17. **Inspection system with uncertain diminishing inspection rate.** (p.78) Consider a new inspection system with initial cost S and end-of-year maintenance cost of C each year, financed at an interest rate of 4%. The system is expected to last N years. In the first year of operation the system performs B inspections, but thereafter the number of yearly inspections decreases by 6% each year. Treat this rate of decrease as a discount rate.
(a) Calculate the benefit-cost ratio. Evaluate it numerically for $S = \$20,000$, $C = \$1,500$, $B = 350,000$ and $N = 10$.
(b) The percentage yearly reduction of the inspection rate is estimated at $i_b = 0.06$. While i_b will be constant over time, its value may err by several tens of percent (plus or minus several times 0.006) or more. Evaluate the robustness of the BCR. Use the robustness to determine a reliable value of BCR, inspections/$.

27. **BCR of a measurement device.** (p.93) Your company is considering the purchase of an expensive measurement device. You will have to take a loan for N years at interest rate i_c per year to cover the cost, with interest and principal payments I_k and P_k, respectively, at the end of each year. The benefit of the device will be assessed by the number, B, of measurements each year with the device. B will be constant over time, but future benefit is discounted at the rate i_b because other competitive technologies will become available.

    (a) Derive an explicit algebraic expression for the benefit-cost ratio (BCR) of the device, recalling that B is constant over time.

    (b) B is constant over time, but its value is a random variable. Assume that B is exponentially distributed with probability density function $p(B) = \lambda \mathrm{e}^{-\lambda B}$, $B \geq 0$. You require that the BCR be no less than $\mathrm{BCR}_{\mathrm{c}}$. Derive an explicit algebraic expression for the probability of violating this requirement.

    (c) The value of the exponential coefficient is estimated as $\widetilde{\lambda}$, but the fractional error of this estimate is unknown (though λ must be positive). You require that the probability of violating the BCR requirement be less than P_{c}. Derive an explicit algebraic expression for the robustness to uncertainty in λ.

Solution to Problem 17, Inspection system with uncertain diminishing inspection rate (p.15).

(a) Present worth of the benefits of are:

$$B_{pw} = \sum_{n=1}^{N}(1+i_b)^{-n}B \quad (229)$$

$$= \frac{1-(1+i_b)^{-N}}{i_b}B \quad (230)$$

$$= \delta_f(i_b)B \quad (231)$$

Present worth of the initial investment and maintenance costs are:

$$C_{pw} = S + \sum_{n=1}^{N}(1+i_c)^{-n}C \quad (232)$$

$$= S + \frac{1-(1+i_c)^{-N}}{i_c}C \quad (233)$$

$$= S + \delta_f(i_c)C \quad (234)$$

The BCR is:

$$\text{BCR} = \frac{B_{pw}}{C_{pw}} \quad (235)$$

$$= \frac{\delta_f(i_b)B}{S+\delta_f(i_c)C} \quad (236)$$

The discount functions are $\delta_f(i_b) = 7.3601$ and $\delta_f(i_c) = 8.1109$. The BCR $= 80.0847$ inspections per dollar, or $1/80.0847 = \$0.0125/$inspection.

(b) The info-gap model for uncertain reduction in inspection rate, i_b, is:

$$\mathcal{U}(h) = \left\{ i_b : \; i_b > -1, \; \left| \frac{i_b - \widetilde{i}_b}{s_b} \right| \le h \right\}, \quad h \ge 0 \quad (237)$$

$\widetilde{i}_b = 0.06$ and $s_b = 0.3\widetilde{i}_b$.

The robustness is:

$$\widehat{h}(\text{BCR}_{\text{c}}, S) = \max \left\{ h : \; \left(\min_{i_b \in \mathcal{U}(h)} \text{BCR}(i_b, S) \right) \ge \text{BCR}_{\text{c}} \right\} \quad (238)$$

Eq.(229) is correct because i_b is constant over time. Hence the inner minimum occurs at $i_b = \widetilde{i}_b + s_b h$, so, from eq.(236):

$$m(h) = \frac{\delta_f(\widetilde{i}_b + s_b h)B}{S+\delta_f(i_c)C} \quad (239)$$

Robustness curves are shown in figs. 6 and 7 for initial investment of $S =$\$20,000. The nominal performance, BCR$_{\text{c}} = 80$ inspection/\$ or 1/BCR$_{\text{c}} = 0.0125$ \$/inspection, has no robustness (the zeroing property). Fig. 6 shows that $\widehat{h}$ equals 1 or 2 at BCR$_{\text{c}} = 73.5$ or 68.0 [inspections/\$] respectively: more robustness entails lower BCR (the trade off property).

What is a reliable number of inspections per \$? The error weight s_h in the info-gap model of eq.(237) is 30% of the nominal discount factor $\widetilde{i}_b$. From fig. 6 we see that $\widehat{h}(\text{BCR}_{\text{c}} = 73.5) = 1$ which means that the BCR will be no less than 73.5 if the rate of decrease of inspections/year is no greater than $\widetilde{i}_b + 1 \times s_b = 1.3\widetilde{i}_b$. Our information is that "The percentage yearly reduction of the inspection rate is estimated at $i_b = 0.06$, but this may err by several tens of percent (plus or minus several times

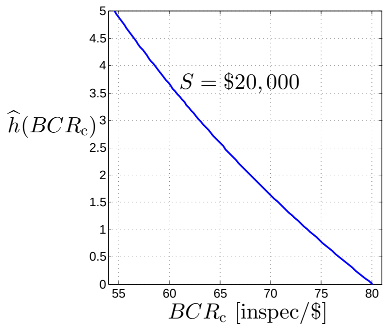

Figure 6: Robustness vs BCR$_c$, problem 17(b), eq.(239).

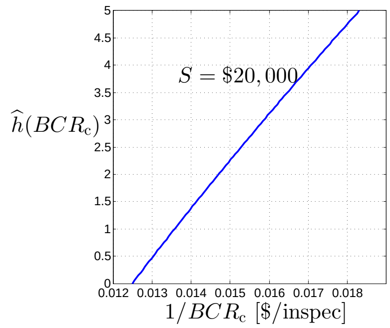

Figure 7: Robustness vs 1/BCR$_c$, problem 17(b), eq.(239).

0.006) or more." Thus a robustness of $\widehat{h} = 1$, meaning immunity again 30% error in $\widetilde{i}_b$, is moderately good. Robustness of $\widehat{h} = 2$, meaning immunity again 60% error in $\widetilde{i}_b$, would be quite good. From fig. 6 we see that $\widehat{h}(\text{BCR}_c = 68.0) = 2$ which means that the BCR will be no less than 68.0 if the rate of decrease of inspections/year is no greater than $\widetilde{i}_b + 2 \times s_b = 1.6\widetilde{i}_b$.

In summary, BCR $= 73.5$ is moderately reliable; BCR $= 68.0$ is quite reliable.

Solution to Problem 27, BCR of a measurement device (p.25).

(27a) The BCR is defined as the ratio of discounted benefit to discounted cost:

$$\text{BCR} = \frac{\sum_{k=1}^{N}(1+i_b)^{-k}B}{\sum_{k=1}^{N}(1+i_c)^{-k}(I_k+P_k)} = QB \tag{335}$$

which defines the quantity Q.

(27b) The probability of violating the BCR requirement is:

$$\begin{aligned} P_{\rm f} &= \text{Prob}(\text{BCR} \le \text{BCR}_{\rm c}) && (336)\\ &= \text{Prob}\left(B \le \frac{\text{BCR}_{\rm c}}{Q}\right) && (337)\\ &= \int_0^{\text{BCR}_c/Q} \lambda \text{e}^{-\lambda B}\,{\rm d}B && (338)\\ &= 1-\text{e}^{-\lambda\,\text{BCR}_c/Q} && (339)\end{aligned}$$

(27c) The info-gap model of uncertainty in λ is:

$$\mathcal{U}(h) = \left\{\lambda:\ \lambda>0,\ \left|\frac{\lambda-\widetilde{\lambda}}{\widetilde{\lambda}}\right| \le h\right\},\quad h\ge 0 \tag{340}$$

The robustness is defined as:

$$\widehat{h} = \max\left\{h:\ \left(\max_{\lambda\in\mathcal{U}(h)} P_{\rm f}(\lambda)\right) \le P_{\rm c}\right\} \tag{341}$$

The inner minimum, $m(h)$, which is the inverse of the robustness function, occurs when λ is maximal at horizon of uncertainty h: $\lambda = (1+h)\widetilde{\lambda}$. Thus:

$$m(h) = 1-\text{e}^{-(1+h)\widetilde{\lambda}\,\text{BCR}_c/Q} \le P_{\rm c} \tag{342}$$

Thus the robustness is:

$$\widehat{h} = -1 - \frac{Q\ln(1-P_{\rm c})}{\widetilde{\lambda}\text{BCR}_{\rm c}} \tag{343}$$

or zero if this is negative.