28. **Loan and investment.** (based on exam, 21.7.2014) (p.91) You will take a loan of \$60,000 at 7% yearly interest, at the start of year $k = 1$. We will consider different repayment schemes.

(a) You will repay \$20,000 of the principal at the end of each year $k = 1, 2, 3$. At the end of each year you will also repay the interest accrued up to that time. How much interest do you pay each year?

(b) Continue part 28a. At the beginning of year k you hold $\$60,000 - (k-1)\$20,000$, for $k = 1, 2, 3$. This sum will be invested with yearly rate of return $i_{\mathrm{inv}} = 0.15$ for the duration of the year. Evaluate the present worth of the returns on the investment (positive) and the interest payments (negative, with interest rate $i = 0.07$) over the 3 years of the loan, at discount rate i_{inv}.

(c) The initial loan, P, is \$60,000, but consider a different repayment scheme from part 28a. You will make 3 equal payments at the end of years $k = 1, 2, 3$. What is the yearly payment if these equal payments will entirely repay the loan and its interest at the end of 3 years? Explain the difference from part 28a.

(d) We now generalize the problem. Consider K yearly interest payments, I_k, $k = 1, \ldots, K$, where the I_k's are negative numbers (payments). The present worth of these payments, with annual discount rate i_{inv}, is denoted PW(i_{inv}). Note that the I_k's are themselves interest payments on a loan, and that the annual discount rate, i_{inv}, is not an interest rate on the loan. Rather, i_{inv} is a rate of return on investments made with the loan whose interest payments are I_k.

The discount rate is constant but uncertain with estimated value $\widetilde{i}_{\mathrm{inv}}$ and positive error estimate s:

$$\mathcal{U}(h) = \left\{ i_{\mathrm{inv}} : \ i_{\mathrm{inv}} \geq 0, \ \left| \frac{i_{\mathrm{inv}} - \widetilde{i}_{\mathrm{inv}}}{s} \right| \leq h \right\}, \quad h \geq 0 \tag{10}$$

Derive an explicit algebraic expression for the minimum (most negative) present worth at horizon of uncertainty h. Denote this result $m(h)$.

(e) Continuing part 28d, we require that the PW be no more negative than the value $\mathrm{PW_c}$. The function $m(h)$ derived in part 28d is the inverse of the robustness function for this requirement, denoted $\widehat{h}(\mathrm{PW_c})$. Schematically (not numerically) sketch the robustness function, $\widehat{h}(\mathrm{PW_c})$ vs. $\mathrm{PW_c}$.

(f) Continuing part 28e, consider two alternative discount rates, and error estimates, for the interest payment scheme in part 28d:

$$\widetilde{i}_{1,\mathrm{inv}} \ < \ \widetilde{i}_{2,\mathrm{inv}} \tag{11}$$

$$\frac{\widetilde{i}_{1,\mathrm{inv}}}{s_1} \ > \ \frac{\widetilde{i}_{2,\mathrm{inv}}}{s_2} \tag{12}$$

Which discount scheme would you prefer if there were no uncertainty in the discount rate? Does this preference hold at all levels of uncertainty? Explain in terms of robustness against uncertainty of the two schemes.

Solution to Problem 28, Loan and investment (p.24).

(28a) The answer is in the fourth column of table 12, according to the following relation:

$$I = iP \tag{339}$$

where $i = 0.07$.

Year	Amount owed at beginning of year, P	Princ. paid at end of year	Interest paid at end of year, I
1	60,000	20,000	4,200
2	40,000	20,000	2,800
3	20,000	20,000	1,400

Table 12: Solution to problem 28a.

(28b) The return on the investment at the end of year k is:

$$R_k = [60,000 - (k-1)20,000]i_{\rm inv} \tag{340}$$

where $i_{\rm inv} = 0.15$. The interest paid at the end of year k is:

$$I_k = [60,000 - (k-1)20,000]i \tag{341}$$

where $i = 0.07$. The present worth is:

$$\begin{aligned}
\text{PW} &= \sum_{k=1}^{3}(1+i_{\rm inv})^{-k}(R_k - I_k) && (342)\\
&= \sum_{k=1}^{3}(1+i_{\rm inv})^{-k}[60,000-(k-1)20,000](i_{\rm inv} - i) && (343)\\
&= \sum_{k=1}^{3}(1.15)^{-k}[60,000-(k-1)20,000](0.08) && (344)\\
&= 0.08\left[1.15^{-1}\times 60,000 + 1.15^{-2}\times 40,000 + 1.15^{-3}\times 20,000\right] && (345)\\
&= 7,645.60 && (346)
\end{aligned}$$

(28c) Define:
$A =$ equal annual payments.
$P =$ principal of loan = \$60,000.
$i =$ annual interest rate of loan = 0.07.
$K =$ number of years of loan = 3.
These are related as:

$$A = \frac{i(1+i)^K}{(1+i)^K - 1}P = 22,863 \tag{347}$$

Thus, the total interest paid is $3 \times (22,863 - 20,000) = 8,589.30$. The total interest paid in part 28a was $4,200 + 2,800 + 1,400 = 8,400$ but spread out differently (less favorably) in time.

(28d) The definition of the PW is:

$$\text{PW} = \sum_{k=1}^{K}(1+i_{\rm inv})^{-k}I_k \tag{348}$$

The PW of the loan payments, I_k, is small if the discount rate (rate of return on the investment), $i_{\rm inv}$, is large. If the rate of return on the investment is large, then distant interest payments have low present worth because, by the time we have to pay that interest, we will have earned lots of money.

We can understand this in more detail as follows. The basic idea is expressed by the following two arguments:

(1) The present worth of future money is LOW if the rate of return on investment is HIGH. That is, if the rate of return is LARGE, then \$1 today is equivalent to many dollars in the future, or \$1 in the future is equivalent to much less than \$1 today. Likewise, if the rate of return is LARGE, then future debt is LESS meaningful in the present because MUCH money will be earned before that future debt is realized.

(2) The present worth of future money is LARGE if the rate of return on investment is LOW. That is, if the rate of return is LOW, then \$1 today is equivalent to only slightly more than \$1 in the future, or \$1 in the future is equivalent to only slightly less than \$1 today. Likewise, if the rate of return is LOW, then future debt is MORE meaningful in the present because LITTLE money will be earned before that future debt is realized.

In problem 28(d) you are asked to find the most negative present worth at horizon of uncertainty h. That is, you must find the condition for which the present worth of future debt is high (very negative). Thus case (2) applies: The present worth of debt is large (very negative) if the rate of return is low, because we won't earn much money before we have to pay that debt.

Stating these two arguments differently: The money-value of time is SMALL if the rate of return is small: A small increment of time has a small impact on money. The money-value of time is LARGE if the rate of return is large: A small increment of time has a large impact on money.

The most negative PW at horizon of uncertainty h is:

$$m(h) = \min_{i_{\rm inv} \in \mathcal{U}(h)} \text{PW} \tag{349}$$

The I_k's in eq.(348) are negative so the minimum (most negative) PW occurs when the discount rate $i_{\rm inv}$ is as small as possible: $i_{\rm inv} = [\widetilde{i}_{\rm inv} - sh]^+$:

$$m(h) = \sum_{k=1}^{K} \left(1 + [\widetilde{i}_{\rm inv} - sh]^+\right)^{-k} I_k \tag{350}$$

(28e)

(28f) Eq.(11), p.24, implies this ranking of the estimated PW's:

$$\text{PW}(\widetilde{i}_{1,\rm inv}) < \text{PW}(\widetilde{i}_{2,\rm inv}) < 0 \tag{351}$$

Thus the first discount scheme has a more negative (less desirable) present worth of the interest payments. Thus, if there were no uncertainty, we would prefer the 2nd scheme.

However, Eq.(12), p.24, implies that the relative error for scheme 1 is lower than for scheme 2. This implies that the cost of robustness for scheme 1 is lower than the cost of robustness for scheme 2. Thus their robustness curves cross, as in fig 10, implying the potential for a reversal of preference between the two schemes. At very negative $\text{PW}_{\rm c}$ we prefer scheme 1 which is more robust, while at less negative $\text{PW}_{\rm c}$ we prefer scheme 2.