15. **Benefit-cost ratio of two design concepts.** (p.72) Consider two design concepts. For both concepts the benefit and maintenance cost at the end of each year is $4,000 and $1,500, respectively. The interest rate is $i = 0.05$.

    (a) Evaluate the benefit-cost ratios of the two design concepts, where the anticipated usable lifetimes of designs 1 and 2 are $N_1 = 3$ years and $N_2 = 5$ years. The initial investments in designs 1 and 2 are $S_1 = \$20,000$ $S_2 = \$33,333$. Which design has a better benefit-cost ratio (BCR)? Why? Note that $S_1/N_1 = S_2/N_2$. So why is the BCR result surprising?

    (b) For what ratio of initial investment is the BCR the same for the two designs? What does this imply about the initial costs of the two designs?

16. **Present worth or benefit-cost ratio?** (p.74) Consider two design concepts for a system with $N = 5$ year expected life, financed at an interest rate of $i = 0.05$. For the jth system, the initial cost is S_j and the benefit and maintenance costs at the end of each year are B_j and C_j respectively. Consider specific values:

    $B_1 = \$1,270.35$, $C_1 = \$461.95$, $S_1 = \$1,000$.

    $B_2 = \$1,154.87$, $C_2 = \$415.75$, $S_2 = \$800$.

    (a) One team of analysts uses the present worth method to compare these concepts, and another team uses the benefit-cost ratio. What do they recommend? Do they agree? What does this imply?

    (b) Now consider fractional uncertainty in benefits and costs with the following info-gap model:

$$\mathcal{U}(h) = \left\{ B_j, C_j : \ \left| \frac{B_j - \widetilde{B}_j}{w_{B,j}} \right| \le h, \ \left| \frac{C_j - \widetilde{C}_j}{w_{C,j}} \right| \le h, \ j = 1, \ 2 \right\}, \quad h \ge 0 \tag{5}$$

The nominal values, $\widetilde{B}_j$ and $\widetilde{C}_j$, take the previous numerical values. The uncertainty weights, $w_{B,j}$ and $w_{C,j}$, are known positive values.

Derive separate robustness functions for satisficing the PW and satisficing the BCR, for each design. You will derive 4 robustness functions. Does the robust prioritization of the designs based on PW, necessarily agree with the robust prioritization based on BCR? Explain.

Solution to Problem 15, Benefit-cost ratio of two design concepts, (p.13).

(a) Present worth of the benefits of design j are:

$$B_{pw}(j) = \sum_{n=1}^{N_j}(1+i)^{-n}B \quad (197)$$
$$= \frac{1-(1+i)^{-N_j}}{i}B \quad (198)$$
$$= \delta_{fj}(i)B \quad (199)$$

Present worth of the initial investment and maintenance costs of design j are:

$$C_{pw}(j) = S + \sum_{n=1}^{N_j}(1+i)^{-n}C \quad (200)$$
$$= S_j + \frac{1-(1+i)^{-N_j}}{i}C \quad (201)$$
$$= S + \delta_{fj}(i)C \quad (202)$$

The BCR of design j is:

$$\text{BCR}(j) = \frac{B_{pw}(j)}{C_{pw}(j)} \quad (203)$$
$$= \frac{\delta_{fj}(i)B}{S_j + \delta_{fj}(i)C} \quad (204)$$

The discount factors for the two designs are:

$$\delta_{f1}(i) = 2.7232, \quad \delta_{f2}(i) = 4.3295 \quad (205)$$

δ_{f1} is less than δ_{f2} because $N_1 < N_2$. However, the ratio is greater than the ratio of the durations:

$$\frac{\delta_{f1}}{\delta_{f2}} = 0.6290 > 0.6 = \frac{N_1}{N_2} \quad (206)$$

The reason: later periods get less weight than earlier periods.

The BCRs of the two designs are:

$$\text{BCR}(1) = \frac{2.7232 \times 4,000}{20,000 + 2.7232 \times 1,500} = 0.4523 \quad (207)$$
$$\text{BCR}(2) = \frac{4.3295 \times 4,000}{33,333 + 4.3295 \times 1,500} = 0.4348 \quad (208)$$

Both the benefits and the costs are lower for design 1 than for design 2, but the BCRs are nearly the same. Nonetheless, design 1 has a better (higher) ratio even though the costs and benefits each period are the same and the initial investment per year is the same ($S_1/N_1 = S_2/N_2$). The reason: The benefits over 5 years from design 2 are discounted disproportionately, as shown in eq.(206). Even though the initial investments look the same for both designs, they are not because future benefits are discounted.

(b) Equate the BCRs and solve for the ratio S_1/S_2:

$$\text{BCR}(1) = \text{BCR}(2) \Longrightarrow \frac{\delta_{f1}B}{S_1 + \delta_{f1}C} = \frac{\delta_{f2}B}{S_2 + \delta_{f2}C} \Longrightarrow (S_2 + \delta_{f2}C)\delta_{f1} = (S_1 + \delta_{f1}C)\delta_{f2} \quad (209)$$
$$\Longrightarrow \frac{S_1}{S_2} = \frac{\delta_{f1}}{\delta_{f2}} \quad (210)$$

The BCRs will be the same if the ratio of initial investments, S_1/S_2, equals the ratio δ_{f1}/δ_{f2}. However, the initial investments are proportional to the lifetimes, $S_1/S_2 = N_1/N_2 = 3/5 = 0.6$, and the discount functions are not: $\delta_{f1}/\delta_{f2} = 0.6290$. In other words, design 1 has a better BCR than design 2 because design 1 is under-priced relative to the initial cost of design 2. Its “discounted fair price”, resulting in the same BCR for both designs, would be:

$$S_1 = \frac{\delta_{f1}}{\delta_{f2}} S_2 = 0.629 \times \$33,333 = \$20,966 > 20,000 \tag{211}$$

Design 1 has a higher benefit-to-cost ratio because its initial cost is low; it is a better buy: more benefit per dollar (of initial investment and discounted maintenance cost).

Solution to Problem 16, Present worth or benefit-cost ratio? (p.14).

(a) Present worth of the benefits of design j are:

$$\begin{aligned} B_{pw}(j) &= \sum_{n=1}^{N} (1+i)^{-n} B_j && (212) \\ &= \frac{1-(1+i)^{-N}}{i} B_j && (213) \\ &= \delta_f(i) B_j && (214) \end{aligned}$$

Present worth of the initial investment and maintenance costs of design j are:

$$\begin{aligned} C_{pw}(j) &= S_j + \sum_{n=1}^{N} (1+i)^{-n} C_j && (215) \\ &= S_j + \frac{1-(1+i)^{-N}}{i} C_j && (216) \\ &= S_j + \delta_f(i) C_j && (217) \end{aligned}$$

The present worth of design j is:

$$\text{PW}_j = B_{pw}(j) - C_{pw}(j) = \delta_f(i)B_j - S_j - \delta_f(i)C_j \qquad (218)$$

The BCR of design j is:

$$\begin{aligned} \text{BCR}(j) &= \frac{B_{pw}(j)}{C_{pw}(j)} && (219) \\ &= \frac{\delta_f(i)B_j}{S_j + \delta_f(i)C_j} && (220) \end{aligned}$$

The discount function is $\delta_f(i) = 4.3295$. Thus:

$B_{pw}(1) = \$5,500$, $S_1 + C_{pw}(1) = \$1,000 + \$2,000 = \$3,000$. Thus PW$(1) = \$2,500$ and BCR$(1) = 1.8333$.

$B_{pw}(2) = \$5,000$, $S_2 + C_{pw}(2) = \$800 + \$1,800 = \$2,600$. Thus PW$(2) = \$2,400$ and BCR$(2) = 1.9231$.

Hence: PW$(1) >$ PW(2) so design 1 is PW-preferred.

But: BCR$(2) >$ BCR(1) so design 2 is BCR-preferred.

The teams disagree. Objective economic analysis is not always unique.

(b) The robustness function for the PW of design j is defined as:

$$\widehat{h}_{pw,j} = \max \left\{ h : \left(\min_{B,C \in \mathcal{U}(h)} \text{PW}_j \right) \ge \text{PW}_\text{c} \right\} \qquad (221)$$

where PW$_j$ is specified by eq.(218).

The robustness function for the BCR of design j is defined as:

$$\widehat{h}_{bcr,j} = \max \left\{ h : \left(\min_{B,C \in \mathcal{U}(h)} \text{BCR}_j \right) \ge \text{BCR}_\text{c} \right\} \qquad (222)$$

where BCR$_j$ is specified by eq.(220).

Let $m_{pw,j}$ denote the inner minimum in eq.(221), which is the inverse of $\widehat{h}_{pw,j}$. Similarly, let $m_{bcr,j}$ denote the inner minimum in eq.(222), which is the inverse of $\widehat{h}_{bcr,j}$.

Both of these inverses occur, at horizon of uncertainty h, for:

$$B_j = \widetilde{B}_j - w_{B,j}h, \quad C_j = \widetilde{C}_j + w_{C,j}h \tag{223}$$

Thus:

$$m_{pw,j} = (\widetilde{B}_j - w_{B,j}h)\delta_f - S_j - (\widetilde{C}_j + w_{C,j}h)\delta_f = \widetilde{\mathrm{PW}}_j - h(w_{B,j} + w_{C,j})\delta_f \geq \mathrm{PW_c} \tag{224}$$

Hence:

$$\boxed{\widehat{h}_{pw,j} = \frac{\widetilde{\mathrm{PW}}_j - \mathrm{PW_c}}{(w_{B,j} + w_{C,j})\delta_f}} \tag{225}$$

Similarly:

$$m_{bcr,j} = \frac{(\widetilde{B}_j - w_{B,j}h)\delta_f}{S_j + (\widetilde{C}_j + w_{C,j}h)\delta_f} \geq \mathrm{BCR_c} \tag{226}$$

Hence:

$$\boxed{\widehat{h}_{bcr,j} = \frac{(\widetilde{\mathrm{BCR}}_j - \mathrm{BCR_c})(S_j + \widetilde{C}_j\delta_f)}{(w_{B,j} + w_{C,j}\mathrm{BCR_c})\delta_f}} \tag{227}$$

The nominal values are the same as before, so the nominal prioritization disagrees between PW and BCR:

$$\widetilde{\mathrm{PW}}_1 > \widetilde{\mathrm{PW}}_2 \quad \text{but} \quad \widetilde{\mathrm{BCR}}_1 < \widetilde{\mathrm{BCR}}_2 \tag{228}$$

However, the costs of robustness (slopes of the robustness functions) for PW and BCR depend differently on the parameters. Thus there may be curve-crossing and preference reversal for one criterion (PW or BCR) but perhaps not for the other. Also, the critical values at which curve crossing occurs may be interpreted differently for the two criteria. Thus the robust prioritization based on PW may or may not agree with the robust prioritization based on BCR.