39. **Present worth, interest, inflation and uncertainty.** (based on exam, 19.7.2016) (p.108).

(a) Consider a project for which the investment at the start of the first year is S, the revenue and cost at the end of each year are constant at R and C, the duration is N years, there is no inflation and the annual discount rate is i. Derive an explicit algebraic expression for the present worth.

(b) Continue part 39a with $S = \$10^4$, $R = \$2,000$, $C = \$1000$ and $i = 0.07$. Find the shortest project duration at which the present worth in non-negative.

(c) Continue part 39a with $R = \$2,000$, $C = \$1000$ and $i = 0.07$. What is the lowest initial investment for which the present worth is **negative** for all project durations?

(d) Continue part 39a and suppose that the discount rate, i, is uncertain with estimate $\widetilde{i}$ and uncertainty weight s_i, both known and positive. The info-gap model is:

$$\mathcal{U}(h) = \left\{ i : \ i \geq 0, \ \left| \frac{i - \widetilde{i}}{s_i} \right| \leq h \right\}, \quad h \geq 0 \tag{29}$$

We require that the present worth be no less than the critical value, PW_c. Derive an explicit algebraic expression for the inverse of the robustness function.

(e) Continue part 39a and suppose that R and C are uncertain as described by this info-gap model:

$$\mathcal{U}(h) = \left\{ R, C : \ \left| \frac{R - \widetilde{R}}{s_R} \right| \leq h, \ \left| \frac{C - \widetilde{C}}{s_C} \right| \leq h \right\}, \quad h \geq 0 \tag{30}$$

where $\widetilde{R}$, $\widetilde{C}$, s_R and s_C are known and positive. We require that the present worth be no less than the critical value, PW_c. Derive an explicit algebraic expression for the robustness function.

(f) Continue part 39a but now introduce constant annual inflation, f. The values R and C are in nominal dollars at the time of the initial investment and i is the nominal annual interest rate. Annual revenue and cost are constant in real dollars. Derive an explicit algebraic expression for the present worth.

(g) Continue part 39f with $R = \$2,000$, $C = \$1000$, $S = \$10,000$ and $f = i$. At what project duration, N, is the present worth equal to zero?

(h) Continue part 39f where i and f are both constant but uncertain with this info-gap model:

$$\mathcal{U}(h) = \left\{ i, f : \ i \geq 0, \ \left| \frac{i - \widetilde{i}}{s_i} \right| \leq h, \ f \geq 0, \ \left| \frac{f - \widetilde{f}}{s_f} \right| \leq h \right\}, \quad h \geq 0 \tag{31}$$

where $\widetilde{i}$, $\widetilde{f}$, s_i and s_f are known and positive. We require that the present worth be no less than the critical value, PW_c. Derive an explicit algebraic expression for the inverse of the robustness function.

(i) Return to part 39a and consider the following dispute. Joe evaluates the project in terms of its present worth, and accepts the project if and only if the present worth is positive. Jane evaluates the project in terms of the benefit-cost ratio, and accepts the project if and only if the BCR exceeds unity. Do they agree or disagree on accepting or rejecting the project? Explain.

(j) Repeat part 39i but now suppose that Jane uses a larger discount rate for the benefits than the value i that both she and Joe use for costs. Joe uses i to discounts benefits as well.

For conditions in which Joe **rejects** the project (using the PW criterion), will Jane **also reject** the project (using the BCR criterion)?

For conditions in which Joe **accepts** the project (using the PW criterion), will Jane **also accept** the project (using the BCR criterion)?

Solution to Problem 39, Present worth, interest, inflation and uncertainty (p.34).

(39a) The present worth is:

$$\text{PW} = -S + \sum_{k=1}^{N}(1+i)^{-k}(R-C) = \boxed{-S + (R-C)\ \underbrace{\frac{1-(1+i)^{-N}}{i}}_{\delta(i)}} \tag{449}$$

(39b) $R > C$, so eq.(449) implies that the present worth increases as the duration, N, increases. This makes sense economically, because longer duration implies more years in which positive net income balances the negative initial investment. Thus, to find the shortest time at which the present worth is non-negative, equate PW to zero and solve for N:

$$\text{PW} = 0 \implies S = (R-C)\delta(i) \implies \frac{1-(1+i)^{-N}}{i} = \frac{S}{R-C} \implies 1 - \frac{Si}{R-C} = (1+i)^{-N} \tag{450}$$

$$\implies N = -\frac{\ln\left(1 - \frac{Si}{R-C}\right)}{\ln(1+i)} = \boxed{17.79} \tag{451}$$

or 18 years if one wants an integer result for the shortest duration with non-negative PW.

(39c) $R > C$, so eq.(449) implies that the present worth increases as the duration, N, increases. Thus the present worth is maximal at infinite duration:

$$\lim_{N\to\infty} \text{PW} = -S + \frac{R-C}{i} \tag{452}$$

Equating this to zero and solving for S yields the lowest initial investment at which the present worth is negative for all finite durations:

$$0 = -S + \frac{R-C}{i} \implies S = \frac{R-C}{i} = \frac{2000-1000}{.07} = \boxed{14,285.71} \tag{453}$$

(39d) The definition of the robustness is:

$$\widehat{h}(\text{PW}_\text{c}) = \max\left\{h:\ \left(\min_{i\in\mathcal{U}(h)} \text{PW}(i)\right) \ge \text{PW}_\text{c}\right\} \tag{454}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function, $\widehat{h}(\text{PW}_\text{c})$. From eq.(449) we see that this minimum occurs when $i = \widetilde{i} + s_i h$. Thus:

$$\boxed{m(h) = -S + (R-C)\delta(\widetilde{i} + s_i h)} \tag{455}$$

(39e) The definition of the robustness is:

$$\widehat{h}(\text{PW}_\text{c}) = \max\left\{h:\ \left(\min_{R,C\in\mathcal{U}(h)} \text{PW}(R,C)\right) \ge \text{PW}_\text{c}\right\} \tag{456}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function, $\widehat{h}(\text{PW}_\text{c})$. From eq.(449) we see that this minimum occurs when R is minimal and C is maximal:

$$R = \widetilde{R} - s_R h, \quad C = \widetilde{C} + s_C h \tag{457}$$

Thus the inner minimum becomes:

$$m(h) = -S + \left(\widetilde{R} - s_R h - \widetilde{C} - s_C h\right)\delta(i) = \text{PW}(\widetilde{R},\widetilde{C}) - (s_R + s_C)\delta(i)h \ge \text{PW}_\text{c} \tag{458}$$

Equating $m(h)$ to PW$_\mathrm{c}$ and solving for h yields the robustness:

$$\boxed{\widehat{h}(\text{PW}_\mathrm{c}) = \frac{\text{PW}(\widetilde{R}, \widetilde{C}) - \text{PW}_\mathrm{c}}{(s_R + s_C)\delta(i)}} \tag{459}$$

or zero if this is negative.

(39f) The nominal net income at the end of year k is $(1+f)^k(R-C)$. Thus the present worth is:

$$\begin{aligned} \text{PW} &= -S + \sum_{k=1}^{N}(1+i)^{-k}(1+f)^k(R-C) &(460)\\ &= -S + (R-C)\sum_{k=1}^{N}\left(\frac{1+f}{1+i}\right)^k &(461)\\ &= \boxed{-S + (R-C)\frac{\left(\frac{1+f}{1+i}\right)^{N+1} - \frac{1+f}{1+i}}{\frac{1+f}{1+i} - 1}} &(462)\end{aligned}$$

(39g) With $f = i$, we see from eq.(461) that:

$$\text{PW} = -S + (R-C)N \implies N = \frac{S}{R-C} = \frac{10^4}{1000} = \boxed{10} \tag{463}$$

(39h) The definition of the robustness is:

$$\widehat{h}(\text{PW}_\mathrm{c}) = \max\left\{h : \left(\min_{i,f\in\mathcal{U}(h)} \text{PW}(i,f)\right) \geq \text{PW}_\mathrm{c}\right\} \tag{464}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function, $\widehat{h}(\text{PW}_\mathrm{c})$. From eq.(460) we see that this minimum occurs when $i = \widetilde{i} + s_i h$ and $f = (\widetilde{f} - s_f f h)^+$. Thus:

$$m(h) = -S + (R-C)\delta\left(\widetilde{i} + s_i h,\, (\widetilde{f} - s_f f h)^+\right) \tag{465}$$

where $\delta(i,f)$ is the fractional expression in eq.(462).

(39i) Let PW$_b$ and PW$_c$ denote the present worth of the benefit and the total cost (including initial investment), respectively. Thus the PW and the BCR of the project are defined as:

$$\begin{aligned} \text{PW} &= \text{PW}_b - \text{PW}_c &(466)\\ \text{BCR} &= \frac{\text{PW}_b}{\text{PW}_c} &(467)\end{aligned}$$

We see that:

$$\text{PW} > 0 \quad \text{if and only if} \quad \text{BCR} > 1 \tag{468}$$

Thus Joe and Jane agree on accepting or rejecting the project.

(39j) Let i_b denote Jane's discount rate for benefits. We know that $i_b > i$, so:

$$\text{PW}_{b,\mathrm{Jane}} = R\sum_{k=1}^{N}(1+i_b)^{-k} \;<\; R\sum_{k=1}^{N}(1+i)^{-k} = \text{PW}_{b,\mathrm{Joe}} \tag{469}$$

Combining this with eqs.(466) and (467) we see that:

$$\text{PW}_{\mathrm{Joe}} < 0 \quad \text{implies} \quad \text{BCR}_{\mathrm{Jane}} < 1 \tag{470}$$

Hence if Joe rejects the project, then so does Jane.

However, we also see that:

$$\text{PW}_{\text{Joe}} > 0 \quad \text{does not imply} \quad \text{BCR}_{\text{Jane}} > 1 \tag{471}$$

Indeed, the left hand condition (causing Joe to accept) can co-exist with the righthand condition in eq.(470) (causing Jane to reject).