9. **Choose quality and lifetime.** (p.60) You must choose between two design concepts. Design 1 has higher quality, higher projected earnings R_1 at the end of each year, longer life, N_1 years, but higher initial cost S_1. Design 2 is the reverse, and has lower quality, lower projected earnings R_2 at the end of each year, shorter life, N_2 years, but lower initial cost S_2. Whichever design you choose, it must be implemented for $2N$ years. The operational lifetimes of designs 1 and 2 are $N_1 = 2N$ and $N_2 = N$, respectively. Thus, if you choose design 2, it must be re-purchased after N years. Which design do you prefer, as a function of the parameters? Suppose a MARR of 15%.

11. **Uncertain compound interest.** (p.64) Invest $\$P$ at the start of N years with estimated yearly rate of return $\widetilde{i}$, where the estimate may err by tens of percent or more. While the actual rate of return, i, is unknown, assume that it is constant over time. Furthermore, assume that i cannot be less than -1 ($i = -1$ implies FW is zero, and any more negative value of i implies debt).
(a) Derive an expression for the robustness of requiring that the future worth of the investment will not be less than the critical value FW_c.
(b) Based on (a) and given that $N = 5$, $\widetilde{i} = 0.06$ and $P = \$10,000$, what is a reliable estimate of the future worth?
(c) Return to (a) and compare two alternatives with different estimated rates of return, $\widetilde{i}_1 < \widetilde{i}_2$. For what values of FW_c do you prefer each alternative, based on the robustness to uncertainty?
(d) Evaluate (c) for $\widetilde{i}_1 = 0.02$, $\widetilde{i}_2 = 0.03$, $N = 5$ and $P = \$10,000$.

Solution to Problem 9, Choose quality and lifetime, (p.7). We will evaluate the present worth (PW) of each design concept. The basic relation between the PW and the net revenue at the end of the periods, F_k, is:

$$\begin{aligned} PW &= (1+i)^0 F_0 + (1+i)^{-1} F_1 + \cdots + (1+i)^{-k} F_k + \cdots + (1+i)^{-N} F_N && (117) \\ &= \sum_{k=0}^{N} (1+i)^{-k} F_k && (118) \end{aligned}$$

Notation for design concepts $m = 1, 2$:

- $N_m =$ number of periods during which this design is operational. $N_1 = 2N_2 = 2N$.
- $S_m =$ initial cost of implementation.
- $R_m =$ net revenue at the end of each year of implementation.
- MARR is 15% so $i = 0.15$.

Design concept 1. Adapting eq.(118), the PW of design concept 1 is:

$$\begin{aligned} PW_1 &= -S_1 + \sum_{k=1}^{N_1} (1+i)^{-k} R_1 && (119) \\ &= \boxed{-S_1 + \frac{1-(1+i)^{-N_1}}{i} R_1} && (120) \end{aligned}$$

Design concept 2. The PW of the first N_2 years of design 2 is analogous to design 1:

$$\begin{aligned} PW_{2,1} &= -S_2 + \sum_{k=1}^{N_2} (1+i)^{-k} R_2 && (121) \\ &= \boxed{-S_2 + \frac{1-(1+i)^{-N_2}}{i} R_2} && (122) \end{aligned}$$

The PW of the second N_2 years of design 2 is just like the first N_2 years, but starting at the end of year N_2. The PW of the second N_2 years is thus:

$$PW_{2,2} = (1+i)^{-N_2} PW_{2,1} \qquad (123)$$

Combining eqs.(122) and (123) we find the PW of design 2:

$$\begin{aligned} \text{PW}_2 &= PW_{2,1} + (1+i)^{-N_2} PW_{2,1} && (124) \\ &= \left(1+(1+i)^{-N_2}\right) \left(-S_2 + \frac{1-(1+i)^{-N_2}}{i} R_2\right) && (125) \\ &= -\left(1+(1+i)^{-N_2}\right) S_2 \; + \; \frac{1-(1+i)^{-2N_2}}{i} R_2 && (126) \\ &= -\left(1+(1+i)^{-N_2}\right) S_2 \; + \; \frac{1-(1+i)^{-N_1}}{i} R_2 && (127) \end{aligned}$$

where eq.(127) results from eq.(126) because $N_1 = 2N_2$.

Note that PW_2 can be derived differently by noting that revenue R_2 is obtained at the end of each year during $2N_2$ years, and that the investment S_2 is made initially and again at the end of year N_2. Thus:

$$\begin{aligned} \text{PW}_2 &= -S_2 - (1+i)^{-N_2} S_2 + \sum_{k=1}^{2N_2} (1+i)^{-k} R_2 && (128) \\ &= -\left(1+(1+i)^{-N_2}\right) S_2 + \frac{1-(1+i)^{-2N_2}}{i} R_2 && (129) \end{aligned}$$

which is precisely eq.(126).

To choose between the design concepts, we must compare eqs.(120) and (127). Note:

• The coefficients of R_1 and R_2 are the same, while $R_2 < R_1$ because the design 2 has lower projected earnings. This mitigates in favor of design 1.

• The coefficient of S_2 is more negative than the coefficient of S_1 (which is -1), but $0 < S_2 < S_1$. It is not clear which design is preferred.

Let's define the following two discount factors:

$$D_1 = \frac{1-(1+i)^{-N_1}}{i} \tag{130}$$
$$D_2 = 1+(1+i)^{-N_2} \tag{131}$$

The PW's of the two designs are:

$$\text{PW}_1 = -S_1 + D_1 R_1 \tag{132}$$
$$\text{PW}_2 = -D_2 S_2 + D_1 R_2 \tag{133}$$

We prefer design 1 if and only if:

$$\text{PW}_1 > \text{PW}_2 \tag{134}$$
$$-S_1 + D_1 R_1 > -D_2 S_2 + D_1 R_2 \tag{135}$$
$$D_1(R_1 - R_2) > S_1 - D_2 S_2 \tag{136}$$
$$D_1 \frac{R_1 - R_2}{S_1} > 1 - D_2 \frac{S_2}{S_1} \tag{137}$$
$$\frac{R_1 - R_2}{S_1} > \frac{1}{D_1} - \frac{D_2}{D_1}\frac{S_2}{S_1} \tag{138}$$

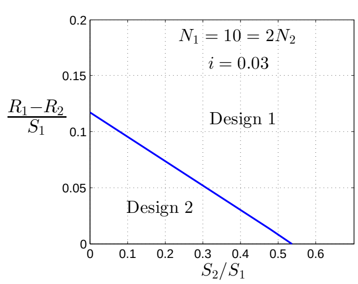

Figure 1: Eq.(138) in problem 9, $i = 0.03$. The curve is a plot of $\frac{1}{D_1} - \frac{D_2 S_2}{D_1 S_1}$ vs. S_2/S_1. $D_1 = 8.5302$, $D_2 = 1.8626$.

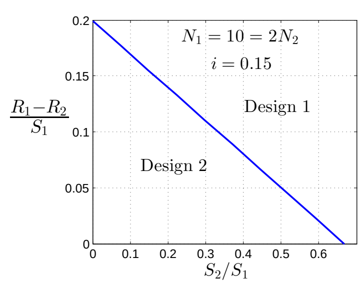

Figure 2: Eq.(138) in problem 9, $i = 0.15$. The curve is a plot of $\frac{1}{D_1} - \frac{D_2 S_2}{D_1 S_1}$ vs. S_2/S_1.

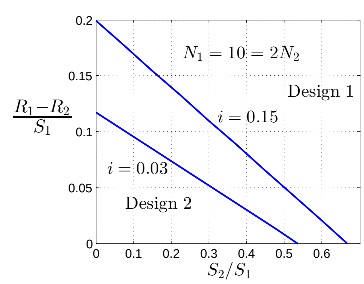

Figure 3: Eq.(138) in problem 9, $i = 0.03$ and $i = 0.015$. The curves are plots of $\frac{1}{D_1} - \frac{D_2 S_2}{D_1 S_1}$ vs. S_2/S_1.

Figs. 1 and 2 show plots of the righthand side of eq.(138) vs S_2/S_1, for two different choices of the interest rate. For each curve, points above the line represent parameter values for which eq.(134) holds and thus design 1 is preferred. Points below the line represent parameter values for which design 2 is preferred. We can understand these curves as follows.

• The ratio $\frac{R_1-R_2}{S_1}$ is positive, and expresses the relative advantage of design 1 over design 2, in terms of greater revenue per year for design 1. At fixed S_2/S_1, design 1 is preferred as $\frac{R_1-R_2}{S_1}$ increases.

• S_2/S_1 is less than one, and is the ratio of initial capital cost of design 2 compared to design 1. At fixed $\frac{R_1-R_2}{S_1}$, design 1 is preferred as S_2/S_1 increases.

• We see in figs. 1 and 2 that, as $\frac{R_1-R_2}{S_1}$ increases, the preference for design 2 requires a compensatory decrease in S_2/S_1. This is because a decrease in revenue from design 2 requires compensation by a decrease in the capital cost of design 2, in order for design 2 to remain preferred over design 1.

i	$1/D_1$	$1/D_2$	$-D_2/D_1$
0.0300	0.1172	0.5369	0.2184
0.0600	0.1359	0.5723	0.2374
0.0900	0.1558	0.6061	0.2571
0.1200	0.1770	0.6380	0.2774
0.1500	0.1993	0.6679	0.2983
0.1800	0.2225	0.6958	0.3198

Table 9: Solution to problem 9.

The vertical and horizontal intercepts of the curves in fig. 1 are $1/D_1$ and $1/D_2$, respectively. The slope is $-D_2/D_1$. Values are shown in table 9. The table and fig. 3 both show that the range of parameter values for which design 2 is preferred increases as the interest rate increases. This is logical because design 2 defers a substantial capital investment to the end of year 5.

Solution to Problem 11, Uncertain compound interest, (p.9).

(a) The FW is:

$$\text{FW}(i) = (1+i)^N P \tag{144}$$

because i is constant over time. The info-gap model is:

$$\mathcal{U}(h) = \left\{ i:\ i \ge -1,\ \left| \frac{i-\widetilde{i}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{145}$$

where s is chosen to equal "tens of percent" of $\widetilde{i}$, for example $s = 0.3\widetilde{i}$. If $i=-1$ then the FW is zero, and any more negative value entails debt. We suppose that debt would terminate the operation through bankruptcy.

The robustness is defined as:

$$\widehat{h}(\text{FW}_\text{c}) = \max \left\{ h:\ \left(\min_{i \in \mathcal{U}(h)} F(i) \right) \ge \text{FW}_\text{c} \right\} \tag{146}$$

Let $m(h)$ denote the inner minimum, which occurs when $i = \widetilde{i} - sh$, for $h \le (1+\widetilde{i})/s$:

$$m(h) = \left(1 + \widetilde{i} - sh \right)^N P \tag{147}$$

Equate this to FW_c and solve for h:

$$\left(1 + \widetilde{i} - sh \right)^N P = \text{FW}_\text{c} \quad \Longrightarrow \quad \boxed{\widehat{h}(\text{FW}_\text{c}) = \frac{1}{s} \left(1 + \widetilde{i} - \left(\frac{\text{FW}_\text{c}}{P} \right)^{1/N} \right)} \tag{148}$$

or zero if this is negative. Note that:

$$\widehat{h}(\text{FW}_\text{c} = 0) = \frac{1+\widetilde{i}}{s} \tag{149}$$

Thus eq.(148) is valid for all non-negative values of FW_c.

(b) We require "several" units of robustness, for instance $\widehat{h} = 3$ and $s = 0.3\widetilde{i} = 0.018$. Solving eq.(148) for FW_c:

$$\text{FW}_\text{c} = (1 + \widetilde{i} - s\widehat{h})^N P = (1 + 0.06 - 0.018 \times 3)^5 10{,}000 = \$10{,}303.62 \tag{150}$$

Thus, even if the estimated rate of return errs by as much as a factor of 3 times s, (implying a lower-than anticipated rate of return) the FW will not be less than $10,303.62.

Compare this with the estimated FW:

$$\text{FW}(\widetilde{i}) = (1+\widetilde{i})^N P = (1+0.06)^5 10{,}000 = \$13{,}382.26 \tag{151}$$

(c) and (d) Choose the uncertainty weights for each case as $s_k = g\widetilde{i}_k$, $k = 1, 2$. Equate the robustnesses of the two alternatives to find the value of FW_c at which the robustness curves cross:

$$\widehat{h}(\text{FW}_\times, \widetilde{i}_1) = \widehat{h}(\text{FW}_\times, \widetilde{i}_2) \quad \Longrightarrow \quad \text{FW}_\times = P \tag{152}$$

The robustness reaches zero at the estimated FW:

$$\widehat{h}(\text{FW}_\text{c}) = 0 \ \text{ if } \ \text{FW}_\text{c} = \text{FW}(\widetilde{i}) = (1+\widetilde{i})^N P \tag{153}$$

Thus $\widehat{h}(\text{FW}_\text{c}, \widetilde{i}_2)$ reaches the FW_c axis to the right of $\widehat{h}(\text{FW}_\text{c}, \widetilde{i}_1)$ because $\widetilde{i}_1 < \widetilde{i}_2$. The robustness curves cross at $\text{FW}_\text{c} = P$, at which value the robustness equals $\widehat{h}(P) = \widetilde{i}_k / s_k = 1/g$. This is illustrated in fig. 4 with $g = 1$, where we see that $\widetilde{i}_2 = 0.03$ is robust-preferred over $\widetilde{i}_1 = 0.02$ for $\text{FW}_\text{c} \ge \$10{,}000$, which is precisely the initial investment, P.

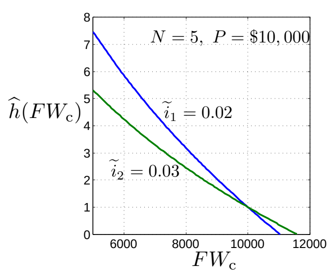

Figure 4: Robustness curves for problem 11 (c), eq.(148).