebraic expression for the present worth.

(g) Continue part 39f with $R = \$2,000$, $C = \$1000$, $S = \$10,000$ and $f = i$. At what project duration, N, is the present worth equal to zero?

(h) Continue part 39f where i and f are both constant but uncertain with this info-gap model:

$$\mathcal{U}(h) = \left\{ i, f : \ i \geq 0, \ \left| \frac{i - \widetilde{i}}{s_i} \right| \leq h, \ f \geq 0, \ \left| \frac{f - \widetilde{f}}{s_f} \right| \leq h \right\}, \quad h \geq 0 \tag{31}$$

where $\widetilde{i}$, $\widetilde{f}$, s_i and s_f are known and positive. We require that the present worth be no less than the critical value, PW_c. Derive an explicit algebraic expression for the inverse of the robustness function.

(i) Return to part 39a and consider the following dispute. Joe evaluates the project in terms of its present worth, and accepts the project if and only if the present worth is positive. Jane evaluates the project in terms of the benefit-cost ratio, and accepts the project if and only if the BCR exceeds unity. Do they agree or disagree on accepting or rejecting the project? Explain.

(j) Repeat part 39i but now suppose that Jane uses a larger discount rate for the benefits than the value i that both she and Joe use for costs. Joe uses i to discounts benefits as well.

For conditions in which Joe **rejects** the project (using the PW criterion), will Jane **also reject** the project (using the BCR criterion)?

For conditions in which Joe **accepts** the project (using the PW criterion), will Jane **also accept** the project (using the BCR criterion)?

40. **Present worth, interest, inflation and uncertainty** (based on exam, 27.9.2016) (p.112)
   (a) Consider an N-year project with constant annual interest i and no inflation. The revenue at the end of year k is $R_k = (1+i)^k R_0$ where R_0 is positive. The cost at the end of year k is C which is positive and constant. The initial investment at the beginning of the first year is S, which is positive.
      i. Derive an explicit algebraic expression for the present worth.
      ii. Given that $S = gC$, $R_0 = C$, $N = 10$ and $i = 0.04$. Find the largest value of g for which the PW is non-negative.
   (b) Consider an N-year project with constant annual interest i and no inflation. The revenue and the cost at the end of each year are R and C which are both positive constants. The initial investment at the beginning of the first year is S, which is positive. Suppose that the present worth equals zero for specific positive values of S, R, C, i and N. Now suppose that i is increased while S and N remain constant. Which of the following must be true In order for the present worth to remain non-negative:
      i. $R - C$ must increase.
      ii. $R - C$ must decrease.
      iii. $R - C$ must remain the same.
      iv. The direction of change in $R - C$ depends on the specific values of S, N and i.
   (c) Consider a project with constant annual interest i and no inflation. The revenue and the cost at the end of each year are R and C which are both positive constants. The initial investment at the beginning of the first year is S, which is positive. The project will run forever, so $N = \infty$. Which of the following statements is true:
      i. The present worth is infinite for any $i > 0$.
      ii. The present worth is infinite only for any $i > 0$ that is also less than some finite value.
      iii. The present worth is finite for any $i > 0$.
      iv. The present worth is finite only for any $i > 0$ that is also less than some finite value.
   (d) Consider an N-year project with constant annual interest i and no inflation. The initial investment at the beginning of the first year is S, which is positive. The revenue and the cost at the end of each year are R and C which are both positive constants, and the revenue is proportional to the cost according to $R = gC$ where g is constant but uncertain according to this info-gap model:

$$\mathcal{U}(h) = \left\{ g : \left| \frac{g - \widetilde{g}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{32}$$

   where $\widetilde{g}$ and w are known positive constants. We require that the present worth be no less than the critical value PW_c. Derive an explicit algebraic expression for the robustness function.
   (e) Repeat problem 40d with the following info-gap model:

$$\mathcal{U}(h) = \left\{ g : \ g \ge 1, \ \left| \frac{g - \widetilde{g}}{w_g} \right| \le h, \ \left| \frac{C - \widetilde{C}}{w_c} \right| \le h \right\}, \quad h \ge 0 \tag{33}$$

   where $\widetilde{g}$, w_g, $\widetilde{C}$ and w_c are known positive constants. Derive an explicit algebraic expression for the inverse of the robustness function.
   (f) Consider constant monthly inflation f. You have a new job and the real value of your monthly salary, at the start of your first month, is R_0. However, the job actually pays

you at the end of each month with a nominal sum whose real value is R_0. What is your nominal salary at the end of the kth month?

(g) Continue part 40f and consider f uncertain according to this info-gap model:

$$\mathcal{U}(h) = \left\{ f : \left| \frac{f - \widetilde{f}}{w} \right| \leq h \right\}, \quad h \geq 0 \tag{34}$$

where $\widetilde{f}$ and w are known positive constants. We require that the nominal salary at the end of the kth month be no less than $S_{\rm c}$, which is positive. Derive an explicit algebraic expression for the robustness function.

(h) You will invest $S = \$1000$ at time $t = 0$ in a project that returns constant nominal annual interest $i_{\rm nom} = 0.08$. The constant annual inflation is $f = 0.06$. What is the nominal value of the investment after $N = 12$ years? What is the real value at that time? What is the real interest rate?

(i) You will invest $\$S$ at time $t = 0$ in a project that returns constant nominal annual interest $i_{\rm nom}$. The constant annual inflation is f. However, both $i_{\rm nom}$ and f are uncertain, as expressed by this info-gap model:

$$\mathcal{U}(h) = \left\{ i_{\rm nom}, f : \ i_{\rm nom} \geq 0, \ \left| \frac{i_{\rm nom} - \widetilde{i}_{\rm nom}}{w_i} \right| \leq h, \ \left| \frac{f - \widetilde{f}}{w_f} \right| \leq h \right\}, \quad h \geq 0 \tag{35}$$

where $\widetilde{i}_{\rm nom}$, w_i, $\widetilde{f}$ and w_f are known positive constants. We require that the real value of the investment at the end of k years be no less than $R_{\rm c}$.

i. Derive an explicit algebraic expression for the inverse of the robustness function.
ii. What is the value of the robustness if $R_{\rm c} = 0$.
iii. What is the value of the robustness if $R_{\rm c} = \left(\dfrac{1 + \widetilde{i}_{\rm nom}}{1 + \widetilde{f}} \right)^k S$?

(j) Consider an N-year project with constant positive revenue R and constant positive cost C at the end of each year. The annual interest rate is i and there is no inflation. The initial investment, at the start of the first year, is S.

i. If $S = 0$, what is the lowest ratio R/C at which the project has a benefit-cost ratio (BCR) no less than one?
ii. If $C = 0$ and $R > 0$, what is the lowest value of S at which the BCR is less than one for all values of N?

41. **Present worth of yearly profit.** (Based on exam 28.5.2018.) (p.115)

(a) The profit at the end of year n is R_n, where $R_1 =$ \$2,500, $R_2 =$ \$3,500, $R_3 =$ \$5,000. The discount rate is $i =$ 0.09. What is the present worth of the total income stream?

(b) The profit at the end of year n is R_n, where $R_1 =$ \$2,500, $R_2 =$ \$3,500, $R_3 =$ \$5,000. At the end of each year you will invest that year's profit, R_n, with yearly rate of return of $i_{\mathrm{a}} =$ 0.15. What is the total accumulated sum at the end of year 3? What is the present worth of that sum using a discount rate of $i =$ 0.09?

(c) The profit at the end of year n is R_n, where the estimated values of these profits are $\widetilde{R}_n$ for $n =$ 1, 2, 3. The uncertainty in these estimates is given by this info-gap model:

$$\mathcal{U}(h) = \left\{ R : \left| \frac{R_n - \widetilde{R}_n}{\widetilde{R}_n} \right| \le h, \ \ n = 1, 2, 3 \right\}, \quad h \ge 0 \tag{36}$$

The discount rate is i. You require that the present worth be no less than the critical value PW_{c}. Derive an explicit algebraic expression for the robustness.

(d) The return on an investment is a random variable, R, in the interval $[R_1, \ R_2]$. The investment is a success if the return exceeds the critical value R_{c}. The probability of success is:

$$P_{\mathrm{s}}(R_{\mathrm{c}}) = \begin{cases} 0 & \text{if } R_{\mathrm{c}} > R_2 \\ \dfrac{R_2 - R_{\mathrm{c}}}{R_2 - R_1}, & \text{if } R_1 \le R_{\mathrm{c}} \le R_2 \\ 1 & R_{\mathrm{c}} < R_1 \end{cases} \tag{37}$$

However, the value of the critical return, R_{c}, is uncertain, as expressed by this info-gap model:

$$\mathcal{U}(h) = \left\{ R_{\mathrm{c}} : \left| \frac{R_{\mathrm{c}} - \widetilde{R}_{\mathrm{c}}}{\widetilde{R}_{\mathrm{c}}} \right| \le h \right\}, \quad h \ge 0 \tag{38}$$

You require that the probability of success be no less than the critical value P_{c}. Derive an explicit algebraic expression for the robustness. Assume that $R_1 \le \widetilde{R}_{\mathrm{c}} \le R_2$.

(e) (Variation on part 41a) The profit at the end of year n is R_n, where $R_1 =$ \$5000, $R_2 =$ \$3,500, $R_3 =$ \$2,500. The discount rate is $i =$ 0.05. What is the present worth of the total income stream?

(f) (Variation of part 41b) The profit at the end of year n is R_n, where $R_1 =$ \$5,000, $R_2 =$ \$3,500, $R_3 =$ \$2,500. At the end of each year you will invest that year's profit, R_n, with yearly rate of return of $i_{\mathrm{a}} =$ 0.1. What is the total accumulated sum at the end of year 3? What is the present worth of that sum using a discount rate of $i =$ 0.04?

(g) (Variation on part 41c) The profit at the end of year n is R_n, where the estimated values of these profits are $\widetilde{R}_n$ for $n =$ 1, 2, 3. The uncertainty in these estimates is given by this info-gap model:

$$\mathcal{U}(h) = \left\{ R : \left| \frac{R_n - \widetilde{R}_n}{w} \right| \le h, \ \ n = 1, 2, 3 \right\}, \quad h \ge 0 \tag{39}$$

where w is a known positive constant. The discount rate is i. You require that the present worth be no less than the critical value PW_{c}. Derive an explicit algebraic expression for the robustness.

(h) (Variation of part 41d) The return on an investment is a random variable, R, in the interval $[R_1,\ R_2]$. The investment is a success if the return exceeds the critical value $R_{\rm c}$. The probability of success is:

$$P_{\rm s}(R_{\rm c}) = \left\{ \begin{array}{cl} 0 & \text{if } R_{\rm c} > R_2 \\ \dfrac{R_2 - R_{\rm c}}{R_2 - R_1}, & \text{if } R_1 \le R_{\rm c} \le R_2 \\ 1 & R_{\rm c} < R_1 \end{array} \right. \tag{40}$$

However, the value of the critical return, $R_{\rm c}$, is uncertain, as expressed by this info-gap model:

$$\mathcal{U}(h) = \left\{ R_{\rm c} : \ \left| \frac{R_{\rm c} - \widetilde{R}_{\rm c}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{41}$$

where w is a known positive constant. You require that the probability of success be no less than the critical value $P_{\rm c}$. Derive an explicit algebraic expression for the robustness. Assume that $R_1 \le \widetilde{R}_{\rm c} \le R_2$.

42. **Time, money and benefit.** (Based on exam 19.7.2018.) (p.117)
   (a) A project will run for 3 years. The investment at the start of the first year is $S = \$10,000$. The income, I_k, at the end of years 1, 2 and 3 is \$4,000, \$3,000 and \$6,000, respectively. The discount rate is $i = 0.09$. There is no inflation. What is the present worth of this project at the start?
   (b) Modify part 42a to include 7% annual inflation: $f = 0.07$. A project will run for 3 years. The investment at the start of the first year is $S = \$10,000$. The nominal income, I_k, at the end of years 1, 2 and 3 is \$4,000, \$3,000 and \$6,000, respectively, where each sum is the nominal value at the time of payment (end of the corresponding year). The real discount rate is $i = 0.09$. What is the real present worth of this project at the start?
   (c) A project will run for 3 years. The income, I_k, at the end of years 1, 2 and 3 is \$20,000, \$15,000 and \$45,000, respectively. In addition, the project causes dis-benefit (a non-monetary cost) at the end of year k in the form of D_k disgruntled customers, whose values are 40, 70, and 50 customers in years 1, 2 and 3, respectively. The discount rate for income is $i = 0.04$, while the discount rate for dis-benefit is $i_{\rm d} = 0.15$. What is the benefit-cost ratio of this project? There is no inflation.
   (d) An Israeli company will invest in a 3-year project in Portugal. Annual inflation in the euro zone is $f_{\rm for}$, and annual inflation of the shekel is $f_{\rm dom}$. The initial investment in Portugal, at time $t = 0$, is S euros. The real income in Portugal, in $t = 0$ euros, at the end of year 3 is $R_{3,\rm for}$ euros. This sum is returned to Israel as shekels at the end of year 3. The exchange rate is r nis/euro. The real discount rate of the shekel is $i_{\rm r,dom}$.
      i. Derive an explicit algebraic expression for the nominal euro income at the end of year 3.
      ii. Derive an explicit algebraic expression for the nominal shekel value of the income at the end of year 3.
      iii. Derive an explicit algebraic expression for the real shekel value of the income at the end of year 3.
      iv. Derive an explicit algebraic expression for the present worth in shekels of the entire project.
      v. What is the numerical value of the present worth if $S = 100,000$ euros, $R_{3,\rm for} = 250,000$ euros, $f_{\rm for} = 0.07$, $f_{\rm dom} = 0.03$, $i_{\rm r,dom} = 0.12$, and $r = 4.2$ nis/euro.
   (e) You currently hold $\$R$, whose future worth in N years is:

$$\text{FW} = (1+i)^N R \tag{42}$$

   where i is the known constant discount rate. However, your holdings, R, are uncertain as specified by this info-gap model:

$$\mathcal{U}(h) = \left\{ R : \left| \frac{R - \widetilde{R}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{43}$$

   where $\widetilde{R}$ and s are both known and positive. You require that the future worth be no less than the critical value, $\text{FW}_{\rm c}$. Derive an explicit algebraic expression for the robustness function.
   (f) Repeat part 42e with one change. R is known and positive, and the discount rate, i, is uncertain as specified by this info-gap model:

$$\mathcal{U}(h) = \left\{ i : \left| \frac{i - \widetilde{i}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{44}$$

where $\widetilde{i}$ and w are both known and positive. Derive an explicit algebraic expression for the robustness function.

(g) You currently hold $\$R$, whose future worth in N years is:

$$\text{FW} = (1+i)^N R \tag{45}$$

where i is the known constant discount rate, while R is uncertain. You must choose between two options, whose info-gap models for uncertainty in R differ.

For option 1 the uncertainty is R is represented as:

$$\mathcal{U}_1(h) = \left\{ R:\ \frac{R-\widetilde{R}_1}{s_1} \ge h \right\}, \quad h \ge 0 \tag{46}$$

where $\widetilde{R}_1$ and s_1 are known positive values. Note that $\mathcal{U}_1(h)$ is **not** an info-gap model of uncertainty.

For option 2 the info-gap model is symmetric:

$$\mathcal{U}_2(h) = \left\{ R:\ \left| \frac{R-\widetilde{R}_2}{s_2} \right| \le h \right\}, \quad h \ge 0 \tag{47}$$

where $\widetilde{R}_2$ and s_2 are known positive values.

Furthermore:

$$\widetilde{R}_2 > \widetilde{R}_1 \tag{48}$$

You require that the future worth be no less than the critical value FW_c. Use the info-gap robustness functions for the two options to determine the range of FW_c for which option 1 is robust-preferred.

(h) Repeat question 42g with one change. Instead of eq.(46) we use the following proper info-gap model:

$$\mathcal{U}_1(h) = \left\{ R:\ \widetilde{R}_1 \le R \le \widetilde{R}_1 + s_1 h \right\}, \quad h \ge 0 \tag{49}$$

where $\widetilde{R}_1$ and s_1 are known positive values.

Use the info-gap robustness functions for the two options to determine the range of FW_c for which option 1 is robust-preferred.

43. **Investments over time.** (Based on exam 4.10.2018.) (p.121)

(a) Consider an N-year project. The investment at the start of the first year is S, the revenue and cost at the end of each year are constant at R and C where $R = (1+\varepsilon)C$, there is no inflation and the annual discount rate is i. Assume that C is positive. Derive an explicit algebraic expression for the present worth at the beginning of the project.

(b) Continuing from part 43a, let S =$10,000, C =$1,000, $i = 0.04$ and $N = 10$. What value of ε results in zero present worth?

(c) Continuing from part 43a, let S =$10,000, C =$1,000, $i = 0.04$ and $\varepsilon = 2$. What is the smallest integer value of N with non-negative present worth?

(d) We now extend part 43a by introducing uncertainty in the coefficient ε, represented by the following info-gap model:

$$\mathcal{U}(h) = \left\{ \varepsilon : \left| \frac{\varepsilon - \widetilde{\varepsilon}}{\widetilde{\varepsilon}} \right| \le h \right\}, \quad h \ge 0 \tag{50}$$

where $\widetilde{\varepsilon}$ is a known positive value. We require that the present worth be no less than the critical value, PW_c. Derive an explicit algebraic expression for the robustness function.

(e) We now extend part 43a by introducing annual inflation f. Everything else remains as in part 43a. Derive an explicit algebraic expression for the present worth at the beginning of the project.

(f) We now continue part 43e and consider two situations with different values of annual discount rate and inflation, and the same values of S, C, R and ε. In situation A the annual discount rate and inflation are $i = 0.04$ and $f = 0.07$, respectively. In situation B the inflation is $f' = 0.10$. Which of the following statements are correct in order for the present worth to be the same in both situations:

i. The discount rate in situation B goes down to the value $i' = 0.0116$.
ii. The discount rate in situation B goes up to the value $i' = 0.0684$.
iii. The discount rate in situation B goes down to the value $i' = 0.0100$.
iv. The discount rate in situation B goes up to the value $i' = 0.10$.

(g) You hold $$R_1$ at time $t = 0$. You could invest that money in a project in the U.S. lasting N years with a constant annual rate of return of $i_\$$. There is no inflation. What is the future worth of this project at its conclusion?

(h) You hold $$R_1$ at time $t = 0$. You could invest that money in a project lasting N years in the euro zone and earn a constant annual real rate of return (in euros) of i_e. The euro zone has constant annual inflation f_e. The exchange rate is r euro/$, and constant over time.

i. What is the euro value of $$R_1$ at time $t = 0$? Call this value R_e.
ii. What is the real future worth of R_e, in euro, at the end of the project?
iii. What is the nominal future worth of R_e, in euro, at the end of the project?
iv. What is the nominal future worth of R_e, in dollars, at the end of the project?

(i) We now combine parts 43g and 43h. Which investment do you prefer, in terms of future worth in dollars? Does euro inflation help or hurt the euro project? Why?

44. **Two investment options.** (Based on exam, 27.5.2019.) (p.123)

(a) \$$C$ is earned at the end of each year, for N years. The earnings are used for operational expenses. The annual interest rate is i. What is the present worth of this income stream if $C = 1000$, $N = 10$ and $i = 0.07$? Why is the value of the present worth less than NC?

(b) We now consider two different plans, each for N years. For plan j, \$$C_j$ is earned at the end of each year, and then used for operational expenses. The annual interest rate is i_j, where:

$$C_1 < C_2 \quad \text{and} \quad i_1 < i_2 \tag{51}$$

What is the intuitive meaning and implication of these two relations? What dilemma do they pose for choosing between the two plans? Given fixed values of i_1 and i_2, for what ratio of C_1 to C_2 do you prefer option 1, based on the present worth of these plans? For example, if $N = 10$, $i_1 = 0.05$ and $i_2 = 0.07$, for what ratio of C_1 to C_2 do you prefer option 1?

(c) Would the choice between the two plans in step 44b be different if one considered the future worth, rather than the present worth?

(d) We now consider two different plans, each with annual interest rate i. For plan j, the duration is N_j, and \$$C_j$ is earned at the end of each year, and then used for operational expenses, where:

$$C_1 < C_2 \tag{52}$$

For fixed values of i, N_2, C_1 and C_2, for what value of N_1 do we prefer plan 1 based on the present worth? For example, if $i = 0.05$, $N_2 = 10$ and $C_2/C_1 = 1.2$, for what values of N_1 do we prefer plan 2?

(e) You must choose between two options, where the value of option j is estimated as $\widetilde{v}_j$, whose error may be $\pm s_j$ or more. The values of $\widetilde{v}_j$ and s_j are known and satisfy the relations:

$$\widetilde{v}_1 < \widetilde{v}_2 \quad \text{and} \quad s_1 < s_2 \tag{53}$$

An info-gap model for uncertainty in the true values, v_j, is:

$$\mathcal{U}(h) = \left\{ v_1, v_2 : \ \left| \frac{v_j - \widetilde{v}_j}{s_j} \right| \le h, \ \ j = 1, 2 \right\}, \quad h \ge 0 \tag{54}$$

You require that the true value be no less than the critical value $v_{\rm c}$. Derive an explicit algebraic expression for the robustness of choice j. For what values of $v_{\rm c}$ do you prefer option 1, according to the principle of robustness?

(f) Uncertainty in the vector of N uncertain values, v, is represented by this ellipsoid-bound info-gap model:

$$\mathcal{U}(h) = \left\{ v : \ (v - \widetilde{v})^T W (v - \widetilde{v}) \le h^2 \right\}, \quad h \ge 0 \tag{55}$$

where W is a known, symmetric, real, positive definite matrix and $\widetilde{v}$ is a known vector. The total return is $r^T v$ where r is a known vector. Derive an explicit algebraic expression for the minimum total return at horizon of uncertainty h.

45. **Investments and earnings.** (based on exam, 16.7.2019) (p.126) At the end of $N = 3$ years from now you will receive a payment, in \$'s, of the fixed sum $A_N = \$10,000$. The general price inflation of \$'s will be f_j in year j for $j = 1, \ldots, N$, where $f_1 = 0.12$, $f_2 = 0.06$, $f_3 = 0.09$. The exchange rate between \$'s and pesos at the end of year N will be $r_N = 40$ peso/\$. The general price inflation of pesos will be ϕ_j in year j for $j = 1, \ldots, N$, where $\phi_1 = 0.24$, $\phi_2 = 0.14$, $\phi_3 = 0.32$.
   (a) What is the real value of A_N in \$'s at the start of year 1?
   (b) What is the real value of A_N in pesos at the start of year 1 if you exchange the payment to pesos when it is received, which is at the end of year N?
   (c) Now suppose that the exchange rate r_N is highly uncertain. Consider the following info-gap model

$$\mathcal{U}(h) = \left\{ r_N : \ r_N \geq 0, \ \left| \frac{r_N - \widetilde{r}_N}{\widetilde{r}_N} \right| \leq h \right\}, \quad h \geq 0 \tag{56}$$

   The value $\widetilde{r}_N$ is a known estimate. You require that the real peso value, at the start of year 1, of the year-N earnings in \$'s, be no less than the non-negative value $R_{\rm c}$. Derive an explicit algebraic expression for the robustness to uncertainty.

46. **Interest, inflation and uncertainty.** (Based on exam 16.7.2019.) (p.127) Consider a project for which the investment at the start of the first year is S, the revenue and cost at the end of each year are constant at R and C where $R - C > 0$, the duration is N years, there is no inflation and the annual discount rate is i.

   (a) Assume that $S = \$7,500$, $R = \$5,000$, $C = \$3000$, $N = 9$, and $i = 0.12$. What is the present worth of this project?

   (b) Continue part 46a, and find the greatest value of the initial investment, S, at which the present worth is non-negative.

   (c) Continue part 46a, and find the shortest project duration, N, at which the present worth is non-negative.

   (d) Return to the generic formulation of the project, and suppose that the discount rate, i, is uncertain with estimate $\widetilde{i}$ and uncertainty weight s_i, both known and positive. The info-gap model is:

$$\mathcal{U}(h) = \left\{ i: \ i \geq 0, \ \left| \frac{i - \widetilde{i}}{s_i} \right| \leq h \right\}, \quad h \geq 0 \tag{57}$$

   We require that the present worth be no less than the critical value, PW_c. Derive an explicit algebraic expression for the inverse of the robustness function.

   (e) Return to the generic formulation of the project, and suppose that R and C are uncertain as described by this info-gap model:

$$\mathcal{U}(h) = \left\{ R, C: \ \left| \frac{R - \widetilde{R}}{s_R} \right| \leq h, \ \left| \frac{C - \widetilde{C}}{s_C} \right| \leq h \right\}, \quad h \geq 0 \tag{58}$$

   where $\widetilde{R}$, $\widetilde{C}$, s_R and s_C are known and positive. We require that the present worth be no less than the critical value, PW_c. Derive an explicit algebraic expression for the robustness function.

   (f) Return to the generic formulation of the project and now introduce constant annual inflation, f. Annual revenue and cost, R and C, are constant in real dollars at the time of the initial investment. i is the nominal annual discount factor. Derive an explicit algebraic expression for the present worth.

47. **Uncertain but correlated investments.** (Based on exam 24.6.2019.) (p.129). Consider N different projects, where the return on investment in project n is estimated to be $\widetilde{u}_n$ dollars for each dollar invested. However, the actual returns are uncertain, but inter-related, according to this info-gap model:

$$\mathcal{U}(h, \widetilde{u}) = \left\{ u: \ (u - \widetilde{u})^T V^{-1} (u - \widetilde{u}) \leq h^2 \right\}, \quad h \geq 0 \tag{59}$$

where V^{-1} is a known, positive definite, real, symmetric matrix. The amount invested in project n is q_n dollars, $n = 1, \ldots, N$. You require that the total return be no less than the critical value R_c. Derive an explicit algebraic expression for the robustness function for investment vector q.

48. **Price indices.** (Based on exam, 3.10.2019.) (p.130) Consider N commodities whose nominal prices, at the end of year j, were $p_1^{(j)}, \ldots, p_N^{(j)}$. The inflation in year j was f_j.

(a) The arithmetic average of the nominal prices in year j is:

$$\overline{p}^{(j)} = \frac{1}{N} \sum_{i=1}^{N} p_i^{(j)} \tag{60}$$

What is the real value, at the start of year 1, of the arithmetic average of these N prices?

(b) The arithmetic average of the nominal price ratios for years 1 and 2 is:

$$\rho = \frac{1}{N} \sum_{i=1}^{N} \frac{p_i^{(2)}}{p_i^{(1)}} \tag{61}$$

What is the real value, at the start of year 1, of the arithmetic average of these price ratios?

(c) The nominal prices in year j are estimated as $\widetilde{p}_i^{(j)}$ but the true nominal prices are uncertain as expressed by this fractional-error info-gap model:

$$\mathcal{U}(h) \ = \ \left\{ p^{(j)} : \ p_i^{(j)} \geq 0, \quad \left| \frac{p_i^{(j)} - \widetilde{p}_i^{(j)}}{w \widetilde{p}_i^{(j)}} \right| \leq h, \ i = 1, \ldots, N \right\}, \quad h \geq 0 \tag{62}$$

where w is a known, positive uncertainty weight. We require that the average price ratio, defined in eq.(61), be no less than the critical value $\rho_{\rm c}$. Derive an explicit algebraic expression for the robustness.

(d) Consider a consumer price index as the weighted sum of the prices of N commodities:

$$CPI(p) = \sum_{i=1}^{N} w_i p_i \tag{63}$$

where the coefficients w_i are known and positive. The best estimate of the price vector is $\widetilde{p}$, but the prices are highly uncertain, as represented by this ellipsoid-bound info-gap model:

$$\mathcal{U}(h) = \left\{ p : \ (p - \widetilde{p})^T \, V^{-1} \, (p - \widetilde{p}) \leq h^2 \right\}, \quad h \geq 0 \tag{64}$$

where V is a known, real, symmetric, positive definite matrix. Derive an explicit algebraic expression for the robustness of $CPI(p)$ exceeding the critical value CPI_c.

(e) The consumer price index for year j is defined as:

$$CPI_j = \sum_{i=1}^{N} w_i p_i^{(j)} \tag{65}$$

where the coefficients w_i are known and positive. The inflation for year j is defined as:

$$f_j = \frac{CPI_j - CPI_{j-1}}{CPI_{j-1}} = \frac{CPI_j}{CPI_{j-1}} - 1 \tag{66}$$

Suppose that the prices for year 1 are well known, denoted by the vector $\widetilde{p}^{(1)}$. However, the prices for year 2 are estimated as $\widetilde{p}^{(2)}$ but uncertain as represented by this fractional-error info-gap model:

$$\mathcal{U}(h) \ = \ \left\{ p^{(2)} : \ p_i^{(2)} \geq 0, \quad \left| \frac{p_i^{(2)} - \widetilde{p}_i^{(2)}}{v_i} \right| \leq h, \ i = 1, \ldots, N \right\}, \quad h \geq 0 \tag{67}$$

where the v_i's are known positive constants. How confident are we that the inflation exceeds a specified critical value, $f_{\rm c}$? Derive an explicit algebraic expression for the inverse of the robustness function for satisfying the requirement $f_2 \geq f_{\rm c}$.

(f) Continue with the definition of inflation in eq.(66). The prices for year 1 are well known, denoted by the vector $\widetilde{p}^{(1)}$. The prices for year 2 are estimated as $\widetilde{p}^{(2)}$, which are uncertain but seem to indicate strong positive inflation. In fact, we suspect that prices in year 2 were greater than $\widetilde{p}^{(2)}$ (which was an early sample), but we don't know by how much. The following asymmetric info-gap model represents the uncertainty in $p^{(2)}$:

$$\mathcal{U}(h) = \left\{ p^{(2)} : \ p_i^{(2)} \ge 0, \quad 0 \le \frac{p_i^{(2)} - \widetilde{p}_i^{(2)}}{v_i} \le h, \ i = 1, \ldots, N \right\}, \quad h \ge 0 \qquad (68)$$

Let $f_2\left(p^{(2)c}, \widetilde{p}^{(1)}\right)$ denote an estimate of inflation, using eq.(66), where we have chosen the vector $p^{(2)c}$ to represent year-2 prices, and we use $\widetilde{p}^{(1)}$ for year-1 prices. We require that $f_2\left(p^{(2)c}, \widetilde{p}^{(1)}\right)$ be greater than the true inflation by no more than δ. That is, the performance requirement is:

$$f_2\left(p^{(2)c}, \widetilde{p}^{(1)}\right) - f_2\left(p^{(2)}, \widetilde{p}^{(1)}\right) \le \delta \qquad (69)$$

Derive an explicit algebraic expression for the robustness function.

(g) Repeat part 48f with this performance requirement rather than eq.(69):

$$f_2\left(p^{(2)}, \widetilde{p}^{(1)}\right) - f_2\left(p^{(2)c}, \widetilde{p}^{(1)}\right) \le \delta \qquad (70)$$

49. **Foreign investment.** (Based on exam, 3.10.2019.) (p.133) Consider an N-year project in which the real peso cost of operations is constant, C peso/year, due at the end of each year of operations. The annual inflation of the peso is constant at f. The exchange rate is constant at r peso/\$. The discount rate for dollars is $i \ne f$. There is no inflation of the dollar.

(a) You will use dollars to buy pesos at the end of each year of operation to cover nominal operating costs. What is the present worth in dollars of this sequence of dollar amounts?

(b) The operations produce goods that are sold at the end of year n for $R_n = (1+\varepsilon)^n R_0$ nominal pesos for $n = 1, \ldots, N$, where ε is a positive constant and $\varepsilon \ne i$. These peso earnings are used to buy dollars at the end of each year. What is the present worth in dollars of this sequence of dollar amounts?

(c) Continuing from parts 49a and 49b, what is the present worth of this project as N approaches infinity if $f < i < \varepsilon$?

(d) Continuing from parts 49a and 49b, what is the present worth of this project as N approaches infinity if $\varepsilon < i < f$?

(e) We modify the problem by considering two countries for which the present worth of the project in country j is:

$$\text{PW}_j = \varepsilon_j \text{PW}_{0j}, \quad j = 1, 2 \qquad (71)$$

where PW_{01} and PW_{02} are known and $\text{PW}_{01} < \text{PW}_{02}$. The coefficient ε_j is estimated to be the same for both countries, $\widetilde{\varepsilon}$, but it is uncertain in both cases according to this info-gap model:

$$\mathcal{U}(h) = \left\{ \varepsilon_j : \ \left| \frac{\varepsilon_j - \widetilde{\varepsilon}}{w_j} \right| \le h, \ j = 1, 2 \right\}, \quad h \ge 0 \qquad (72)$$

where w_1 and w_2 are known and $0 < w_1 < w_2$. Thus country 2 is putatively better but more uncertain than country 1. We require that the present worth be no less than the critical value PW_c. For what values of PW_c do we prefer country 2 according to the method of robust-satisficing?

50. **Future worth.** (Based on midterm exam, 26.6.2023.) (p.135)

(a) You will invest $P = \$100,000$ now with annual interest rate of $i = 0.15$. What is the future worth after $N = 25$ years?

(b) Consider two alternative investments of the same initial sum, P. The annual interest rates of these two investments are related as:

$$0 < i_1 < i_2 \tag{73}$$

The durations of these investments are related as:

$$N_1 > N_2 > 0 \tag{74}$$

For what values of the ratio $\frac{N_1}{N_2}$ is the first investment preferred, in terms of future worth?

(c) You will invest the sum $\$P$ for N years with annual interest rate i. However, the value of P is uncertain, with estimated value $\widetilde{P}$ and estimated error w, where $w > 0$. The following info-gap model represents the uncertainty in P:

$$\mathcal{U}(h) = \left\{ P : \left| \frac{P - \widetilde{P}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{75}$$

We require that the future worth be no less than $F_{\rm c}$. Derive an explicit algebraic expression for the robustness to uncertainty.

51. **Forecasting.** (p.136) Consider an economic variable, such as inflation or unemployment, whose discrete-time dynamics for 1 step into the future are represented by:

$$y_1 = \lambda_1 y_0 \tag{76}$$

where y_0 is known and positive, but λ_1 is uncertain. The best available estimate of λ_1 is $\widetilde{\lambda}$, but we are confident that this is an underestimate because the economy is changing — heating up or cooling down — in unknown ways. However, we don't have any reliable upper bound for λ_1.

(a) Given the information available, explain why the following info-gap model is a plausible representation of the uncertainty about λ_1:

$$\mathcal{U}(h) = \left\{ \lambda_1 : \ \widetilde{\lambda} \le \lambda_1 \le (1+h)\widetilde{\lambda} \right\}, \quad h \ge 0 \tag{77}$$

(b) Suppose we have additional information, namely, that $\widetilde{\lambda}$ may err by as much as the known positive value w or more. How would one incorporate this additional information in a modification of the info-gap model of eq.(77)?

(c) Consider the following slope-adjusted forecasting model:

$$y_1^{\rm s} = \ell y_0 \tag{78}$$

where we are free to choose the coefficient ℓ to express our understanding that $\widetilde{\lambda}$ is an underestimate of λ_1. Let us assume:

$$\ell > \widetilde{\lambda} \tag{79}$$

The error of this forecast is:

$$\varepsilon = |y_1^{\rm s} - y_1| \tag{80}$$

We require that this forecasting error not exceed a critical value, $\varepsilon_{\rm c}$:

$$\varepsilon \le \varepsilon_{\rm c} \tag{81}$$

Derive an explicit algebraic expression for the inverse of the robustness to uncertainty for satisfying this forecasting requirement. Explain how this robustness function assists in choosing the value of ℓ.

(d) The info-gap model for uncertainty in eq.(77) is non-probabilistic. What sort of information would we need in order to represent the uncertainty in λ_1 probabilistically? Suppose that we have some information of this sort. Denote the estimated probability density function (pdf) by $\widetilde{p}(\lambda_1)$. However, this estimated pdf is highly uncertain because it is based on past data, while the economy is changing dynamically. How might we represent the info-gap uncertainty in this pdf?