12. **Investment with uncertain costs and returns.** (p.70) Consider the following project. The initial cost is S. The income is estimated to be $\widetilde{R}$ at the end of each year. The annual cost of operating the new system is estimated to be $\widetilde{C}$ at the end of each year. These estimates may err substantially. Use a fractional error info-gap model for costs and returns:

$$\mathcal{U}(h)\left\{R, C: \left|\frac{R_k-\widetilde{R}}{\varepsilon_R \widetilde{R}}\right| \le h,\ \left|\frac{C_k-\widetilde{C}}{\varepsilon_C \widetilde{C}}\right| \le h,\ k=1,\ldots,N\right\}, \quad h \ge 0 \tag{2}$$

The company's minimal acceptable rate of return (MARR) is i. The system will operate for N years.

(a) Derive the robustness function of the PW.

(b) Compare two realizations of this system with the following characteristics:

$$\text{PW}(\widetilde{R}_1, \widetilde{C}_1) \quad < \quad \text{PW}(\widetilde{R}_2, \widetilde{C}_2) \tag{3}$$

$$\varepsilon_{1,R}\widetilde{R}_1 + \varepsilon_{1,C}\widetilde{C}_1 \quad < \quad \varepsilon_{2,R}\widetilde{R}_2 + \varepsilon_{2,C}\widetilde{C}_2 \tag{4}$$

For what values of the PW do you prefer option 1? Provide an intuitive explanation of the results.

Solution to Problem 12, Investment with uncertain costs and returns, (p.10).

(a)

- $S =$ Initial cost of the project.
- $\widetilde{R} =$ estimated revenue at the end of each period.
- $\widetilde{C} =$ estimated operating cost at the end of each period.
- $N =$ number of periods.
- MARR = i.

The PW is:

$$PW(R,C) = -S + \sum_{k=1}^{N}(1+i)^{-k}R_k - \sum_{k=1}^{N}(1+i)^{-k}C_k \tag{162}$$

The robustness is defined as:

$$\widehat{h} = \max\left\{h : \left(\min_{R,C\in\mathcal{U}(h)} \text{PW}(R,C)\right) \geq \text{PW}_{\text{c}}\right\} \tag{163}$$

Let $m(h)$ denote the inner minimum, which occurs when:

$$R_k = \widetilde{R} - \varepsilon_R\widetilde{R}h = \widetilde{R}(1-\varepsilon_R h), \quad C_k = \widetilde{C} + \varepsilon_C\widetilde{C}h = \widetilde{C}(1+\varepsilon_C h) \tag{164}$$

Thus:

$$\begin{aligned} m(h) &= -S + \sum_{k=1}^{N}(1+i)^{-k}\widetilde{R}(1-\varepsilon_R h) - \sum_{k=1}^{N}(1+i)^{-k}\widetilde{C}(1+\varepsilon_C h) && (165)\\ &= \text{PW}(\widetilde{R},\widetilde{C}) - (\varepsilon_R\widetilde{R}+\varepsilon_C\widetilde{C})h\sum_{k=1}^{N}(1+i)^{-k} && (166)\\ &= \text{PW}(\widetilde{R},\widetilde{C}) - (\varepsilon_R\widetilde{R}+\varepsilon_C\widetilde{C})h\frac{1-(1+i)^{-N}}{i} && (167) \end{aligned}$$

Equate this to PW_{c} and solve for h to obtain the robustness:

$$\boxed{\widehat{h} = \frac{\text{PW}(\widetilde{R},\widetilde{C}) - \text{PW}_{\text{c}}}{\varepsilon_R\widetilde{R}+\varepsilon_C\widetilde{C}}\,\frac{i}{1-(1+i)^{-N}}} \tag{168}$$

or zero if this is negative.

(b) The conditions of eqs.(3) and (4) cause the robustness curves of the two options to cross (see fig. 5), where option 1 is nominally worse but with lower cost of robustness (steeper robustness curve). Thus, the robustness criterion prefers option 1 when PW_{c} is less than the value, $\text{PW}_\times$, at which the robustness curves cross one another. This is obtained by equating the robustness functions:

$$\begin{aligned} \widehat{h}_1 = \widehat{h}_2 \quad &\Longrightarrow \quad \frac{\text{PW}(\widetilde{R}_1,\widetilde{C}_1) - \text{PW}_\times}{\varepsilon_{1,R}\widetilde{R}_1+\varepsilon_{1,C}\widetilde{C}_1} = \frac{\text{PW}(\widetilde{R}_2,\widetilde{C}_2) - \text{PW}_\times}{\varepsilon_{2,R}\widetilde{R}_2+\varepsilon_{2,C}\widetilde{C}_2} && (169)\\ &\Longrightarrow \quad \text{PW}_\times = \frac{\widetilde{\text{PW}}_1 E_2 - \widetilde{\text{PW}}_2 E_1}{E_2 - E_1} && (170) \end{aligned}$$

which is positive if:

$$\frac{\widetilde{\text{PW}}_1}{E_1} > \frac{\widetilde{\text{PW}}_2}{E_2} \tag{171}$$

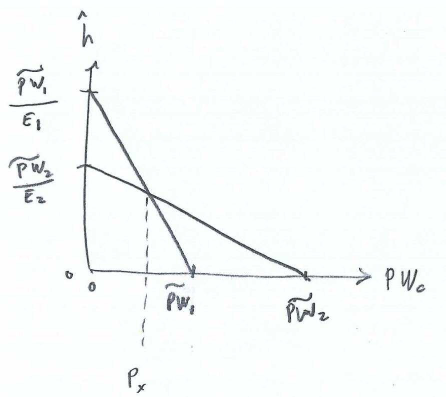

Figure 5: Robustnesses vs critical present worth, problem 12(b).

where we have defined:

$$E_i = \varepsilon_{i,R}\widetilde{R}_i + \varepsilon_{i,C}\widetilde{C}_i, \quad \widetilde{\mathsf{PW}}_i = \mathsf{PW}(\widetilde{R}_i, \widetilde{C}_i) \tag{172}$$

In summary:

$$\text{option 1} \succ \text{option 2} \quad \text{if} \quad \mathsf{PW}_{\mathrm{c}} < \mathsf{PW}_{\times} \tag{173}$$