40. **Present worth, interest, inflation and uncertainty** (based on exam, 27.9.2016) (p.99)
   (a) Consider an N-year project with constant annual interest i and no inflation. The revenue at the end of year k is $R_k = (1+i)^k R_0$ where R_0 is positive. The cost at the end of year k is C which is positive and constant. The initial investment at the beginning of the first year is S, which is positive.
      i. Derive an explicit algebraic expression for the present worth.
      ii. Given that $S = gC$, $R_0 = C$, $N = 10$ and $i = 0.04$. Find the largest value of g for which the PW is non-negative.
   (b) Consider an N-year project with constant annual interest i and no inflation. The revenue and the cost at the end of each year are R and C which are both positive constants. The initial investment at the beginning of the first year is S, which is positive. Suppose that the present worth equals zero for specific positive values of S, R, C, i and N. Now suppose that i is increased while S and N remain constant. Which of the following must be true In order for the present worth to remain non-negative:
      i. $R - C$ must increase.
      ii. $R - C$ must decrease.
      iii. $R - C$ must remain the same.
      iv. The direction of change in $R - C$ depends on the specific values of S, N and i.
   (c) Consider a project with constant annual interest i and no inflation. The revenue and the cost at the end of each year are R and C which are both positive constants. The initial investment at the beginning of the first year is S, which is positive. The project will run forever, so $N = \infty$. Which of the following statements is true:
      i. The present worth is infinite for any $i > 0$.
      ii. The present worth is infinite only for any $i > 0$ that is also less than some finite value.
      iii. The present worth is finite for any $i > 0$.
      iv. The present worth is finite only for any $i > 0$ that is also less than some finite value.
   (d) Consider an N-year project with constant annual interest i and no inflation. The initial investment at the beginning of the first year is S, which is positive. The revenue and the cost at the end of each year are R and C which are both positive constants, and the revenue is proportional to the cost according to $R = gC$ where g is constant but uncertain according to this info-gap model:

$$\mathcal{U}(h) = \left\{ g : \left| \frac{g - \widetilde{g}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{32}$$

   where $\widetilde{g}$ and w are known positive constants. We require that the present worth be no less than the critical value PW_c. Derive an explicit algebraic expression for the robustness function.
   (e) Repeat problem 40d with the following info-gap model:

$$\mathcal{U}(h) = \left\{ g : \ g \ge 1, \ \left| \frac{g - \widetilde{g}}{w_g} \right| \le h, \ \left| \frac{C - \widetilde{C}}{w_c} \right| \le h \right\}, \quad h \ge 0 \tag{33}$$

   where $\widetilde{g}$, w_g, $\widetilde{C}$ and w_c are known positive constants. Derive an explicit algebraic expression for the inverse of the robustness function.
   (f) Consider constant monthly inflation f. You have a new job and the real value of your monthly salary, at the start of your first month, is R_0. However, the job actually pays

you at the end of each month with a nominal sum whose real value is R_0. What is your nominal salary at the end of the kth month?

(g) Continue part 40f and consider f uncertain according to this info-gap model:

$$\mathcal{U}(h) = \left\{ f : \left| \frac{f - \widetilde{f}}{w} \right| \le h \right\}, \quad h \ge 0 \tag{34}$$

where $\widetilde{f}$ and w are known positive constants. We require that the nominal salary at the end of the kth month be no less than $S_{\rm c}$, which is positive. Derive an explicit algebraic expression for the robustness function.

(h) You will invest $S = \$1000$ at time $t = 0$ in a project that returns constant nominal annual interest $i_{\rm nom} = 0.08$. The constant annual inflation is $f = 0.06$. What is the nominal value of the investment after $N = 12$ years? What is the real value at that time? What is the real interest rate?

(i) You will invest $\$S$ at time $t = 0$ in a project that returns constant nominal annual interest $i_{\rm nom}$. The constant annual inflation is f. However, both $i_{\rm nom}$ and f are uncertain, as expressed by this info-gap model:

$$\mathcal{U}(h) = \left\{ i_{\rm nom}, f : \ i_{\rm nom} \ge 0, \ \left| \frac{i_{\rm nom} - \widetilde{i}_{\rm nom}}{w_i} \right| \le h, \ \left| \frac{f - \widetilde{f}}{w_f} \right| \le h \right\}, \quad h \ge 0 \tag{35}$$

where $\widetilde{i}_{\rm nom}$, w_i, $\widetilde{f}$ and w_f are known positive constants. We require that the real value of the investment at the end of k years be no less than $R_{\rm c}$.

i. Derive an explicit algebraic expression for the inverse of the robustness function.
ii. What is the value of the robustness if $R_{\rm c} = 0$.
iii. What is the value of the robustness if $R_{\rm c} = \left(\dfrac{1 + \widetilde{i}_{\rm nom}}{1 + \widetilde{f}} \right)^k S$?

(j) Consider an N-year project with constant positive revenue R and constant positive cost C at the end of each year. The annual interest rate is i and there is no inflation. The initial investment, at the start of the first year, is S.

i. If $S = 0$, what is the lowest ratio R/C at which the project has a benefit-cost ratio (BCR) no less than one?
ii. If $C = 0$ and $R > 0$, what is the lowest value of S at which the BCR is less than one for all values of N?

Solution to Problem 40, Present worth, interest, inflation and uncertainty (p.33).

(40(a)i) The present worth is:

$$\begin{aligned} \text{PW} &= -S + \sum_{k=1}^{N} (1+i)^{-k} (R_k - C_k) && (455) \\ &= -S + R_0 \sum_{k=1}^{N} (1+i)^{-k} (1+i)^k - C \sum_{k=1}^{N} (1+i)^{-k} && (456) \\ &= \boxed{-S + R_0 N - \underbrace{\frac{1-(1+i)^{-N}}{i}}_{\delta(i)} C} && (457) \end{aligned}$$

(40(a)ii) From eq.(457) we find:

$$\text{PW} = -gC + NC - \delta(i)C \geq 0 \quad \Longrightarrow \quad g \leq N - \delta(i) = 10 - \frac{1-1.04^{-10}}{0.04} = \boxed{1.88910} \tag{458}$$

(40b) From eq.(455), the present worth is:

$$\text{PW} = -S + \delta(i)(R - C) \tag{459}$$

where $\delta(i)$ is defined in eq.(457). The parameters for which the PW is zero are all positive, which implies that $R - C > 0$. From the geometric series that defines $\delta(i)$, e.g. the righthand sum in eq.(456), we conclude that:

$$\frac{\partial \delta(i)}{\partial i} < 0 \tag{460}$$

Hence, from eq.(459) we conclude that:

$$\frac{\partial \text{PW}}{\partial i} < 0 \tag{461}$$

Finally, we conclude that, if i increases from a constellation of parameters at which PW $= 0$, we see that $R - C$ must increase in order for the PW to remain positive.

Thus statement 40(b)i is the only true statement.

(40c) The present worth is:

$$\text{PW} = -S + \delta(i)(R - C) \tag{462}$$

We note that:

$$\lim_{N\to\infty} \delta(i) = \frac{1}{i} \tag{463}$$

Hence the PW is:

$$\text{PW} = -S + \frac{R - C}{i} \tag{464}$$

This is finite for any positive i. Thus statement 40(c)iii is the only true statement.

(40d) The present worth is:

$$\text{PW} = -S + \delta(i)(gC - C) = -S + \delta(i)(g-1)C \tag{465}$$

The robustness is defined as:

$$\widehat{h}(\text{PW}_\text{c}) = \max \left\{ h : \left(\min_{g \in \mathcal{U}(h)} \text{PW} \right) \geq \text{PW}_\text{c} \right\} \tag{466}$$

Let $m(h)$ denote the inner minimum, which occurs for $g = \widetilde{g} - wh$, so:

$$m(h) = -S + \delta(i) \left[(\widetilde{g} - wh)C - C \right] \geq \text{PW}_\text{c} \tag{467}$$

Hence:

$$\boxed{\widehat{h}(\mathsf{PW}_{\rm c}) = \frac{-S + \delta(i)(\widetilde{g}C - C) - \mathsf{PW}_{\rm c}}{\delta(i) w C} = \frac{\mathsf{PW}(\widetilde{g}) - \mathsf{PW}_{\rm c}}{\delta(i) w C}} \tag{468}$$

or zero if this is negative.

(40e) The present worth is specified in eq.(465). The robustness is defined as:

$$\widehat{h}(\mathsf{PW}_{\rm c}) = \max\left\{h : \left(\min_{g,C\in\mathcal{U}(h)} \mathsf{PW}\right) \ge \mathsf{PW}_{\rm c}\right\} \tag{469}$$

Let $m(h)$ denote the inner minimum. The info-gap model requires that $g \ge 1$, so define $x^\star$ to equal x if $x \ge 1$ and to equal 1 otherwise. Thus $m(h)$ occurs for:

$$g = (\widetilde{g} - w_g h)^\star, \quad C = \widetilde{C} - w_c h \tag{470}$$

Thus the inverse of the robustness function is:

$$\boxed{m(h) = -S + \delta(i)\left[(\widetilde{g} - w_g h)^\star - 1\right](\widetilde{C} - w_c h)} \tag{471}$$

(40f) Your nominal salary at the end of the kth month is:

$$\boxed{S_k = (1+f)^k R_0} \tag{472}$$

(40g) The robustness is defined as:

$$\widehat{h}(\mathsf{PW}_{\rm c}) = \max\left\{h : \left(\min_{f\in\mathcal{U}(h)} S_k\right) \ge S_{\rm c}\right\} \tag{473}$$

Let $m(h)$ denote the inner minimum which occurs for $f = \widetilde{f} - wf$:

$$m(h) = (1 + \widetilde{f} - wh)^k R_0 \ge S_{\rm c} \implies \boxed{\widehat{h}(S_{\rm c}) = \frac{1}{w}\left[1 + \widetilde{f} - \left(\frac{S_{\rm c}}{R_0}\right)^{1/k}\right]} \tag{474}$$

or zero if this is negative.

(40h) The nominal value at the end of k years is:

$$A_k = (1 + i_{\rm nom})^k S \implies \boxed{A_{12} = 1.08^{12} \times 1000 = \$2,518.17} \tag{475}$$

The real value at the end of k years is:

$$R_k = (1+f)^{-k} A_k = (1+f)^{-k}(1+i_{\rm nom})^k S \implies \boxed{R_{12} = \left(\frac{1.08}{1.06}\right)^{12} \times 1000 = \$1,251.45} \tag{476}$$

The real interest rate is defined by:

$$R_k = (1 + i_{\rm r})^k S \tag{477}$$

Comparing this with the left hand part of eq.(476) we see that the real interest rate, $i_{\rm r}$, is related to the inflation, f, and the nominal interest rate, $i_{\rm nom}$, by:

$$(1+i_{\rm r})^k = (1+f)^{-k}(1+i_{\rm nom})^k \implies (1+i_{\rm r})^{-k}(1+f)^{-k} = (1+i_{\rm nom})^{-k} \implies (1+i_{\rm r})(1+f) = 1+i_{\rm nom} \tag{478}$$

Hence:

$$i_{\rm r} = \frac{1+i_{\rm nom}}{1+f} - 1 = \frac{i_{\rm nom}-f}{1+f} \quad \Longrightarrow \quad \boxed{i_{\rm r} = \frac{0.02}{1.06} = 0.018867} \tag{479}$$

(40(i)i) The real value of the investment at the end of k years is, from eq.(476):

$$R_k(i_{\rm nom}, f) = \left(\frac{1+i_{\rm nom}}{1+f}\right)^k S \tag{480}$$

The robustness is defined as:

$$\widehat{h}(R_{\rm c}) = \max\left\{h:\ \left(\min_{i_{\rm nom}, f \in \mathcal{U}(h)} R_k\right) \ge R_{\rm c}\right\} \tag{481}$$

Let $m(h)$ denote the inner minimum, which occurs for $i_{\rm nom} = (\widetilde{i}_{\rm nom} - w_i h)^+$ and $f = \widetilde{f} + w_f h$. Thus the inverse of the robustness function is:

$$\boxed{m(h) = \left(\frac{1+(\widetilde{i}_{\rm nom} - w_i h)^+}{1+\widetilde{f}+w_f h}\right)^k S} \tag{482}$$

(40(i)ii) From eq.(482) we see that $m(h)$ is positive for all finite positive h. Also:

$$\lim_{h\to\infty} m(h) = 0 \tag{483}$$

$\boxed{\text{Thus } m(h) = R_{\rm c} = 0 \text{ implies that } \widehat{h}(0) = \infty.}$

(40(i)iii) From eq.(482) we see that $m(h) = R_{\rm c} = \left(\frac{1+\widetilde{i}_{\rm nom}}{1+\widetilde{f}}\right)^k S$ if $h = 0$. $\boxed{\text{Thus } \widehat{h}(R_{\rm c}) = 0}$ for this value of $R_{\rm c}$.

(40(j)i) The BCR is:

$$\text{BCR} = \frac{\text{PW}_b}{\text{PW}_c} \quad \text{where} \quad \text{PW}_b = \sum_{k=1}^{N}(1+i)^{-k}R = \delta(i)R, \quad \text{PW}_c = S + \sum_{k=1}^{N}(1+i)^{-k}C = S + \delta(i)C \tag{484}$$

$S = 0$ implies that BCR $= R/C$. Thus:

$$\boxed{\text{BCR} \ge 1 \quad \text{if and only if} \quad \frac{R}{C} \ge 1} \tag{485}$$

(40(j)ii) $C = 0$ and eq.(484) imply:

$$\text{BCR} = \frac{\delta(i)R}{S} \tag{486}$$

$\delta(i)$ is:

$$\delta(i) = \sum_{k=1}^{N}(1+i)^{-k} = \frac{1-(1+i)^{-N}}{i} \tag{487}$$

From these relations we see that:

$$\frac{\partial \delta(i)}{\partial N} > 0 \quad \text{and} \quad \lim_{N\to\infty} \delta(i) = \frac{1}{i} \tag{488}$$

Hence:

$$\max_N \text{BCR} = \frac{R}{iS} \tag{489}$$

We require that $\frac{R}{iS} < 1$, so we can assert:

$$\boxed{\text{BCR} < 1 \quad \text{for all } N, \quad \text{if and only if} \quad S > \frac{R}{i}} \tag{490}$$