**Solution to Problem 1, Compound interest,** (p.3). The basic relation is:

$$F = (1+i)^N P \tag{109}$$

In our case, $i = 0.12$, $N = 6$ and $P = 1,500$. Thus $F = \$2,960.73$.

**Solution to Problem 2, Compound interest,** (p.3). See table 4, which explains that:

$200 interest is paid each year for years 1–4.
$100 interest is paid each year for years 5–8.
$1,200 is the total interest paid.

| Year | Amount owed at beginning of year | Interest accrued for year | Payment at end of year |
|---|---|---|---|
| 1 | 2,000 | 200 | 200 |
| 2 | 2,000 | 200 | 200 |
| 3 | 2,000 | 200 | 200 |
| 4 | 2,000 | 200 | 1,200 |
| 5 | 1,000 | 100 | 100 |
| 6 | 1,000 | 100 | 100 |
| 7 | 1,000 | 100 | 100 |
| 8 | 1,000 | 100 | 1,100 |
| Total: | | 1,200 | 3,200 |

Table 4: Solution to problem 2.

**Solution to Problem 3, Compound interest,** (p.3). We can immediately obtain the answer from the following relation:

$$F = (1+i)^N P = 1.1^8 \times 2000 = 4287.17 \tag{110}$$

However, it is interesting to compare the details of the result, in comparison to table 4 from problem 2. See table 5.

• 2nd column (amount owed at beginning of each year):
Compound interest on principal: row $n$ = $1.1^{n-1}\times$ row 1.

• 3rd column (interest accrued for year):
3rd column = 0.1 $\times$ 2nd column.

• Thus the total interest paid is $2,287.17, which is much greater than in problem 2 because of (1) compounding (2) deferred repayment of all principal to year 8.

| Year | Amount owed at beginning of year | Interest accrued for year | Payment at end of year |
|---|---|---|---|
| 1 | 2,000 | 200 | 0 |
| 2 | 2,200 | 220 | 0 |
| 3 | 2,420 | 242 | 0 |
| 4 | 2,662 | 266.20 | 0 |
| 5 | 2,928.20 | 292.82 | 0 |
| 6 | 3,221.02 | 322.10 | 0 |
| 7 | 3,543.12 | 354.31 | 0 |
| 8 | 3,897.43 | 389.74 | 4,287.17 |
| Total: | | 2,287.17 | 4,287.17 |

Table 5: Solution to problem 3.

**Solution to Problem 4, Equivalent annual payment,** (p.3). The basic relation is:

$$A = \frac{i(1+i)^N}{(1+i)^N - 1} P \tag{111}$$

In our case: $i = 0.1$, $N = 5$, $P = 20,000$. Thus:

$$A = 0.263797 \times 20,000 = \$5,275.95 \tag{112}$$

**Solution to Problem 5, Compound interest,** (p.3). See table 6.

- 2nd column (amount owed at beginning of each year):
  Remaining principal minus last year's payment of $4,000.
- 3rd column (interest accrued for year):
  3rd column = 0.1 $\times$ 2nd column.
- Thus the total interest paid is $6,000.
- The total payment in problem 4 is $5 \times 5,275.95 = \$26,379.75$. Thus the interest paid in problem 4 is $6,379.75.
- The interest paid in problem 4 is greater than in problem 5 because of repayment of principal during the loan in problem 5.

| Year | Amount owed at beginning of year | Interest accrued for year | Payment at end of year |
|---|---|---|---|
| 1 | 20,000 | 2,000 | 6,000 |
| 2 | 16,000 | 1,600 | 5,600 |
| 3 | 12,000 | 1,200 | 5,200 |
| 4 | 8,000 | 800 | 4,800 |
| 5 | 4,000 | 400 | 4,400 |
| Total: | | 6,000 | 26,000 |

Table 6: Solution to problem 5.

**Solution to Problem 6, Investment and return,** (p.4). We will evaluate the present worth (PW) of the new system. The basic relation between the PW and the net revenue at the end of the periods, $F_k$, is:

$$\begin{aligned} PW &= (1+i)^0 F_0 + (1+i)^{-1} F_1 + \cdots + (1+i)^{-k} F_k + \cdots + (1+i)^{-N} F_N && (113) \\ &= \sum_{k=0}^{N} (1+i)^{-k} F_k && (114) \end{aligned}$$

In our problem:

- $S =$ Initial cost of the project = \$640,000.
- $R_k =$ revenue at the end of $k$th period = \$180,000.
- $C_k =$ operating cost at the end of $k$th period.
  = \$44,000 for $k = 1, 2$.
  = $\$44,000 - 4,000(k-2)$ for $k = 3, \ldots, 8$.
- $F_0 = -S$.
- $F_k = R_k - C_k$, $k = 1, \ldots, N$.
- $N =$ number of periods = 8.
- $M =$ re-sale value of equipment at end of project = 0.
- MARR = 15%, so $i = 0.15$.

Adapting eq.(114), the PW is:

$$\begin{aligned} PW &= -S + \sum_{k=1}^{N} (1+i)^{-k} R_k - \sum_{k=1}^{N} (1+i)^{-k} C_k + (1+i)^{-N} M && (115) \\ &= -640,000 + \underbrace{\underbrace{\frac{1-(1+i)^{-N}}{i}}_{4.487322} \times 180,000}_{807{,}717.87} \\ &\quad \underbrace{-\underbrace{\left( \underbrace{\frac{1}{1+i}}_{0.869565} + \underbrace{\frac{1}{(1+i)^2}}_{0.756144} \right)}_{1.625709} 44,000}_{-71{,}531.20} \underbrace{- \sum_{k=3}^{8} \frac{1}{(1+i)^k} (44,000 - 4,000(k-2))}_{-90{,}459.12} + 0 && (116) \\ &= 5,727.55 && (117) \end{aligned}$$

We can also calculate the equivalent future worth (FW) from the relation:

$$\text{FW} = \underbrace{(1+i)^N}_{3.059023} \text{PW} = \$17,520.71 \qquad (118)$$

At the MARR of 15%, this project seems worthwhile.

**Solution to Problem 7, Size of a grant,** (p.5). We will evaluate the present worth (PW) of the cost of the project. That is the size of loan we need at the start of the project.

The basic relation between the PW and the net revenue at the end of the periods, $F_k$, is:

$$PW = (1+i)^{-1}F_1 + \cdots + (1+i)^{-k}F_k + \cdots + (1+i)^{-N}F_N \quad (119)$$

$$= \sum_{k=1}^{N}(1+i)^{-k}F_k \quad (120)$$

In our problem:

- $N =$ number of periods = 10.
- $C_k =$ operating cost at the end of $k$th period = \$15,000 = $C$. $k = 1, \ldots, N$.
- $M =$ re-sale value of equipment at end of project = \$3,000.
- $F_k = -C_k$, $k = 1, \ldots, N-1$.
- $F_N = -C_N + M$.
- $i = 0.06$.

Eq.(120), the PW of the cost, is:

$$PW = -\sum_{k=1}^{N}(1+i)^{-k}C_k + (1+i)^{-N}M \quad (121)$$

$$= -\frac{1-(1+i)^{-N}}{i}C + (1+i)^{-N}M \quad (122)$$

$$= -7.360087C + 0.558395M \quad (123)$$

$$= -110,401.31 + 1,675.19 \quad (124)$$

$$= -108,726.12 \quad (125)$$

The smallest loan that will cover the costs is $S = \$108,726.12$. The cost terms are shown in table 7.

| $k$ | $(1+i)^{-k}C$ |
|---|---|
| 1 | 14,150.94 |
| 2 | 13,349.94 |
| 3 | 12,594.28 |
| 4 | 11,881.40 |
| 5 | 11,208.87 |
| 6 | 10,574.40 |
| 7 | 9,975.85 |
| 8 | 9,411.18 |
| 9 | 8,878.47 |
| 10 | 8,375.92 |
| Total | 110,401.30 |

Table 7: Solution to problem 7.

The PW of the cost payments, \$110,401.30, is less than the actual cost payments, $10\times$\$15,000 = \$150,000. Thus the loan itself is much less than the actual cost payments. The explanation is that the un-used funds from the loan are invested at 6% until they are needed. This is accounted for by calculating the PW of the cost payments based on 6% interest. In addition there is the end-of-life salvage value.

**Solution to Problem 8, Present worth of a loan,** (p.6). The basic relation between the PW and the net annual payments at the end of the periods, $A_k$, is:

$$\begin{aligned}\text{PW} &= (1+i)^{-1}A_1 + \cdots + (1+i)^{-k}A_k + \cdots + (1+i)^{-N}A_N \qquad (126)\\ &= \sum_{k=1}^{N}(1+i)^{-k}A_k \qquad (127)\end{aligned}$$

In our problem:

- $N =$ number of periods = 10.
- $A_k =$ annual repayment of loan, $P$, at end of $k$th period, $k = 1, \ldots, N$, containing two parts:
  - $\frac{P}{N} =$ repayment of principal.
  - $\left(P - (k-1)\frac{P}{N}\right) i =$ payment of last year's interest on remaining loan.

**(a)** Adapting eq.(127), the PW is:

$$\begin{aligned}\text{PW} &= \sum_{k=1}^{N}(1+i)^{-k}A_k \qquad (128)\\ &= \sum_{k=1}^{N}(1+i)^{-k}\left[\frac{P}{N} + \left(P - (k-1)\frac{P}{N}\right) i\right] \qquad (129)\\ &= \underbrace{P\sum_{k=1}^{N}(1+i)^{-k}\overbrace{\left[\frac{1}{N} + \left(1 - (k-1)\frac{1}{N}\right) i\right]}^{q_k}}_{q=1} \qquad (130)\\ &= P \qquad (131)\end{aligned}$$

The terms in eq.(130) for part (a) are shown in table 8.

| $k$ | $(1+i)^{-k}$ | $q_k$ |
|---|---|---|
| 1 | 0.9434 | 0.1600 |
| 2 | 0.8900 | 0.1540 |
| 3 | 0.8396 | 0.1480 |
| 4 | 0.7921 | 0.1420 |
| 5 | 0.7473 | 0.1360 |
| 6 | 0.7050 | 0.1300 |
| 7 | 0.6651 | 0.1240 |
| 8 | 0.6274 | 0.1180 |
| 9 | 0.5919 | 0.1120 |
| 10 | 0.5584 | 0.1060 |

Table 8: Solution to problem 8(a).

**(b)** One could use an identity for arithmetic-geometric progressions (Gradshteyn and Ryzhik, p.1, item 0.113).

**Solution to Problem 9, Choose quality and lifetime,** (p.7). We will evaluate the present worth (PW) of each design concept. The basic relation between the PW and the net revenue at the end of the periods, $F_k$, is:

$$PW = (1+i)^0 F_0 + (1+i)^{-1} F_1 + \cdots + (1+i)^{-k} F_k + \cdots + (1+i)^{-N} F_N \quad (132)$$

$$= \sum_{k=0}^{N} (1+i)^{-k} F_k \quad (133)$$

Notation for design concepts $m = 1, 2$:

- $N_m =$ number of periods during which this design is operational. $N_1 = 2N_2 = 2N$.
- $S_m =$ initial cost of implementation.
- $R_m =$ net revenue at the end of each year of implementation.
- MARR is 15% so $i = 0.15$.

**Design concept 1.** Adapting eq.(133), the PW of design concept 1 is:

$$\text{PW}_1 = -S_1 + \sum_{k=1}^{N_1} (1+i)^{-k} R_1 \quad (134)$$

$$= \boxed{-S_1 + \frac{1-(1+i)^{-N_1}}{i} R_1} \quad (135)$$

**Design concept 2.** The PW of the first $N_2$ years of design 2 is analogous to design 1:

$$\text{PW}_{2,1} = -S_2 + \sum_{k=1}^{N_2} (1+i)^{-k} R_2 \quad (136)$$

$$= -S_2 + \frac{1-(1+i)^{-N_2}}{i} R_2 \quad (137)$$

The PW of the second $N_2$ years of design 2 is just like the first $N_2$ years, but starting at the end of year $N_2$. The PW of the second $N_2$ years is thus:

$$\text{PW}_{2,2} = (1+i)^{-N_2} PW_{2,1} \quad (138)$$

Combining eqs.(137) and (138) we find the PW of design 2:

$$\text{PW}_2 = PW_{2,1} + (1+i)^{-N_2} PW_{2,1} \quad (139)$$

$$= \left(1 + (1+i)^{-N_2}\right) \left(-S_2 + \frac{1-(1+i)^{-N_2}}{i} R_2\right) \quad (140)$$

$$= -\left(1 + (1+i)^{-N_2}\right) S_2 \; + \; \frac{1-(1+i)^{-2N_2}}{i} R_2 \quad (141)$$

$$= \boxed{-\left(1 + (1+i)^{-N_2}\right) S_2 \; + \; \tfrac{1-(1+i)^{-N_1}}{i} R_2} \quad (142)$$

where eq.(142) results from eq.(141) because $N_1 = 2N_2$.

Note that $\text{PW}_2$ can be derived differently by noting that revenue $R_2$ is obtained at the end of each year during $2N_2$ years, and that the investment $S_2$ is made initially and again at the end of year $N_2$. Thus:

$$\text{PW}_2 = -S_2 - (1+i)^{-N_2} S_2 + \sum_{k=1}^{2N_2} (1+i)^{-k} R_2 \quad (143)$$

$$= -\left(1 + (1+i)^{-N_2}\right) S_2 + \frac{1-(1+i)^{-2N_2}}{i} R_2 \quad (144)$$

which is precisely eq.(141).

To choose between the design concepts, we must compare eqs.(135) and (142). Note:

• The coefficients of $R_1$ and $R_2$ are the same, while $R_2 < R_1$ because the design 2 has lower projected earnings. This mitigates in favor of design 1.

• The coefficient of $S_2$ is more negative than the coefficient of $S_1$ (which is $-1$), but $0 < S_2 < S_1$. It is not clear which design is preferred.

Let's define the following two discount factors:

$$D_1 = \frac{1-(1+i)^{-N_1}}{i} \quad (145)$$

$$D_2 = 1+(1+i)^{-N_2} \quad (146)$$

The PW's of the two designs are:

$$\text{PW}_1 = -S_1 + D_1 R_1 \quad (147)$$

$$\text{PW}_2 = -D_2 S_2 + D_1 R_2 \quad (148)$$

We prefer design 1 if and only if:

$$\text{PW}_1 > \text{PW}_2 \quad (149)$$

$$-S_1 + D_1 R_1 > -D_2 S_2 + D_1 R_2 \quad (150)$$

$$D_1(R_1 - R_2) > S_1 - D_2 S_2 \quad (151)$$

$$D_1 \frac{R_1 - R_2}{S_1} > 1 - D_2 \frac{S_2}{S_1} \quad (152)$$

$$\frac{R_1 - R_2}{S_1} > \frac{1}{D_1} - \frac{D_2}{D_1}\frac{S_2}{S_1} \quad (153)$$

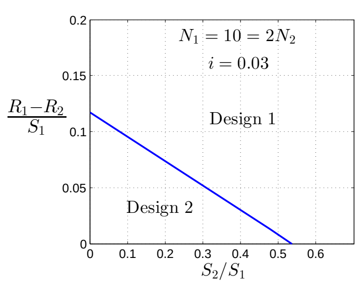


Figure 1: Eq.(153) in problem 9, $i = 0.03$. The curve is a plot of $\frac{1}{D_1} - \frac{D_2 S_2}{D_1 S_1}$ vs. $S_2/S_1$. $D_1 = 8.5302$, $D_2 = 1.8626$.

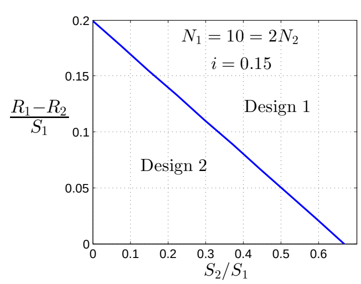


Figure 2: Eq.(153) in problem 9, $i = 0.15$. The curve is a plot of $\frac{1}{D_1} - \frac{D_2 S_2}{D_1 S_1}$ vs. $S_2/S_1$.

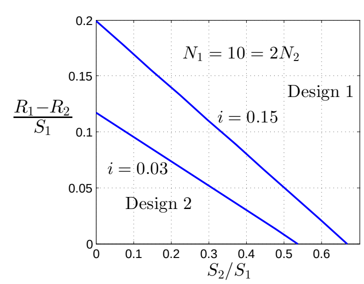


Figure 3: Eq.(153) in problem 9, $i = 0.03$ and $i = 0.015$. The curves are plots of $\frac{1}{D_1} - \frac{D_2 S_2}{D_1 S_1}$ vs. $S_2/S_1$.

Figs. 1 and 2 show plots of the righthand side of eq.(153) vs $S_2/S_1$, for two different choices of the interest rate. For each curve, points above the line represent parameter values for which eq.(149) holds and thus design 1 is preferred. Points below the line represent parameter values for which design 2 is preferred. We can understand these curves as follows.

• The ratio $\frac{R_1-R_2}{S_1}$ is positive, and expresses the relative advantage of design 1 over design 2, in terms of greater revenue per year for design 1. At fixed $S_2/S_1$, design 1 is preferred as $\frac{R_1-R_2}{S_1}$ increases.

• $S_2/S_1$ is less than one, and is the ratio of initial capital cost of design 2 compared to design 1. At fixed $\frac{R_1-R_2}{S_1}$, design 1 is preferred as $S_2/S_1$ increases.

• We see in figs. 1 and 2 that, as $\frac{R_1-R_2}{S_1}$ increases, the preference for design 2 requires a compensatory decrease in $S_2/S_1$. This is because a decrease in revenue from design 2 requires compensation by a decrease in the capital cost of design 2, in order for design 2 to remain preferred over design 1.

| $i$ | $1/D_1$ | $1/D_2$ | $-D_2/D_1$ |
|---|---|---|---|
| 0.0300 | 0.1172 | 0.5369 | 0.2184 |
| 0.0600 | 0.1359 | 0.5723 | 0.2374 |
| 0.0900 | 0.1558 | 0.6061 | 0.2571 |
| 0.1200 | 0.1770 | 0.6380 | 0.2774 |
| 0.1500 | 0.1993 | 0.6679 | 0.2983 |
| 0.1800 | 0.2225 | 0.6958 | 0.3198 |

Table 9: Solution to problem 9.

The vertical and horizontal intercepts of the curves in fig. 1 are $1/D_1$ and $1/D_2$, respectively. The slope is $-D_2/D_1$. Values are shown in table 9. The table and fig. 3 both show that the range of parameter values for which design 2 is preferred increases as the interest rate increases. This is logical because design 2 defers a substantial capital investment to the end of year 5.

**Solution to Problem 10, Uncertain simple interest,** (p.8).

**(a)** The robustness is defined as:

$$\widehat{h}(F_{\rm c}) = \max \left\{ h : \ \left( \max_{i \in \mathcal{U}(h)} F(i) \right) \le F_{\rm c} \right\} \tag{154}$$

Let $m(h)$ denote the inner maximum, which occurs when $i = \widetilde{i} + s_i h$:

$$m(h) = \left[ 1 + (\widetilde{i} + s_i h) N \right] P \tag{155}$$

Equate this to $F_{\rm c}$ and solve for $h$:

$$\left[ 1 + (\widetilde{i} + s_i h) N \right] P = F_{\rm c} \quad \Longrightarrow \quad \widehat{h}(F_{\rm c}) = \frac{F_{\rm c} - (1 + \widetilde{i} N) P}{s_i N P} \tag{156}$$

or zero if this is negative.

**(b)** We require "several" units of robustness, for instance $\widehat{h} = 3$. Solving eq.(156) for $F_{\rm c}$:

$$F_{\rm c} = s_i N P \widehat{h} + (1 + \widetilde{i} N) P = 0.02 \times 5 \times 10,000 \times 3 + (1 + 0.06 \times 5) \times 10,000 = 16,000 \tag{157}$$

The estimated interest rate, $\widetilde{i}$, can err by 3 times the estimated error, $s_i$, and the repayment will not exceed $16,000.

The nominal repayment is:

$$F_{\rm c} = (1 + \widetilde{i} N) P = (1 + 0.06 \times 5) \times 10,000 = 13,000 \tag{158}$$

**Solution to Problem 11, Uncertain compound interest,** (p.9).

**(a)** The FW is:

$$\text{FW}(i) = (1+i)^N P \tag{159}$$

because $i$ is constant over time. The info-gap model is:

$$\mathcal{U}(h) = \left\{ i:\ i \ge -1,\ \left| \frac{i-\widetilde{i}}{s} \right| \le h \right\}, \quad h \ge 0 \tag{160}$$

where $s$ is chosen to equal "tens of percent" of $\widetilde{i}$, for example $s = 0.3\widetilde{i}$. If $i = -1$ then the FW is zero, and any more negative value entails debt. We suppose that debt would terminate the operation through bankruptcy.

The robustness is defined as:

$$\widehat{h}(\text{FW}_\text{c}) = \max \left\{ h:\ \left( \min_{i \in \mathcal{U}(h)} F(i) \right) \ge \text{FW}_\text{c} \right\} \tag{161}$$

Let $m(h)$ denote the inner minimum, which occurs when $i = \widetilde{i} - sh$, for $h \le (1+\widetilde{i})/s$:

$$m(h) = \left(1 + \widetilde{i} - sh\right)^N P \tag{162}$$

Equate this to $\text{FW}_\text{c}$ and solve for $h$:

$$\left(1 + \widetilde{i} - sh\right)^N P = \text{FW}_\text{c} \quad \Longrightarrow \quad \boxed{\widehat{h}(\text{FW}_\text{c}) = \frac{1}{s}\left(1 + \widetilde{i} - \left(\frac{\text{FW}_\text{c}}{P}\right)^{1/N}\right)} \tag{163}$$

or zero if this is negative. Note that:

$$\widehat{h}(\text{FW}_\text{c} = 0) = \frac{1+\widetilde{i}}{s} \tag{164}$$

Thus eq.(163) is valid for all non-negative values of $\text{FW}_\text{c}$.

**(b)** We require "several" units of robustness, for instance $\widehat{h} = 3$ and $s = 0.3\widetilde{i} = 0.018$. Solving eq.(163) for $\text{FW}_\text{c}$:

$$\text{FW}_\text{c} = (1 + \widetilde{i} - s\widehat{h})^N P = (1 + 0.06 - 0.018 \times 3)^5 10{,}000 = \$10{,}303.62 \tag{165}$$

Thus, even if the estimated rate of return errs by as much as a factor of 3 times $s$, (implying a lower-than anticipated rate of return) the FW will not be less than \$10,303.62.

Compare this with the estimated FW:

$$\text{FW}(\widetilde{i}) = (1+\widetilde{i})^N P = (1+0.06)^5 10{,}000 = \$13{,}382.26 \tag{166}$$

**(c) and (d)** Choose the uncertainty weights for each case as $s_k = g\widetilde{i}_k$, $k = 1, 2$. Equate the robustnesses of the two alternatives to find the value of $\text{FW}_\text{c}$ at which the robustness curves cross:

$$\widehat{h}(\text{FW}_\times, \widetilde{i}_1) = \widehat{h}(\text{FW}_\times, \widetilde{i}_2) \quad \Longrightarrow \quad \text{FW}_\times = P \tag{167}$$

The robustness reaches zero at the estimated FW:

$$\widehat{h}(\text{FW}_\text{c}) = 0 \ \text{ if } \ \text{FW}_\text{c} = \text{FW}(\widetilde{i}) = (1+\widetilde{i})^N P \tag{168}$$

Thus $\widehat{h}(\text{FW}_\text{c}, \widetilde{i}_2)$ reaches the $\text{FW}_\text{c}$ axis to the right of $\widehat{h}(\text{FW}_\text{c}, \widetilde{i}_1)$ because $\widetilde{i}_1 < \widetilde{i}_2$. The robustness curves cross at $\text{FW}_\text{c} = P$, at which value the robustness equals $\widehat{h}(P) = \widetilde{i}_k / s_k = 1/g$. This is illustrated in fig. 4 with $g = 1$, where we see that $\widetilde{i}_2 = 0.03$ is robust-preferred over $\widetilde{i}_1 = 0.02$ for $\text{FW}_\text{c} \ge \$10{,}000$, which is precisely the initial investment, $P$.

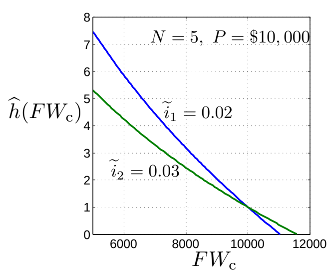


Figure 4: Robustness curves for problem 11 (c), eq.(163).

**Solution to Problem 12, Investment with uncertain costs and returns,** (p.10).

**(a)**

- $S =$ Initial cost of the project.
- $\widetilde{R} =$ estimated revenue at the end of each period.
- $\widetilde{C} =$ estimated operating cost at the end of each period.
- $N =$ number of periods.
- MARR = $i$.

The PW is:

$$PW(R,C) = -S + \sum_{k=1}^{N}(1+i)^{-k}R_k - \sum_{k=1}^{N}(1+i)^{-k}C_k \tag{169}$$

The robustness is defined as:

$$\widehat{h} = \max\left\{h : \left(\min_{R,C\in\mathcal{U}(h)} \text{PW}(R,C)\right) \ge \text{PW}_{\text{c}}\right\} \tag{170}$$

Let $m(h)$ denote the inner minimum, which occurs when:

$$R_k = \widetilde{R} - \varepsilon_R\widetilde{R}h = \widetilde{R}(1-\varepsilon_R h), \quad C_k = \widetilde{C} + \varepsilon_C\widetilde{C}h = \widetilde{C}(1+\varepsilon_C h) \tag{171}$$

Thus:

$$\begin{aligned} m(h) &= -S + \sum_{k=1}^{N}(1+i)^{-k}\widetilde{R}(1-\varepsilon_R h) - \sum_{k=1}^{N}(1+i)^{-k}\widetilde{C}(1+\varepsilon_C h) && (172)\\ &= \text{PW}(\widetilde{R},\widetilde{C}) - (\varepsilon_R\widetilde{R}+\varepsilon_C\widetilde{C})h\sum_{k=1}^{N}(1+i)^{-k} && (173)\\ &= \text{PW}(\widetilde{R},\widetilde{C}) - (\varepsilon_R\widetilde{R}+\varepsilon_C\widetilde{C})h\frac{1-(1+i)^{-N}}{i} && (174)\end{aligned}$$

Equate this to $\text{PW}_{\text{c}}$ and solve for $h$ to obtain the robustness:

$$\boxed{\widehat{h} = \frac{\text{PW}(\widetilde{R},\widetilde{C}) - \text{PW}_{\text{c}}}{\varepsilon_R\widetilde{R}+\varepsilon_C\widetilde{C}}\,\frac{i}{1-(1+i)^{-N}}} \tag{175}$$

or zero if this is negative.

**(b)** The conditions of eqs.(3) and (4) cause the robustness curves of the two options to cross (see fig. 5), where option 1 is nominally worse but with lower cost of robustness (steeper robustness curve). Thus, the robustness criterion prefers option 1 when $\text{PW}_{\text{c}}$ is less than the value, $\text{PW}_{\times}$, at which the robustness curves cross one another. This is obtained by equating the robustness functions:

$$\begin{aligned} \widehat{h}_1 = \widehat{h}_2 &\implies \frac{\text{PW}(\widetilde{R}_1,\widetilde{C}_1) - \text{PW}_{\times}}{\varepsilon_{1,R}\widetilde{R}_1+\varepsilon_{1,C}\widetilde{C}_1} = \frac{\text{PW}(\widetilde{R}_2,\widetilde{C}_2) - \text{PW}_{\times}}{\varepsilon_{2,R}\widetilde{R}_2+\varepsilon_{2,C}\widetilde{C}_2} && (176)\\ &\implies \text{PW}_{\times} = \frac{\widetilde{\text{PW}}_1 E_2 - \widetilde{\text{PW}}_2 E_1}{E_2 - E_1} && (177)\end{aligned}$$

which is positive if:

$$\frac{\widetilde{\text{PW}}_1}{E_1} > \frac{\widetilde{\text{PW}}_2}{E_2} \tag{178}$$

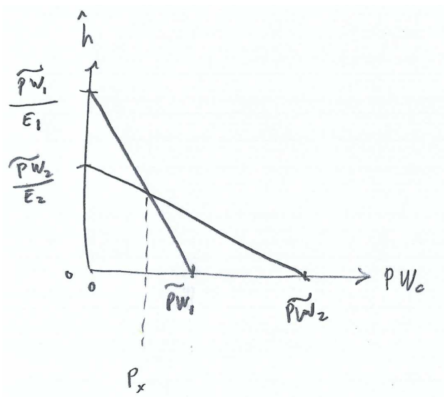

Figure 5: Robustnesses vs critical present worth, problem 12(b).

where we have defined:

$$E_i = \varepsilon_{i,R}\widetilde{R}_i + \varepsilon_{i,C}\widetilde{C}_i, \quad \widetilde{\mathsf{PW}}_i = \mathsf{PW}(\widetilde{R}_i, \widetilde{C}_i) \tag{179}$$

In summary:

$$\text{option 1} \succ \text{option 2} \quad \text{if} \quad \mathsf{PW}_{\rm c} < \mathsf{PW}_{\times} \tag{180}$$

**Solution to Problem 13, Investment with uncertain probabilistic returns,** (p.11).

**(a)**

- $S =$ Initial cost of the project.
- $R =$ estimated revenue at the end of each period.
- $C =$ estimated operating cost at the end of each period.
- $N =$ number of periods.
- MARR = $i$.

The PW is:

$$PW(R) = -S + \sum_{k=1}^{N}(1+i)^{-k}R - \sum_{k=1}^{N}(1+i)^{-k}C \tag{181}$$

$$= -S + (R-C)\underbrace{\frac{1-(1+i)^{-N}}{i}}_{\delta} \tag{182}$$

The probability of failure is:

$$P_{\rm f} = \text{Prob}(\text{PW} \le \text{PW}_{\rm c}) \tag{183}$$

$$= \text{Prob}(-S + (R-C)\delta \le \text{PW}_{\rm c}) \tag{184}$$

$$= \text{Prob}\left(R \le \frac{\text{PW}_{\rm c} + S + \delta C}{\delta}\right) \tag{185}$$

$$= \text{Prob}\,(R \le R_{\rm c}) \quad \text{(which defines } R_{\rm c}) \tag{186}$$

$$= \int_0^{R_{\rm c}} p(R)\,{\rm d}R \tag{187}$$

$$= 1 - {\rm e}^{-\lambda R_{\rm c}} \tag{188}$$

**(b)** The info-gap model for uncertain exponential distribution is:

$$\mathcal{U}(h) = \left\{p(R) = \lambda{\rm e}^{-\lambda R} :\ \lambda \ge 0,\ \left|\frac{\lambda - \widetilde{\lambda}}{\widetilde{\lambda}}\right| \le h\right\}, \quad h \ge 0 \tag{189}$$

The robustness is defined as:

$$\widehat{h} = \max\left\{h :\ \left(\max_{p\in\mathcal{U}(h)} P_{\rm f}(p)\right) \le P_{\rm fc}\right\} \tag{190}$$

Let $m(h)$ denote the inner maximum, which occurs when $\lambda$ is as large as possible at horizon of uncertainty $h$: $\lambda = (1+h)\widetilde{\lambda}$. Thus:

$$m(h) = 1 - \exp\left[-(1+h)\widetilde{\lambda}R_{\rm c}\right] \tag{191}$$

We require:

$$1 - \exp\left[-(1+h)\widetilde{\lambda}R_{\rm c}\right] \le P_{\rm fc} \tag{192}$$

Solve for $h$ to obtain the robustness:

$$\widehat{h}(P_{\rm fc}) = -1 - \frac{\ln(1-P_{\rm fc})}{\widetilde{\lambda}R_{\rm c}} \tag{193}$$

We see that $\widehat{h}(P_{\rm fc}) = 0$ when $P_{\rm fc} = P_{\rm f}(\widetilde{p})$. Also, $\widehat{h} > 0$ when $P_{\rm fc} > P_{\rm f}(\widetilde{p})$.

**(c)** Let $\mathcal{P}$ denote the set of all mathematically legitimate pdf's. The info-gap model is:

$$\mathcal{U}(h) = \{p(R) :\ p(R) \in \mathcal{P},\ |p(R) - \widetilde{p}(R)| \le h\}, \quad h \ge 0 \tag{194}$$

The robustness is the same as eq.(190) with the new info-gap model.

The estimated probability of failure, from eq.(188), is:

$$P_{\rm f}(\widetilde{p}) = 1 - {\rm e}^{-\widetilde{\lambda} R_{\rm c}} \tag{195}$$

We are told that $P_{\rm f}(\widetilde{p}) \ll 1$, so:

$$R_{\rm c} \ll 1/\widetilde{\lambda} \tag{196}$$

Thus, from eq.(187), we see that the inner maximum, $m(h)$, occurs when $p(R)$ is a large as possible in the interval $0 \le R \le R_{\rm c}$:

$$p(R) = \widetilde{p}(R) + h, \quad R \le R_{\rm c} \tag{197}$$

Because of eq.(196) we will be able to normalize this pdf by reducing the tail for $R > R_{\rm c}$.

Now we find that:

$$m(h) = \int_0^{R_{\rm c}} (\widetilde{p}(R) + h)\, {\rm d}R = \underbrace{1 - {\rm e}^{-\widetilde{\lambda} R_{\rm c}}}_{P_{\rm f}(\widetilde{p})} + h\, R_{\rm c} \tag{198}$$

Equating this to $P_{\rm fc}$ and solving for $h$ yields the robustness:

$$P_{\rm f}(\widetilde{p}) + h\, R_{\rm c} = P_{\rm fc} \quad \Longrightarrow \quad \widehat{h}(P_{\rm fc}) = \frac{P_{\rm fc} - P_{\rm f}(\widetilde{p})}{R_{\rm c}} \tag{199}$$

or zero if this is negative.

**Solution to Problem 14, Investment with uncertain costs and returns, revisited,** (p.12).

- $S =$ Initial cost of the project = \$640,000.
- $\widetilde{R}_k =$ estimated revenue at the end of $k$th period = \$180,000.
- $\widetilde{C}_k =$ estimated operating cost at the end of $k$th period.
  = \$44,000 for $k = 1,\, 2$.
  = $\$44,000 - 4,000(k-2)$ for $k = 3,\, \ldots,\, 8$.
- $N =$ number of periods = 8.
- MARR = 15%, so $i = 0.15$.

The PW is:

$$PW(R,C) = -S + \sum_{k=1}^{N} (1+i)^{-k} R_k - \sum_{k=1}^{N} (1+i)^{-k} C_k \tag{200}$$

The info-gap model for cost and return uncertainty:

$$\mathcal{U}(h) \left\{ R, C : \left| \frac{R_k - \widetilde{R}_k}{\varepsilon_R \widetilde{R}_k} \right| \le h,\ \left| \frac{C_k - \widetilde{C}_k}{\varepsilon_C \widetilde{C}_k} \right| \le h,\ k = 1,\, \ldots,\, N \right\}, \quad h \ge 0 \tag{201}$$

where $\varepsilon_R = 0.3$ and $\varepsilon_C = 0.1$.

The robustness is defined as:

$$\widehat{h} = \max \left\{ h : \left( \min_{R,C \in \mathcal{U}(h)} \mathrm{PW}(R,C) \right) \ge \mathrm{PW_c} \right\} \tag{202}$$

Let $m(h)$ denote the inner minimum, which occurs when:

$$R_k = \widetilde{R}_k - \varepsilon_R \widetilde{R}_k h = \widetilde{R}_k (1 - \varepsilon_R h), \quad C_k = \widetilde{C}_k + \varepsilon_C \widetilde{C}_k h = \widetilde{C}_k (1 + \varepsilon_C h) \tag{203}$$

Thus:

$$m(h) = -S + \sum_{k=1}^{N} (1+i)^{-k} \widetilde{R}_k (1 - \varepsilon_R h) - \sum_{k=1}^{N} (1+i)^{-k} \widetilde{C}_k (1 + \varepsilon_C h) \tag{204}$$

$$= \mathrm{PW}(\widetilde{R}, \widetilde{C}) - h \underbrace{\sum_{k=1}^{N} (1+i)^{-k} (\widetilde{R}_k \varepsilon_R + \widetilde{C}_k \varepsilon_C)}_{Q} \tag{205}$$

Equate to $\mathrm{PW_c}$ and solve for $h$ to obtain the robustness:

$$\widehat{h} = \frac{\mathrm{PW}(\widetilde{R}, \widetilde{C}) - \mathrm{PW_c}}{Q} \tag{206}$$

or zero if this is negative.

**Solution to Problem 15, Benefit-cost ratio of two design concepts,** (p.13).

**(a)** Present worth of the benefits of design $j$ are:

$$B_{pw}(j) = \sum_{n=1}^{N_j}(1+i)^{-n}B \quad (207)$$

$$= \frac{1-(1+i)^{-N_j}}{i}B \quad (208)$$

$$= \delta_{fj}(i)B \quad (209)$$

Present worth of the initial investment and maintenance costs of design $j$ are:

$$C_{pw}(j) = S + \sum_{n=1}^{N_j}(1+i)^{-n}C \quad (210)$$

$$= S_j + \frac{1-(1+i)^{-N_j}}{i}C \quad (211)$$

$$= S + \delta_{fj}(i)C \quad (212)$$

The BCR of design $j$ is:

$$\text{BCR}(j) = \frac{B_{pw}(j)}{C_{pw}(j)} \quad (213)$$

$$= \frac{\delta_{fj}(i)B}{S_j + \delta_{fj}(i)C} \quad (214)$$

The discount factors for the two designs are:

$$\delta_{f1}(i) = 2.7232, \quad \delta_{f2}(i) = 4.3295 \quad (215)$$

$\delta_{f1}$ is less than $\delta_{f2}$ because $N_1 < N_2$. However, the ratio is greater than the ratio of the durations:

$$\frac{\delta_{f1}}{\delta_{f2}} = 0.6290 > 0.6 = \frac{N_1}{N_2} \quad (216)$$

The reason: later periods get less weight than earlier periods.

The BCRs of the two designs are:

$$\text{BCR}(1) = \frac{2.7232 \times 4,000}{20,000 + 2.7232 \times 1,500} = 0.4523 \quad (217)$$

$$\text{BCR}(2) = \frac{4.3295 \times 4,000}{33,333 + 4.3295 \times 1,500} = 0.4348 \quad (218)$$

Both the benefits and the costs are lower for design 1 than for design 2, but the BCRs are nearly the same. Nonetheless, design 1 has a better (higher) ratio even though the costs and benefits each period are the same and the initial investment per year is the same ($S_1/N_1 = S_2/N_2$). The reason: The benefits over 5 years from design 2 are discounted disproportionately, as shown in eq.(216). Even though the initial investments look the same for both designs, they are not because future benefits are discounted.

**(b)** Equate the BCRs and solve for the ratio $S_1/S_2$:

$$\text{BCR}(1) = \text{BCR}(2) \Longrightarrow \frac{\delta_{f1}B}{S_1 + \delta_{f1}C} = \frac{\delta_{f2}B}{S_2 + \delta_{f2}C} \Longrightarrow (S_2 + \delta_{f2}C)\delta_{f1} = (S_1 + \delta_{f1}C)\delta_{f2} \quad (219)$$

$$\Longrightarrow \frac{S_1}{S_2} = \frac{\delta_{f1}}{\delta_{f2}} \quad (220)$$

The BCRs will be the same if the ratio of initial investments, $S_1/S_2$, equals the ratio $\delta_{f1}/\delta_{f2}$. However, the initial investments are proportional to the lifetimes, $S_1/S_2 = N_1/N_2 = 3/5 = 0.6$, and the discount functions are not: $\delta_{f1}/\delta_{f2} = 0.6290$. In other words, design 1 has a better BCR than design 2 because design 1 is under-priced relative to the initial cost of design 2. Its "discounted fair price", resulting in the same BCR for both designs, would be:

$$S_1 = \frac{\delta_{f1}}{\delta_{f2}} S_2 = 0.629 \times \$33,333 = \$20,966 > 20,000 \tag{221}$$

Design 1 has a higher benefit-to-cost ratio because its initial cost is low; it is a better buy: more benefit per dollar (of initial investment and discounted maintenance cost).

**Solution to Problem 16, Present worth or benefit-cost ratio?** (p.14).

**(a)** Present worth of the benefits of design $j$ are:

$$\begin{aligned} B_{pw}(j) &= \sum_{n=1}^{N}(1+i)^{-n}B_j & (222)\\ &= \frac{1-(1+i)^{-N}}{i}B_j & (223)\\ &= \delta_f(i)B_j & (224)\end{aligned}$$

Present worth of the initial investment and maintenance costs of design $j$ are:

$$\begin{aligned} C_{pw}(j) &= S_j+\sum_{n=1}^{N}(1+i)^{-n}C_j & (225)\\ &= S_j+\frac{1-(1+i)^{-N}}{i}C_j & (226)\\ &= S_j+\delta_f(i)C_j & (227)\end{aligned}$$

The present worth of design $j$ is:

$$\text{PW}_j = B_{pw}(j)-C_{pw}(j)=\delta_f(i)B_j-S_j-\delta_f(i)C_j \quad (228)$$

The BCR of design $j$ is:

$$\begin{aligned} \text{BCR}(j) &= \frac{B_{pw}(j)}{C_{pw}(j)} & (229)\\ &= \frac{\delta_f(i)B_j}{S_j+\delta_f(i)C_j} & (230)\end{aligned}$$

The discount function is $\delta_f(i)=4.3295$. Thus:

$B_{pw}(1)=\$5,500$, $S_1+C_{pw}(1)=\$1,000+\$2,000=\$3,000$. Thus PW$(1)=\$2,500$ and BCR$(1)=1.8333$.

$B_{pw}(2)=\$5,000$, $S_2+C_{pw}(2)=\$800+\$1,800=\$2,600$. Thus PW$(2)=\$2,400$ and BCR$(2)=1.9231$.

Hence: PW$(1)>$ PW$(2)$ so design 1 is PW-preferred.

But: BCR$(2)>$ BCR$(1)$ so design 2 is BCR-preferred.

The teams disagree. Objective economic analysis is not always unique.

**(b)** The robustness function for the PW of design $j$ is defined as:

$$\widehat{h}_{pw,j}=\max\left\{h:\ \left(\min_{B,C\in\mathcal{U}(h)}\text{PW}_j\right)\geq \text{PW}_{\rm c}\right\} \quad (231)$$

where PW$_j$ is specified by eq.(228).

The robustness function for the BCR of design $j$ is defined as:

$$\widehat{h}_{bcr,j}=\max\left\{h:\ \left(\min_{B,C\in\mathcal{U}(h)}\text{BCR}_j\right)\geq \text{BCR}_{\rm c}\right\} \quad (232)$$

where BCR$_j$ is specified by eq.(230).

Let $m_{pw,j}$ denote the inner minimum in eq.(231), which is the inverse of $\widehat{h}_{pw,j}$. Similarly, let $m_{bcr,j}$ denote the inner minimum in eq.(232), which is the inverse of $\widehat{h}_{bcr,j}$.

Both of these inverses occur, at horizon of uncertainty $h$, for:

$$B_j = \widetilde{B}_j - w_{B,j}h, \quad C_j = \widetilde{C}_j + w_{C,j}h \tag{233}$$

Thus:

$$m_{pw,j} = (\widetilde{B}_j - w_{B,j}h)\delta_f - S_j - (\widetilde{C}_j + w_{C,j}h)\delta_f = \widetilde{\mathrm{PW}}_j - h(w_{B,j} + w_{C,j})\delta_f \ge \mathrm{PW_c} \tag{234}$$

Hence:

$$\boxed{\widehat{h}_{pw,j} = \frac{\widetilde{\mathrm{PW}}_j - \mathrm{PW_c}}{(w_{B,j} + w_{C,j})\delta_f}} \tag{235}$$

Similarly:

$$m_{bcr,j} = \frac{(\widetilde{B}_j - w_{B,j}h)\delta_f}{S_j + (\widetilde{C}_j + w_{C,j}h)\delta_f} \ge \mathrm{BCR_c} \tag{236}$$

Hence:

$$\boxed{\widehat{h}_{bcr,j} = \frac{(\widetilde{\mathrm{BCR}}_j - \mathrm{BCR_c})(S_j + \widetilde{C}_j\delta_f)}{(w_{B,j} + w_{C,j}\mathrm{BCR_c})\delta_f}} \tag{237}$$

The nominal values are the same as before, so the nominal prioritization disagrees between PW and BCR:

$$\widetilde{\mathrm{PW}}_1 > \widetilde{\mathrm{PW}}_2 \quad \text{but} \quad \widetilde{\mathrm{BCR}}_1 < \widetilde{\mathrm{BCR}}_2 \tag{238}$$

However, the costs of robustness (slopes of the robustness functions) for PW and BCR depend differently on the parameters. Thus there may be curve-crossing and preference reversal for one criterion (PW or BCR) but perhaps not for the other. Also, the critical values at which curve crossing occurs may be interpreted differently for the two criteria. Thus the robust prioritization based on PW may or may not agree with the robust prioritization based on BCR.

**Solution to Problem 17, Inspection system with uncertain diminishing inspection rate** (p.15).

**(a)** Present worth of the benefits of are:

$$B_{pw} = \sum_{n=1}^{N}(1+i_b)^{-n}B \quad (239)$$
$$= \frac{1-(1+i_b)^{-N}}{i_b}B \quad (240)$$
$$= \delta_f(i_b)B \quad (241)$$

Present worth of the initial investment and maintenance costs are:

$$C_{pw} = S+\sum_{n=1}^{N}(1+i_c)^{-n}C \quad (242)$$
$$= S+\frac{1-(1+i_c)^{-N}}{i_c}C \quad (243)$$
$$= S+\delta_f(i_c)C \quad (244)$$

The BCR is:

$$\text{BCR} = \frac{B_{pw}}{C_{pw}} \quad (245)$$
$$= \frac{\delta_f(i_b)B}{S+\delta_f(i_c)C} \quad (246)$$

The discount functions are $\delta_f(i_b) = 7.3601$ and $\delta_f(i_c) = 8.1109$. The BCR $= 80.0847$ inspections per dollar, or $1/80.0847 = \$0.0125/$inspection.

**(b)** The info-gap model for uncertain reduction in inspection rate, $i_b$, is:

$$\mathcal{U}(h) = \left\{ i_b : \; i_b > -1, \; \left| \frac{i_b - \widetilde{i}_b}{s_b} \right| \le h \right\}, \quad h \ge 0 \quad (247)$$

$\widetilde{i}_b = 0.06$ and $s_b = 0.3\widetilde{i}_b$.

The robustness is:

$$\widehat{h}(\text{BCR}_\text{c}, S) = \max \left\{ h : \; \left( \min_{i_b \in \mathcal{U}(h)} \text{BCR}(i_b, S) \right) \ge \text{BCR}_\text{c} \right\} \quad (248)$$

Eq.(239) is correct because $i_b$ is constant over time. Hence the inner minimum occurs at $i_b = \widetilde{i}_b + s_b h$, so, from eq.(246):

$$m(h) = \frac{\delta_f(\widetilde{i}_b + s_b h)B}{S+\delta_f(i_c)C} \quad (249)$$

Robustness curves are shown in figs. 6 and 7 for initial investment of $S =$\$20,000. The nominal performance, BCR$_\text{c} = 80$ inspection/\$ or $1/$BCR$_\text{c} = 0.0125$ \$/inspection, has no robustness (the zeroing property). Fig. 6 shows that $\widehat{h}$ equals 1 or 2 at BCR$_\text{c} = 73.5$ or 68.0 [inspections/\$] respectively: more robustness entails lower BCR (the trade off property).

What is a reliable number of inspections per \$? The error weight $s_h$ in the info-gap model of eq.(247) is 30% of the nominal discount factor $\widetilde{i}_b$. From fig. 6 we see that $\widehat{h}(\text{BCR}_\text{c} = 73.5) = 1$ which means that the BCR will be no less than 73.5 if the rate of decrease of inspections/year is no greater than $\widetilde{i}_b + 1 \times s_b = 1.3\widetilde{i}_b$. Our information is that "The percentage yearly reduction of the inspection rate is estimated at $i_b = 0.06$, but this may err by several tens of percent (plus or minus several times

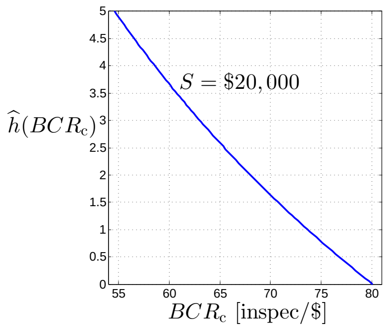


Figure 6: Robustness vs BCR$_c$, problem 17(b), eq.(249).

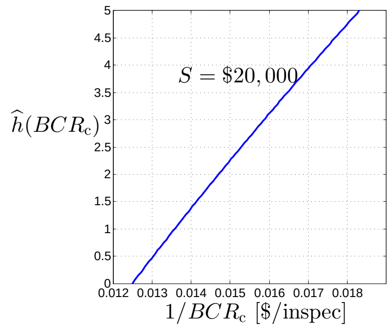


Figure 7: Robustness vs 1/BCR$_c$, problem 17(b), eq.(249).

0.006) or more." Thus a robustness of $\widehat{h} = 1$, meaning immunity again 30% error in $\widetilde{i}_b$, is moderately good. Robustness of $\widehat{h} = 2$, meaning immunity again 60% error in $\widetilde{i}_b$, would be quite good. From fig. 6 we see that $\widehat{h}(\text{BCR}_c = 68.0) = 2$ which means that the BCR will be no less than 68.0 if the rate of decrease of inspections/year is no greater than $\widetilde{i}_b + 2 \times s_b = 1.6\widetilde{i}_b$.

In summary, BCR $= 73.5$ is moderately reliable; BCR $= 68.0$ is quite reliable.

**Solution to Problem 18, General inflation** (p.15).

¶ The nominal sum $A_0$ buys a basket of goods, $B$, in year zero. The annual price inflation of these goods is $f$. Thus the nominal sum required to purchase $B$ in year $k$ is:

$$A_k = (1+f)^k A_0 \tag{250}$$

The purchasing power of the currency is reduced by $\frac{1}{2}$ if twice as many dollars are need in year $k$, so:

$$A_k = 2A_0 \tag{251}$$

Thus:

$$\frac{1}{2} = \frac{A_0}{A_k} = (1+f)^{-k} \implies \ln\frac{1}{2} = -k\ln(1+f) \implies k = \frac{\ln 2}{\ln(1+f)} = 9.006 \sim 9 \tag{252}$$

¶ Here is another way of looking at the problem (and getting the same answer). Let $A_k$ be the nominal value (number of dollar bills) needed in year $k$ to buy the basket of goods $B$. The real value in year $k$ is:

$$R_k = (1+f)^{-k} A_k \tag{253}$$

The real value of $B$ is constant:

$$R_k = R_0 \tag{254}$$

Real and nominal values are equal in the base year:

$$R_0 = A_0 \tag{255}$$

So eq.(253) is:

$$A_0 = (1+f)^{-k} A_k \tag{256}$$

We seek the year, $k$, in which the purchasing power of $1 is reduced by half, so that twice as many dollars are needed to purchase $B$:

$$A_k = 2A_0 \tag{257}$$

Thus eq.(256) becomes:

$$\frac{1}{2} = (1+f)^{-k} \tag{258}$$

We now continue as in eq.(252).

¶ Find $k$ for which $A_0/A_k = 1/n$ (see fig. 8):

$$\frac{1}{n} = (1+f)^{-k} \implies \ln\frac{1}{n} = -k\ln(1+f) \implies k = \frac{\ln n}{\ln(1+f)} \tag{259}$$

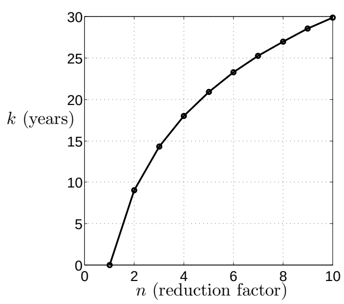


Figure 8: $k$ (years) vs $n$ (reduction factor), problem 18, eq.(259).

**Solution to Problem 19, Comparing alternatives under inflation** (p.16).

First consider alternative A. The real and nominal value of the expense in year $k$ are related as:

$$R_k^{\mathrm{A}} = (1+f)^{-k} A_k^{\mathrm{A}} \tag{260}$$

The PW of alternative A is:

$$\begin{aligned} \text{PW}_{\mathrm{A}} &= \sum_{k=1}^{N} (1+i_{\mathrm{r}})^{-k} R_k^{\mathrm{A}} && (261) \\ &= \sum_{k=1}^{N} (1+i_{\mathrm{r}})^{-k} (1+f)^{-k} A_k^{\mathrm{A}} && (262) \\ &= -388,476.01 && (263) \end{aligned}$$

Now consider alternative B. The PW of alternative B is:

$$\begin{aligned} \text{PW}_{\mathrm{B}} &= \sum_{k=1}^{N} (1+i_{\mathrm{r}})^{-k} R_k^{\mathrm{B}} && (264) \\ &= -369,085.21 && (265) \end{aligned}$$

Alternative B has lower PW of its expenses and thus, from this point of view, is preferred.

**Solution to Problem 20, Salary erosion from inflation** (p.17).

**(20a)** The year 0 real salaries are calculated as follows. See results in table 10 on p.86.

• **Nominal income from end of year 1:** The year 0 nominal equivalent of the nominal income in year 1, correcting for inflation in year 1, is:

$$A_{0,1} = (1+f_1)^{-1} A_1 \tag{266}$$

Nominal and real income in year-0 are the same, so the **real year 0 income from year 1 is:**

$$R_{0,1} = A_{0,1} = (1+f_1)^{-1} A_1 \tag{267}$$

• **Nominal income from end of year 2:** The year 1 nominal equivalent of the nominal income in year 2, correcting for inflation in year 2, is:

$$A_{1,2} = (1+f_2)^{-1} A_2 \tag{268}$$

The year 0 nominal equivalent of nominal income $A_{1,2}$, correcting for inflation in year 1, is:

$$A_{0,2} = (1+f_1)^{-1} A_{1,2} = (1+f_1)^{-1}(1+f_2)^{-1} A_2 \tag{269}$$

Nominal and real income in year 0 are the same, so the **real year 0 income from year 2 is:**

$$R_{0,2} = A_{0,2} = (1+f_1)^{-1}(1+f_2)^{-1} A_2 \tag{270}$$

• **Nominal income from end of year $k$:** Generalizing eq.(269), the nominal income in year 0 from the income in year $k$ is:

$$A_{0,k} = A_k \prod_{j=1}^{k} (1+f_j)^{-1} \tag{271}$$

Nominal and real income in year-0 are the same, so the **real year 0 income from year $k$ is:**

$$R_{0,k} = A_{0,k} = A_k \prod_{j=1}^{k} (1+f_j)^{-1} \tag{272}$$

The nominal and real salaries are shown in table 10.

| Year, $k$ | $\prod_{j=1}^{k}(1+f_j)^{-1}$ | $A_k$ | $R_k$ |
|---|---|---|---|
| 1 | 0.9337 | 34,000 | 31,746 |
| 2 | 0.8859 | 36,200 | 32,068 |
| 3 | 0.8135 | 38,800 | 31,563 |
| 4 | 0.7315 | 41,500 | 30,359 |

Table 10: Solution to problem 20a.

**(20b)** An info-gap model for uncertain inflation is:

$$\mathcal{U}(h) = \left\{ f :\ f_k > -1,\ \left| \frac{f_k - \widetilde{f}_k}{s_k} \right| \le h,\ k = 1, \ldots, 4 \right\}, \quad h \ge 0 \tag{273}$$

where $\widetilde{f}_k$ is the estimated inflation in year $k$ and $s_k = \varepsilon \widetilde{f}_k$.

The performance requirement is:

$$R_{0,k} \ge R_{kc} \tag{274}$$

The robustness for year $k$ is defined as:

$$\widehat{h}_k = \max\left\{h:\ \left(\min_{f\in\mathcal{U}(h)} R_{0,k}(f)\right) \ge R_{k\mathrm{c}}\right\} \tag{275}$$

The inner minimum, $m_k(h)$, is the inverse of the robustness and occurs when each $f_k$ is as large as possible at horizon of uncertainty $h$: $f_k = \widetilde{f}_k + s_k h = \widetilde{f}_k + \varepsilon \widetilde{f}_k h = (1+\varepsilon h)\widetilde{f}_k$. Thus, from eq.(272):

$$m_k = A_k \prod_{j=1}^{k} \left[1 + (1+\varepsilon h)\widetilde{f}_j\right]^{-1} \tag{276}$$

**Solution to Problem 21, Exchange rate devaluation** (p.18). An initial US$ investment $S$ has returns $R_{k,\text{dom}}$ in US$ in years $k = 1, \ldots, N$, or returns $R_{k,\text{for}}$ in the foreign currency, where:

$$R_{k,\text{dom}} = r_k R_{k,\text{for}} \tag{277}$$

and:

$$r_k = (1+\varepsilon)^{-k} r_0 \tag{278}$$

Recall that $r_0 = 1$ US$ per unit of foreign currency. For (a) $\varepsilon = 0.08$, and for (b) $\varepsilon = -0.06$.

The PW$_{\text{dom}}$, calculated with domestic currency, is:

$$\text{PW}_{\text{dom}} = \sum_{k=1}^{N} (1 + i_{\text{dom}})^{-k} R_{k,\text{dom}} \tag{279}$$

where $i_{\text{dom}} = 0.26$.

The PW$_{\text{for}}$, calculated with foreign currency, is:

$$\text{PW}_{\text{for}} = \sum_{k=1}^{N} (1 + i_{\text{for}})^{-k} R_{k,\text{for}} \tag{280}$$

where $i_{\text{for}}$ must be determined.

[Note: The PW terms in eqs.(279) and (280) do not include the initial investment. If one includes them then they become:

$$\text{PW}_{\text{dom}} = -S + \sum_{k=1}^{N} (1 + i_{\text{dom}})^{-k} R_{k,\text{dom}} \tag{281}$$

$$\text{PW}_{\text{for}} = -r_0 S + \sum_{k=1}^{N} (1 + i_{\text{for}})^{-k} R_{k,\text{for}} \tag{282}$$

Recall that $r_0 = 1$. Thus the same "$-S$" terms on both sides of eq.(283) cancel out.]

The PW's in eqs.(279) and (280) must be equal, after exchanging one of the currencies. Thus, using eqs.(277) and (278):

$$\sum_{k=1}^{N} (1 + i_{\text{dom}})^{-k} (1+\varepsilon)^{-k} R_{k,\text{for}} = \sum_{k=1}^{N} (1 + i_{\text{for}})^{-k} R_{k,\text{for}} \tag{283}$$

This relation holds if:

$$(1 + i_{\text{dom}})^{-k} (1+\varepsilon)^{-k} = (1 + i_{\text{for}})^{-k} \quad \text{for all } k \tag{284}$$

Thus:

$$i_{\text{for}} = (1 + i_{\text{dom}})(1+\varepsilon) - 1 \tag{285}$$

For (a):

$$i_{\text{for}} = (1 + 0.26)(1 + 0.08) - 1 = 0.3608 \tag{286}$$

For (b):

$$i_{\text{for}} = (1 + 0.26)(1 - 0.06) - 1 = 0.1844 \tag{287}$$

**Solution to Problem 22, Real and nominal preferences with inflation and interest** (p.19).

**(a)** Take $b = 0$ as the baseline year. The present worth for nominal series $A_k^{(i)}$ is:

$$\text{PW}_i = \sum_{k=1}^{N} (1 + i_{\text{nom}})^{-k} A_k^{(i)} \tag{288}$$

where:

$$i_{\text{nom}} = i_{\text{r}} + (1 + i_{\text{r}})f \quad \text{or} \quad 1 + i_{\text{nom}} = (1 + i_{\text{r}})(1 + f) \tag{289}$$

This is also the present real worth in the base period, because real and nominal worths are equal in the base period. To see this, calculate the real value in each period and then calculate the PW of these real values:

$$R_k^{(i)} = (1 + f)^{-k} A_k^{(i)} \tag{290}$$

The PW is:

$$\text{PW}_i = \sum_{k=1}^{N} (1 + i_{\text{r}})^{-k} R_k^{(i)} \tag{291}$$

$$= \sum_{k=1}^{N} (1 + i_{\text{r}})^{-k} (1 + f)^{-k} A_k^{(i)} \tag{292}$$

$$= \sum_{k=1}^{N} (1 + i_{\text{nom}})^{-k} A_k^{(i)} \tag{293}$$

which is eq.(288). Thus, using the definition of nominal interest, $i_{\text{nom}}$, in eq.(289), the present real and nominal worths are equal.

**(b)** Since the nominal and real present worths are the same, the preference between the two series is the same.

**(c)** We will show how to choose the time-varying nominal interest rates, $i_{\text{nom},k}$, so that the PW in the baseline period is the same for both the real and nominal the values. This will then cause the prioritization to be the same. In other words, we will derive the time-varying generalization of eq.(289).

- The PW in the baseline period, of the nominal value $A_k^{(i)}$, is:

$$\text{PW}_i = A_k^{(i)} \underbrace{\prod_{j=1}^{k} (1 + i_{\text{nom},j})^{-1}}_{I_{\text{nom},k}^{-1}} \tag{294}$$

where the nominal interest rates, $i_{\text{nom},j}$, are not yet defined.

- The real value of $A_k^{(i)}$ in period $k$ is:

$$R_k^{(i)} = A_k^{(i)} \underbrace{\prod_{j=1}^{k} (1 + f_j)^{-1}}_{F_k^{-1}} \tag{295}$$

- The PW in the baseline period, of $R_k^{(i)}$, is:

$$\text{PW}_i = R_k^{(i)} \underbrace{\prod_{j=1}^{k} (1 + i_{\text{r},j})^{-1}}_{I_{\text{r},k}^{-1}} \tag{296}$$

$$= I_{\text{r},k}^{-1} F_k^{-1} A_k^{(i)} \tag{297}$$

• We want eqs.(294) and (297) to be equal because they are in the baseline period, so:

$$I_{\mathrm{nom},k}^{-1} = I_{\mathrm{r},k}^{-1} F_k^{-1} \tag{298}$$

which defines the nominal interest rates introduced in eq.(294). Note that this is the same as eq.(289) if $k = 1$ or if both the inflation and the real interest rate are time-independent.

Now we can calculate the PW from nominal and real values of the entire series.

• From eq.(294), the PW in the baseline period, for the nominal series, is:

$$\mathrm{PW}_i = \sum_{k=1}^{N} I_{\mathrm{nom},k}^{-1} A_k^{(i)} \tag{299}$$

• From eqs.(295)–(298), the PW in the baseline period, for the real series, is:

$$\begin{aligned} \mathrm{PW}_i &= \sum_{k=1}^{N} I_{\mathrm{r},k}^{-1} R_k^{(i)} && (300) \\ &= \sum_{k=1}^{N} I_{\mathrm{r},k}^{-1} F_k^{-1} A_k^{(i)} && (301) \\ &= \sum_{k=1}^{N} I_{\mathrm{nom},k}^{-1} A_k^{(i)} && (302) \end{aligned}$$

• Eqs.(299) and (302) are the same. We have proven that baseline nominal and real PWs are the same if the nominal interest rate is defined with eq.(298).

**(d)** We can calculate the PW with either eq.(292), which depends on both $f$ and $i_{\mathrm{r}}$, or with eq.(291) which depends only on $i_{\mathrm{r}}$. The robustnesses of these two alternatives may be different, resulting in different preferences. We will first derive the robustness with eq.(292). The derivation with eq.(291) is analogous.

The PW of series $i$ given by eq.(292) is:

$$\mathrm{PW}_i = \sum_{k=1}^{N} (1 + i_{\mathrm{r}})^{-k} (1 + f)^{-k} A_k^{(i)} \tag{303}$$

Suppose, without loss of generality, that:

$$\mathrm{PW}_1 > \mathrm{PW}_2 \tag{304}$$

with the estimated real interest rate and inflation, $\widetilde{i}_r$ and $\widetilde{f}$, so we nominally prefer series 1. How robust is this preference to uncertainty in $\widetilde{i}_r$ and $\widetilde{f}$? How wrong can we be in our estimates, $\widetilde{i}_r$ and $\widetilde{f}$, and the choice of series 1 over series 2 does not change?

Consider the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ (i_{\mathrm{r}}, f) :\ i_{\mathrm{r}} > -1,\ \left| \frac{i_{\mathrm{r}} - \widetilde{i}_r}{s_r} \right| \le h,\ f > -1,\ \left| \frac{f - \widetilde{f}}{s_f} \right| \le h \right\}, \quad h \ge 0 \tag{305}$$

The robustness is the greatest horizon of uncertainty up to which eq.(304) holds:

$$\widehat{h}_{292} = \max \left\{ h :\ \left( \min_{i_{\mathrm{r}}, f \in \mathcal{U}(h)} [\mathrm{PW}_1 - \mathrm{PW}_2] \right) \ge 0 \right\} \tag{306}$$

Let $m_{292}(h)$ denote the inner minimum, which is the inverse of the robustness function. From eq.(303) we can write:

$$\text{PW}_1 - \text{PW}_2 = \sum_{k=1}^{N} (1+i_\text{r})^{-k}(1+f)^{-k}\left(A_k^{(1)} - A_k^{(2)}\right) \tag{307}$$

One can use a numerical procedure to evaluate and plot $m_{292}(h)$ vs. $h$. The value of $h$ at which $m_{292}(h) = 0$ is the robustness of the preference for series 1.

Now consider the other representation of PW.

The PW of series $i$ given by eq.(291) is:

$$\text{PW}_i = \sum_{k=1}^{N} (1+i_\text{r})^{-k} R_k^{(i)} \tag{308}$$

The robustness is the greatest horizon of uncertainty up to which eq.(304) holds:

$$\widehat{h}_{291} = \max\left\{h:\ \left(\min_{i_\text{r}\in\mathcal{U}(h)} [\text{PW}_1 - \text{PW}_2]\right) \geq 0\right\} \tag{309}$$

Let $m_{291}(h)$ denote the inner minimum, which is the inverse of the robustness function. From eq.(308) we can write:

$$\text{PW}_1 - \text{PW}_2 = \sum_{k=1}^{N} (1+i_\text{r})^{-k}\left(R_k^{(1)} - R_k^{(2)}\right) \tag{310}$$

One can use a numerical procedure to evaluate and plot $m_{291}(h)$ vs. $h$. The value of $h$ at which $m_{291}(h) = 0$ is the robustness of the preference for series 1. The function $m_{291}(h)$ is different from the function $m_{292}(h)$, so the robustnesses are different.

**(e)** We can easily generalize part (d). The definition of the robustness is very similar to eq.(306):

$$\widehat{h}(\delta) = \max\left\{h:\ \left(\min_{i_\text{r},f\in\mathcal{U}(h)} [\text{PW}_1 - \text{PW}_2]\right) \geq \delta\right\} \tag{311}$$

We calculate the inner minimum, $m(h)$, as before, as explained in connection with eqs.(307) and (310). This yields the inverse of $\widehat{h}(\delta)$: a plot of $h$ vs. $m(h)$ is the same as a plot of $\widehat{h}(\delta)$ vs. $\delta$.

**(f)** We proceed similarly to part (d). The PW of series $i$ is given by eq.(303). Suppose, without loss of generality, that eq.(6), p.19, holds with the estimates series, so we putatively prefer series 1. How robust is this preference to uncertainty in the nominal values $A_k^{(i)}$?

Consider the following fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{A_k^{(i)}:\ \left|\frac{A_k^{(i)} - \widetilde{A}_k^{(i)}}{s_k^{(i)}}\right| \leq h,\ k=1,\ldots,N, i=1,2\right\},\quad h\geq 0 \tag{312}$$

The robustness is the greatest horizon of uncertainty up to which eq.(304) holds:

$$\widehat{h}(\delta) = \max\left\{h:\ \left(\min_{A_k^{(i)}\in\mathcal{U}(h)} [\text{PW}_1 - \text{PW}_2]\right) \geq \delta\right\} \tag{313}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function. From eq.(303) we can write:

$$\text{PW}_1 - \text{PW}_2 = \sum_{k=1}^{N} (1+i_\text{r})^{-k}(1+f)^{-k}\left(A_k^{(1)} - A_k^{(2)}\right) \tag{314}$$

It is evident that $m(h)$ occurs for:

$$A_k^{(1)} = \widetilde{A}_k^{(1)} - s_k^{(1)} h \tag{315}$$

$$A_k^{(2)} = \widetilde{A}_k^{(2)} + s_k^{(2)} h \tag{316}$$

Put these expressions into eq.(314), equate to $\delta$, and solve for $h$ to obtain the robustness. We find:

$$\begin{aligned} m(h) &= \sum_{k=1}^{N} (1+i_{\mathrm{r}})^{-k}(1+f)^{-k} \left( \widetilde{A}_k^{(1)} - \widetilde{A}_k^{(2)} \right) - h \sum_{k=1}^{N} (1+i_{\mathrm{r}})^{-k}(1+f)^{-k} \left( s_k^{(1)} + s_k^{(2)} \right) & (317) \\ &= \Delta_{\widetilde{A}} - h\Delta_s & (318) \end{aligned}$$

which defines $\Delta_{\widetilde{A}}$ and $\Delta_s$. Equating this to $\delta$ and solving for $h$ yields the robustness:

$$\widehat{h}(\delta) = \frac{\Delta_{\widetilde{A}} - \delta}{\Delta_s} \tag{319}$$

or zero if this is negative.

**Solution to Problem 23, Project financing** (p.21).

**(23a)** Define the following quantities:

$t_0 = 0 =$ start time.

$t_n =$ duration of the $n$th stage, for $n = 1,\ 2,\ 3$.

$\theta_n = \sum_{j=1}^{n-1} t_j =$ the year in which stage $n$ starts. Thus: $\theta_1 = 0,\ \ \theta_2 = t_1,\ \ \theta_3 = t_1 + t_2$.

$T_n = \sum_{j=n}^{4} t_j =$ duration from start of stage $n$ to $t_4$ years after the end of the project. Thus:

$T_1 = t_1 + t_2 + t_3 + t_4,\ \ T_2 = t_2 + t_3 + t_4,\ \ T_3 = t_3 + t_4,\ \ T_4 = t_4$

Note that $T_1 = \theta_n + T_n$.

Stage $n$ starts $\theta_n$ years after the beginning of the project. The cost of stage $n$, in $t = 0$ shekels, is $c_n$. This cost inflates at the annual rate $f$. Thus at the start of stage $n$ we take a loan of $(1+f)^{\theta_n} c_n$ shekels. We pay this loan at nominal interest rate $i_{\rm nom}$ after $T_n$ years. This nominal payment is:

$$A_n = (1 + i_{\rm nom})^{T_n} (1 + f)^{\theta_n} c_n \tag{320}$$

The total nominal payment is:

$$A = \sum_{n=1}^{3} A_n = \sum_{n=1}^{3} (1 + i_{\rm nom})^{T_n} (1 + f)^{\theta_n} c_n \tag{321}$$

The real shekel value, in project-start-time ($t = 0$) shekels, is:

$$R = (1 + f)^{-T_1} A \tag{322}$$

$$= (1 + f)^{-T_1} \sum_{n=1}^{3} (1 + i_{\rm nom})^{T_n} (1 + f)^{\theta_n} c_n \tag{323}$$

$$= \sum_{n=1}^{3} (1 + i_{\rm nom})^{T_n} (1 + f)^{\theta_n - T_1} c_n \tag{324}$$

$$= \sum_{n=1}^{3} (1 + i_{\rm nom})^{T_n} (1 + f)^{-T_n} c_n \tag{325}$$

$$= \sum_{n=1}^{3} \left( \frac{1 + i_{\rm nom}}{1 + f} \right)^{T_n} c_n \tag{326}$$

**(23b)** Nominal income $I_3$ is received at the end of each of $t_3$ years, invested at interest $i_{\rm nom}$ and held an additional $t_4$ years after completion:

End of year 1: $I_3$ invested for $t_3 - 1 + t_4$ years $\Longrightarrow$ $(1 + i_{\rm nom})^{t_3 - 1 + t_4} I_3$ when loan repaid.

End of year 2: $I_3$ invested for $t_3 - 2 + t_4$ years $\Longrightarrow$ $(1 + i_{\rm nom})^{t_3 - 2 + t_4} I_3$ when loan repaid.

...

End of year $t_3$: $I_3$ invested for $t_3 - t_3 + t_4$ years $\Longrightarrow$ $(1 + i_{\rm nom})^{t_3 - t_3 + t_4} I_3$ when loan repaid.

Hence the total nominal cumulative income at repayment is:

$$I = I_3 \sum_{j=1}^{t_3} (1 + i_{\rm nom})^{t_3 - j + t_4} \tag{327}$$

**(23c)** The solution is the same as part (23a) **except** that we must calculate $c_1$ from the euro cost $\gamma_1$ as: $c_1 = r\gamma_1$. Now the solution is the same eqs.(321) and (326) with $t_1$ replaced by $\tau_1$.

**(23d)** The definition of the robustness is:

$$\widehat{h}(R_{\rm c}) = \max\left\{h:\ \left(\max_{c\in\mathcal{U}(h)} R(c)\right) \le R_{\rm c}\right\} \tag{328}$$

where $R(c)$ is the real shekel cost of the project from eq.(326). Let $m(h)$ denote the inner maximum, which is the inverse of the robustness. This maximum occurs when $c_n = \widetilde{c}_n + hw_n$. From eq.(326), where we define the coefficient of $c_n$ in that expression as $b_n$, we find:

$$m(h) = \sum_{n=1}^{3} b_n(\widetilde{c}_n + hw_n) = R(\widetilde{c}) + hR(w) \le R_{\rm c} \quad\Longrightarrow\quad \widehat{h}(R_{\rm c}) = \frac{R_{\rm c} - R(\widetilde{c})}{R(w)} \tag{329}$$

or zero if this is negative.

**(23e)** The definition of the robustness is:

$$\widehat{h}(R_{\rm c}) = \max\left\{h:\ \left(\max_{f\in\mathcal{U}(h)} R(f)\right) \le R_{\rm c}\right\} \tag{330}$$

where $R(f)$ is the real shekel cost of the project from eq.(326). Let $m(h)$ denote the inner maximum, which is the inverse of the robustness. This maximum occurs when $f = \widetilde{f} - hs$. From eq.(326) we find:

$$m(h) = \sum_{n=1}^{3} \left(\frac{1+i_{\rm nom}}{1+\widetilde{f}-hs}\right)^{T_n} c_n \tag{331}$$

This is the inverse of the robustness function, $\widehat{h}(R_{\rm c})$.

**(23f)** The probability of default is defined as:

$$P_{\rm d} = \text{Prob}(R \ge R_{\rm c}) \tag{332}$$

From eq.(326), where we define the coefficient of $c_n$ in that expression as $b_n$, we find:

$$R = \sum_{n=1}^{3} b_n c_n = b_1 c_1 + g \tag{333}$$

where defines $g$ whose value is known. Thus the probability of default becomes:

$$P_{\rm d} = \text{Prob}(b_1c_1 + g \ge R_{\rm c}) = \text{Prob}\left(c_1 \ge \frac{R_{\rm c}-g}{b_1}\right) = \text{Prob}\left(\frac{c_1-\widetilde{c}_1}{\sigma_1} \ge \frac{R_{\rm c}-g}{b_1\sigma_1} - \frac{\widetilde{c}_1}{\sigma_1}\right) \tag{334}$$

$$= 1 - \Phi\left(\frac{R_{\rm c}-g}{b_1\sigma_1} - \frac{\widetilde{c}_1}{\sigma_1}\right) \tag{335}$$

**Solution to Problem 24, Compound interest** (p.23). See table 11, which explains that:

$420 interest is paid at the end of year 1.
$350 interest is paid at the end of year 2.
$210 interest is paid at the end of year 3.

| Year | Amount owed at beginning of year | Interest paid at end of year | Principal paid at end of year |
|---|---|---|---|
| 1 | 6,000 | 420 | 1000 |
| 2 | 5,000 | 350 | 2000 |
| 3 | 3,000 | 210 | 3000 |

Table 11: Solution to problem 24.

**Solution to Problem 25, Start up** (p.23). Let $I_k$ and $P_k$ denote the interest and principal payments at the end of year $k$. The revenue at the end of year $k$ is $R_k$. The PW with interest $i = 0.07$ is:

$$\text{PW} = \sum_{k=1}^{3}(1+i)^{-k}\left(-I_k - P_k + R_k\right) \tag{336}$$
$$= 1.07^{-1}(\underbrace{-420-1000+800}_{-620}) \tag{337}$$
$$+1.07^{-2}(\underbrace{-350-2000+3200}_{+850}) \tag{338}$$
$$+1.07^{-3}(\underbrace{-210-3000+7200}_{+3990}) \tag{339}$$
$$= -579.44 + 742.42 + 3257.03 \tag{340}$$
$$= +3420.01 \tag{341}$$

**Solution to Problem 26, Start up, continued** (p.23). The present worth is:

$$\text{PW}(R) = \sum_{k=1}^{N}(1+i)^{-k}\left(-I_k - P_k + R_k\right) \tag{342}$$

The robustness is defined as:

$$\widehat{h}(\text{PW}_\text{c}) = \max\left\{h:\ \left(\min_{R\in\mathcal{U}(h)} \text{PW}(R)\right) \ge \text{PW}_\text{c}\right\} \tag{343}$$

The inner minimum, denoted $m(h)$, is the inverse of the robustness function. $m(h)$ occurs for $R_k = \widetilde{R}_k - s_k h$:

$$m(h) = \sum_{k=1}^{N}(1+i)^{-k}\left(-I_k - P_k + \widetilde{R}_k - s_k h\right) \tag{344}$$
$$= \text{PW}(\widetilde{R}) - h\sum_{k=1}^{N}(1+i)^{-k}s_k \tag{345}$$

Equating this to $\text{PW}_\text{c}$ and solving for $h$ yields the robustness:

$$\widehat{h}(\text{PW}_\text{c}) = \frac{\text{PW}(\widetilde{R}) - \text{PW}_\text{c}}{\sum_{k=1}^{N}(1+i)^{-k}s_k} \tag{346}$$

or zero if this is negative.

**Solution to Problem 27, BCR of a measurement device** (p.23).

**(27a)** The BCR is defined as the ratio of discounted benefit to discounted cost:

$$\text{BCR} = \frac{\sum_{k=1}^{N}(1+i_b)^{-k}B}{\sum_{k=1}^{N}(1+i_c)^{-k}(I_k+P_k)} = QB \tag{347}$$

which defines the quantity $Q$.

**(27b)** The probability of violating the BCR requirement is:

$$\begin{aligned} P_{\rm f} &= \text{Prob}(\text{BCR} \le \text{BCR}_{\rm c}) && (348)\\ &= \text{Prob}\left(B \le \frac{\text{BCR}_{\rm c}}{Q}\right) && (349)\\ &= \int_0^{\text{BCR}_c/Q} \lambda \text{e}^{-\lambda B}\,\text{d}B && (350)\\ &= 1-\text{e}^{-\lambda\,\text{BCR}_c/Q} && (351)\end{aligned}$$

**(27c)** The info-gap model of uncertainty in $\lambda$ is:

$$\mathcal{U}(h) = \left\{\lambda:\ \lambda > 0,\ \left|\frac{\lambda-\widetilde{\lambda}}{\widetilde{\lambda}}\right| \le h\right\},\quad h \ge 0 \tag{352}$$

The robustness is defined as:

$$\widehat{h} = \max\left\{h:\ \left(\max_{\lambda\in\mathcal{U}(h)} P_{\rm f}(\lambda)\right) \le P_{\rm c}\right\} \tag{353}$$

The inner minimum, $m(h)$, which is the inverse of the robustness function, occurs when $\lambda$ is maximal at horizon of uncertainty $h$: $\lambda = (1+h)\widetilde{\lambda}$. Thus:

$$m(h) = 1-\text{e}^{-(1+h)\widetilde{\lambda}\,\text{BCR}_c/Q} \le P_{\rm c} \tag{354}$$

Thus the robustness is:

$$\widehat{h} = -1 - \frac{Q\ln(1-P_{\rm c})}{\widetilde{\lambda}\text{BCR}_{\rm c}} \tag{355}$$

or zero if this is negative.

**Solution to Problem 28, Loan and investment** (p.24).

**(28a)** The answer is in the fourth column of table 12, according to the following relation:

$$I = iP \tag{356}$$

where $i = 0.07$.

| Year | Amount owed at beginning of year, $P$ | Princ. paid at end of year | Interest paid at end of year, $I$ |
|---|---|---|---|
| 1 | 60,000 | 20,000 | 4,200 |
| 2 | 40,000 | 20,000 | 2,800 |
| 3 | 20,000 | 20,000 | 1,400 |

Table 12: Solution to problem 28a.

**(28b)** The return on the investment at the end of year $k$ is:

$$R_k = [60,000 - (k-1)20,000]i_{\rm inv} \tag{357}$$

where $i_{\rm inv} = 0.15$. The interest paid at the end of year $k$ is:

$$I_k = [60,000 - (k-1)20,000]i \tag{358}$$

where $i = 0.07$. The present worth is:

$$\text{PW} = \sum_{k=1}^{3}(1+i_{\rm inv})^{-k}(R_k - I_k) \tag{359}$$

$$= \sum_{k=1}^{3}(1+i_{\rm inv})^{-k}[60,000 - (k-1)20,000](i_{\rm inv} - i) \tag{360}$$

$$= \sum_{k=1}^{3}(1.15)^{-k}[60,000 - (k-1)20,000](0.08) \tag{361}$$

$$= 0.08\left[1.15^{-1}\times 60,000 + 1.15^{-2}\times 40,000 + 1.15^{-3}\times 20,000\right] \tag{362}$$

$$= 7,645.60 \tag{363}$$

**(28c)** Define:
$A =$ equal annual payments.
$P =$ principal of loan = $60,000.
$i =$ annual interest rate of loan = 0.07.
$K =$ number of years of loan = 3.
These are related as:

$$A = \frac{i(1+i)^K}{(1+i)^K - 1}P = 22,863 \tag{364}$$

Thus, the total interest paid is $3\times(22,863 - 20,000) = 8,589.30$. The total interest paid in part 28a was $4,200 + 2,800 + 1,400 = 8,400$ but spread out differently (less favorably) in time.

**(28d)** The definition of the PW is:

$$\text{PW} = \sum_{k=1}^{K}(1+i_{\rm inv})^{-k}I_k \tag{365}$$

The PW of the loan payments, $I_k$, is small if the discount rate (rate of return on the investment), $i_{\text{inv}}$, is large. If the rate of return on the investment is large, then distant interest payments have low present worth because, by the time we have to pay that interest, we will have earned lots of money.

We can understand this in more detail as follows. The basic idea is expressed by the following two arguments:

(1) The present worth of future money is LOW if the rate of return on investment is HIGH. That is, if the rate of return is LARGE, then $1 today is equivalent to many dollars in the future, or $1 in the future is equivalent to much less than $1 today. Likewise, if the rate of return is LARGE, then future debt is LESS meaningful in the present because MUCH money will be earned before that future debt is realized.

(2) The present worth of future money is LARGE if the rate of return on investment is LOW. That is, if the rate of return is LOW, then $1 today is equivalent to only slightly more than $1 in the future, or $1 in the future is equivalent to only slightly less than $1 today. Likewise, if the rate of return is LOW, then future debt is MORE meaningful in the present because LITTLE money will be earned before that future debt is realized.

In problem 28(d) you are asked to find the most negative present worth at horizon of uncertainty h. That is, you must find the condition for which the present worth of future debt is high (very negative). Thus case (2) applies: The present worth of debt is large (very negative) if the rate of return is low, because we won't earn much money before we have to pay that debt.

Stating these two arguments differently: The money-value of time is SMALL if the rate of return is small: A small increment of time has a small impact on money. The money-value of time is LARGE if the rate of return is large: A small increment of time has a large impact on money.

The most negative PW at horizon of uncertainty $h$ is:

$$m(h) = \min_{i_{\text{inv}} \in \mathcal{U}(h)} \text{PW} \tag{366}$$

The $I_k$'s in eq.(365) are negative so the minimum (most negative) PW occurs when the discount rate $i_{\text{inv}}$ is as small as possible: $i_{\text{inv}} = [\widetilde{i}_{\text{inv}} - sh]^+$:

$$m(h) = \sum_{k=1}^{K} \left(1 + [\widetilde{i}_{\text{inv}} - sh]^+\right)^{-k} I_k \tag{367}$$

**(28e)** See fig. 9 on p.99.

**(28f)** Eq.(11), p.24, implies this ranking of the estimated PW's:

$$\text{PW}(\widetilde{i}_{1,\text{inv}}) < \text{PW}(\widetilde{i}_{2,\text{inv}}) < 0 \tag{368}$$

Thus the first discount scheme has a more negative (less desirable) present worth of the interest payments. Thus, if there were no uncertainty, we would prefer the 2nd scheme.

However, Eq.(12), p.24, implies that the relative error for scheme 1 is lower than for scheme 2. This implies that the cost of robustness for scheme 1 is lower than the cost of robustness for scheme 2. Thus their robustness curves cross, as in fig 10, implying the potential for a reversal of preference between the two schemes. At very negative $\text{PW}_\text{c}$ we prefer scheme 1 which is more robust, while at less negative $\text{PW}_\text{c}$ we prefer scheme 2.

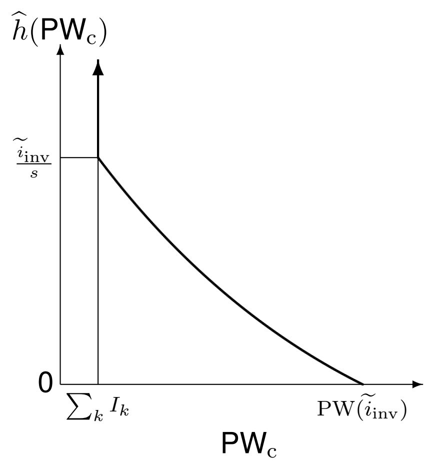


Figure 9: Solution for problem 28e.

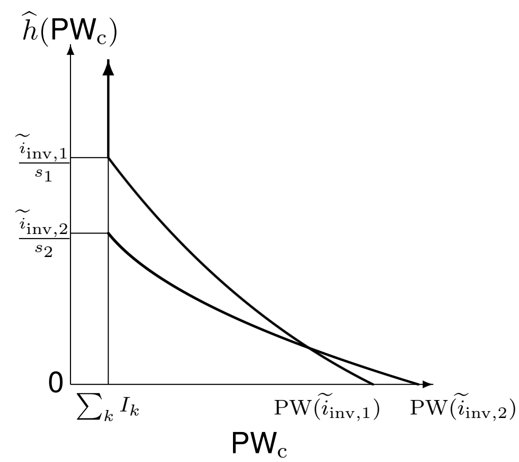


Figure 10: Solution for problem 28f.

**Solution to Problem 29, Future foreign earnings,** (p.25).

**(29a)** The real value at time 0, in \$'s, of the \$ payment $A_k$ at the end of year $k$ is, from eq.(272) (see the derivation there, p.86):

$$R_{0,k} = A_{0,k} = A_k \prod_{j=1}^{k} (1 + f_j)^{-1} \tag{369}$$

**(29b)** The foreign nominal value in year $k$ is:

$$A_{k,\text{for}} = r_k A_k \tag{370}$$

Eq.(369) applies to pesos when using the peso inflation rates, so the real value at time 0, in peso's, of the \$ payment $A_k$ after transfering to pesos at the end of year $k$ is:

$$R_{0,k,\text{for}} = A_{0,k,\text{for}} = A_{k,\text{for}} \prod_{j=1}^{k} (1 + \phi_j)^{-1} \tag{371}$$

$$= r_k A_k \prod_{j=1}^{k} (1 + \phi_j)^{-1} \tag{372}$$

**(29c)** The uncertainty in the \$ inflation is irrelevant, because the \$ inflation rate does not influence the peso value. This is because the value of $A_k$ is fixed in dollars. However, the uncertainty in the exchange rate is relevant. The robustness is:

$$\widehat{h} = \max \left\{ h : \left( \min_{r_k \in \mathcal{U}(h)} R_{0,k,\text{for}}(r_k) \right) \geq R_{\text{c}} \right\} \tag{373}$$

Let $m(h)$ denote the inner minimum, which occurs for the lowest exchange rate at horizon of uncertainty $h$: $r_k = (1-h)^{+} \widetilde{r}_k$. From eq.(372):

$$m(h) = (1-h)^{+} R_{0,k,\text{for}}(\widetilde{r}_k) \tag{374}$$

Equating this to $R_{\text{c}}$ and solving for $h$ yields the robustness:

$$\widehat{h} = 1 - \frac{R_{\text{c}}}{R_{0,k,\text{for}}(\widetilde{r}_k)} \tag{375}$$

or zero if this is negative.

**Solution to Problem 30, Interest payment,** (p.26).

**(30a)** The solution appears in column 3 and last entry of column 4 of table 13. Examples of 2nd column calculations:

row $k = 2$: Amount owed at beginning of year 2 is $40,000(1 + 0.05) - 11,000 = 31,000$.
row $k = 3$: Amount owed at beginning of year 3 is $31,000(1 + 0.05) - 11,000 = 21,550$.

| Year | Amount owed at beginning of year | Interest accrued for year | Payment at end of year |
|---|---|---|---|
| 1 | 40,000 | 2,000 | 11,000 |
| 2 | 31,000 | 1,550 | 11,000 |
| 3 | 21,550 | 1,077.5 | 11,000 |
| 4 | 11,627.5 | 581.375 | 12,208.875 |

Table 13: Solution to problem 30a.

**(30b)** Each of the 3 missed investments was $I =$\$11,000. The annual rate of return is $i = 0.07$.

The value of the first investment at the end of year 4, made at the start of year 2, would have been:

$$(1 + i)^3 I = 1.225 \times 11,000 = \$13,475.473 \tag{376}$$

The value of the second investment at the end of year 4, made at the start of year 3, would have been:

$$(1 + i)^2 I = 1.1449 \times 11,000 = \$12,593.9 \tag{377}$$

The value of the third investment at the end of year 4, made at the start of year 4, would have been:

$$(1 + i)^1 I = 1.07 \times 11,000 = \$11,770 \tag{378}$$

**(30c)** The solution combines eqs.(376)–(378). If the rate of return is $i$, then the total return at the end of year 3, from the 3 combined investments, is:

$$R = I \sum_{j=1}^{3} (1 + i)^j \tag{379}$$

The minimum occurs when $i$ is as small as possible at horizon of uncertainty $h$: $i = \min[0, \ \widetilde{i} - h]$. Call this minimum $m(h)$. Thus:

$$m(h) = \begin{cases} I \sum_{j=1}^{3} (1 + \widetilde{i} - h)^j & \text{if } h \le \widetilde{i} \\ 3I & \text{else} \end{cases} \tag{380}$$

**Solution to Problem 31, Random returns,** (p.27).

**(31a)** The PW is:

$$\text{PW} = \sum_{k=1}^{N}(1+i)^{-k}R = \underbrace{\frac{1-(1+i)^{-N}}{i}}_{\eta} R \quad (381)$$

which defines the quantity $\eta$. The probability that the PW is less than $\text{PW}_\text{c}$ is:

$$P_\text{f} = \text{Prob}(\text{PW} \le \text{PW}_\text{c}) = \text{Prob}(\eta R \le \text{PW}_\text{c}) = \text{Prob}\left(R \le \frac{\text{PW}_\text{c}}{\eta}\right) \quad (382)$$

$$= \begin{cases} 0 & \text{if } \frac{\text{PW}_\text{c}}{\eta} \le R_1 \\ \frac{(\text{PW}_\text{c}/\eta) - R_1}{R_2 - R_1} & \text{if } R_1 < \frac{\text{PW}_\text{c}}{\eta} < R_2 \\ 1 & \text{if } \frac{\text{PW}_\text{c}}{\eta} \ge R_2 \end{cases} \quad (383)$$

**(31b)** The robustness is defined as:

$$\widehat{h} = \max\left\{h : \left(\max_{R_2 \in \mathcal{U}(h)} P_\text{f}\right) \le P_\text{c}\right\} \quad (384)$$

Denote the inner maximum by $m(h)$. From the middle line of eq.(383) we see that $m(h)$ occurs when $R_2$ is as small as possible at horizon of uncertainty $h$: $R_2 = \max[R_1, \ \widetilde{R}_2 - s_2 h]$.

Consider first the case:

$$\widetilde{R}_2 - s_2 h \ge R_1 \iff h \le \frac{\widetilde{R}_2 - R_1}{s_2} \quad (385)$$

We find:

$$m(h) = \frac{(\text{PW}_\text{c}/\eta) - R_1}{\widetilde{R}_2 - s_2 h - R_1} \quad (386)$$

Equating this to $P_\text{c}$ and solving for $h$ yields the robustness:

$$m(h) = P_\text{c} \iff \frac{\widetilde{R}_2 - s_2 h - R_1}{(\text{PW}_\text{c}/\eta) - R_1} = \frac{1}{P_\text{c}} \iff \widetilde{R}_2 - s_2 h - R_1 = \frac{(\text{PW}_\text{c}/\eta) - R_1}{P_\text{c}} \quad (387)$$

Thus:

$$\widehat{h} = \frac{\widetilde{R}_2 - R_1}{s_2} - \frac{(\text{PW}_\text{c}/\eta) - R_1}{s_2 P_\text{c}} \quad (388)$$

or zero if this is negative. We see that the condition in eq.(385) holds for all non-negative robustnesses. Note that the robustness is zero if:

$$P_\text{c} \ge \frac{(\text{PW}_\text{c}/\eta) - R_1}{\widetilde{R}_2 - R_1} \quad (389)$$

where the RHS is the nominal value of $P_\text{f}$.

**(31c)** The nominal peso value obtained by exchanging $\$R_k$ is:

$$A_k = r_k R_k \quad (390)$$

The real value at the start of year 1 of this nominal peso value is:

$$\pi_k = A_k \prod_{j=1}^{k}(1+f_j)^{-1} = r_k R_k \prod_{j=1}^{k}(1+f_j)^{-1} \quad (391)$$

**(31d)** Let $m(h)$ denote the lowest value of $\pi_k$ at horizon of uncertainty $h$. From eq.(391) we see that this minimum occurs when $r_k$ is small and each $f_j$ is large. Hence:

$$m(h) = (1-h)^{+} \widetilde{r}_k R_k \prod_{j=1}^{k}\left(1+(1+h)\widetilde{f}_j\right)^{-1} \quad (392)$$

**Solution to Problem 32, Loan with interest,** (p.28).

**(32a)**

| Year | Loan | Amount owed at beginning of year | Interest accrued for year | Payment at end of year |
|---|---|---|---|---|
| 1 | 5,000 | 5,000 | 300 | 300 |
| 2 | 3,000 | 8,000 | 480 | 480 |
| 3 | 1,000 | 9,000 | 540 | 540 |

Table 14: Solution to problem 32a.

**(32b)**

| Year | Loan | Amount owed at beginning of year | Interest accrued for year | Payment at end of year |
|---|---|---|---|---|
| 1 | 5,000 | 5,000 | 300 | 0 |
| 2 | 3,000 | 8,300 | 498 | 0 |
| 3 | 1,000 | 9,798 | 587.88 | 10,385.88 |

Table 15: Solution to problem 32b.

Explanation of line 1: 300 = 5,000 $\times$ 0.06.
Explanation of line 2: 8,300 = 5,000 + 3,000 + 300. 498 = 8,300 $\times$ 0.06.
Explanation of line 3: 9,798 = 8,300 + 1,000 + 300 + 498. 587.88 = 9,798 $\times$ 0.06.
Total payment at end of year 3 is 10,385.88 = 9,798 + 587.88

A different route to the same answer is to treat each of the 3 loans separately:
Total payment is: 10,385.88 = $5,000 \times (1+i)^3 + 3,000 \times (1+i)^2 + 1000 \times (1+i)$.

**(32c)**

$$\text{PW} = -S + \sum_{k=1}^{N} (1+i)^{-k} \widetilde{R} = -S + \widetilde{R} \underbrace{\frac{1-(1+i)^{-N}}{i}}_{Q} \tag{393}$$

**(32d)** The definition of the robustness is:

$$\widehat{h} = \max \left\{ h : \left( \min_{R \in \mathcal{U}(h)} \text{PW}(R) \right) \geq \text{PW}_{\text{c}} \right\} \tag{394}$$

Let $m(h)$ denote the inner minimum which occurs for $R = (1-h)^{+} \widetilde{R}$.

For $h \leq 1$:

$$m(h) = -S + (1-h)^{+} Q \widetilde{R} \quad \Longrightarrow \quad \widehat{h} = \frac{Q\widetilde{R} - S - \text{PW}_{\text{c}}}{Q\widetilde{R}} \tag{395}$$

where $Q$ is defined in eq.(393).

For $h > 1$:

$$m(h) = -S \tag{396}$$

Combining eqs.(395) and (396) we obtain:

$$\widehat{h} = \begin{cases} \infty & \text{if } \mathsf{PW}_{\rm c} \le -S \\ \dfrac{-S + Q\widetilde{R} - \mathsf{PW}_{\rm c}}{Q\widetilde{R}} & \text{if } -S < \mathsf{PW}_{\rm c} \le -S + Q\widetilde{R} \\ 0 & \text{if } -S + Q\widetilde{R} < \mathsf{PW}_{\rm c} \end{cases} \tag{397}$$

Note that $\mathsf{PW}(\widetilde{R}) = -S + Q\widetilde{R}$.

**(32e)**

$$P_{\rm f} = \mathsf{Prob}(R \le R_{\rm c}) = \int_0^{R_{\rm c}} p(R)\,{\rm d}R = \boxed{1 - {\rm e}^{-\lambda R_{\rm c}}} \tag{398}$$

**(32f)** The definition of the robustness is:

$$\widehat{h} = \max\left\{ h: \ \left( \max_{\lambda \in \mathcal{U}(h)} P_{\rm f}(\lambda) \right) \le P_{\rm fc} \right\} \tag{399}$$

Let $m(h)$ denote the inner maximum which occurs for $\lambda = (1+h)\widetilde{\lambda}$. Thus:

$$m(h) = 1 - {\rm e}^{-(1+h)\widetilde{\lambda} R_{\rm c}} \le P_{\rm c} \quad \Longleftrightarrow \quad \boxed{\widehat{h} = -1 - \frac{1}{\widetilde{\lambda} R_{\rm c}} \ln(1 - P_{\rm c})} \tag{400}$$

or zero.

**Solution to Problem 33, Earnings and investment,** (p.29).

**(33a)** If $\$F$ is earned at the end of year $k$, then the present worth of this income, with interest $i$, is $PW = (1+i)^{-k}F$. In our case, $k = 3$, $i = 0.06$ and $F = 150,000$. Thus PW $= 1.06^{-3} \times 150,000 =$ $\boxed{\text{125,942.90.}}$

**(33b)** If $\$F$ is earned at the end of year 1, then the present worth of this income, with interest $i$, is $PW = (1+i)^{-1}F$.

Similarly, if $\$F$ is earned at the end of year 2, then the present worth of this income, with interest $i$, is $PW = (1+i)^{-2}F$.

The present worth of the entire income stream is the sum of the individual PWs:

$$PW = \sum_{k=1}^{N}(1+i)^{-k}F = \frac{1-(1+i)^{-N}}{i}F \tag{401}$$

For $F =$ 30,000, $i =$ 0.06 and $N =$ 5 we find $\boxed{PW = 126,370.91}$.

**(33c)** The peso interest rate is $i_p =$ 0.06. The US inflation rate is $f_{us} =$ 0.04.

The nominal dollar earnings at the end of year $k$ are $(1+f_{us})^kF$.

The peso value of these earnings are $r(1+f_{us})^kF$.

The sum of the peso holdings at the end of $N$ years is:

$$T_p = rF\sum_{k=1}^{N}(1+f_{us})^k = rF\frac{(1+f_{us})^{N+1}-(1+f_{us})}{f_{us}} \tag{402}$$

The present worth of the sum of pesos is:

$$PW = (1+i_p)^{-N}T_p = rF(1+i_p)^{-N}\frac{(1+f_{us})^{N+1}-(1+f_{us})}{f_{us}} \tag{403}$$

The numerical value is:

$$PW = 20 \times 30000 \times 1.06^{-5}\frac{1.04^6-1.04}{0.04} = \boxed{2,525,572} \tag{404}$$

The following procedure is **not** correct because it presumes that each yearly earning is re-invested.

The nominal dollar earnings at the end of year $k$ are $(1+f_{us})^kF$. (Correct)

The peso value of these earnings are $r(1+f_{us})^kF$. (Correct)

Hence the present worth in pesos is:

$$PW = \sum_{k=1}^{N}(1+i_p)^{-k}r(1+f_{us})^kF = rF\sum_{k=1}^{N}\left(\frac{1+f_{us}}{1+i_p}\right)^k = rF\sum_{k=1}^{N}\rho^k = rF\frac{\rho^{N+1}-\rho}{\rho-1} \tag{405}$$

$\rho =$ 1.04 / 1.06 = 0.982231. Hence the PW is:

$$PW = 20 \times 30,000 \times \frac{0.982231^6 - 09.82231}{0.982231-1} = 2,834,382.905 \tag{406}$$

**(33d)** The peso and US interest rates are $i_p =$ 0.08 and $i_{us} =$ 0.06. The US inflation rate is $f_{us} =$ 0.04. The real annual earnings are $F =$ \$30,000 and the duration is $N =$ 5 years.

The nominal dollar sum earned at the end of year $k$ is $(1+f_{us})^kF$.

The nominal peso sum obtained at the end of year $k$ is $r(1+f_{us})^kF$.

The pesos obtained at the end of year $k$ earn interest $i_p$ for $N-k$ years. Thus, at the end of the $N$ years, the pesos from year $k$ obtain the value $(1+i_p)^{N-k}r(1+f_{us})^kF$.

Thus the total peso balance at the end of $N$ years is:

$$\begin{aligned} T_p &= \sum_{k=1}^{N}(1+i_p)^{N-k} r(1+f_{us})^k F & (407)\\ &= (1+i_p)^N rF \sum_{k=1}^{N}\left(\frac{1+f_{us}}{1+i_p}\right)^k & (408)\\ &= (1+i_p)^N rF \sum_{k=1}^{N}\rho^k & (409)\\ &= (1+i_p)^N rF \frac{\rho^{N+1}-\rho}{\rho-1} & (410)\\ &= \boxed{3.9417\times 10^6} & (411) \end{aligned}$$

The nominal dollar value obtained by exchanging $T_p$ pesos is:

$$\boxed{A_{us} = \frac{T_p}{r} = 1.9709\times 10^5} \quad (412)$$

The real value of $A_{us}$ is:

$$R_{us} = (1+f_{us})^{-N} A_{us} \quad (413)$$

The PW of these dollars is:

$$\boxed{\mathrm{PW} = (1+i_{us})^{-N}(1+f_{us})^{-N} A_{us} = 1.2105\times 10^5} \quad (414)$$

**(33e)** From eq.(401), the PW is:

$$\mathrm{PW} = \sum_{k=1}^{N}(1+i)^{-k} F = \frac{1-(1+i)^{-N}}{i} F \quad (415)$$

The definition of the robustness is:

$$\widehat{h} = \max\left\{h : \left(\min_{F\in\mathcal{U}(h)} \mathrm{PW}\right) \ge \mathrm{PW_c}\right\} \quad (416)$$

Let $m(h)$ denote the inner minimum, which occurs for $F = \widetilde{F} - sh$ so:

$$m(h) = \underbrace{\frac{i-(1+i)^{-N}}{i}}_{\rho}(\widetilde{F} - sh) \ge \mathrm{PW_c} \quad (417)$$

Thus the robustness is:

$$\boxed{\widehat{h} = \frac{1}{s}\left(\widetilde{F} - \frac{\mathrm{PW_c}}{\rho}\right)} \quad (418)$$

or zero if this is negative.

**(33f)** We will use the robustness function from eq.(418). The robustness curves for the two plans intersect at $\mathrm{PW}_{\times}$:

$$\widehat{h}_1 = \widehat{h}_2 \implies \frac{1}{s_1}\left(\widetilde{F}_1 - \frac{\mathrm{PW}_{\times}}{\rho}\right) = \frac{1}{s_2}\left(\widetilde{F}_2 - \frac{\mathrm{PW}_{\times}}{\rho}\right) \implies \frac{\widetilde{F}_1}{s_1} - \frac{\widetilde{F}_2}{s_2} = \frac{\mathrm{PW_c}}{\rho}\left(\frac{1}{s_1} - \frac{1}{s_2}\right) \implies \quad (419)$$

$$\boxed{\mathrm{PW}_{\times} = \frac{\rho \widetilde{F}_1 s_1 \left(\dfrac{s_2}{s_1} - \dfrac{\widetilde{F}_2}{\widetilde{F}_1}\right)}{s_2 - s_1}} \quad (420)$$

We conclude from the two inequalities in eq.(20) that $s_2 < s_1$. Hence the term on the right in eq.(420) is positive.

The horizontal intercept of the robustness curve is at $\mathrm{PW_c} = \rho\widetilde{F}$. Thus plan 1 is more robust than, and hence preferred over, plan 2 if and only if:

$$\boxed{\mathrm{PW}_\times < \mathrm{PW_c} < \rho\widetilde{F}_1} \tag{421}$$

The economic interpretation for this preference reversal is as follows.

First, $\widetilde{F}_1 > \widetilde{F}_2$ in eq.(20) implies that plan 1 nominally has greater present worth than plan 2:

$$\mathrm{PW}(\widetilde{F}_1) = \rho\widetilde{F}_1 > \rho\widetilde{F}_2 = \mathrm{PW}(\widetilde{F}_2) \tag{422}$$

However, $\frac{s_2}{s_1} < \frac{\widetilde{F}_2}{\widetilde{F}_1}$ in eq.(20) implies that plan 2 is less uncertain than plan 1, and this more than compensates for plan 1's nominally greater PW. Consequently, the cost of robustness for plan 2 is lower than for plan 1, and their robustness curves cross. We will robust-prefer plan 1 only at large $\mathrm{PW_c}$ where its nominal predominance still entails greater robustness than plan 2.

**Solution to Problem 34, Great idea for a start-up**(p.30).

**(34a)** The present worth if the project succeeds and if it fails is:

$$\mathsf{PW}_{\text{success}} = -C + (1+i)^{-N} R \tag{423}$$

$$\mathsf{PW}_{\text{fail}} = -C \tag{424}$$

The expected value of the present worth is:

$$\mathsf{E}(\mathsf{PW}) = p\mathsf{PW}_{\text{success}} + (1-p)\mathsf{PW}_{\text{fail}} = -C + p(1+i)^{-N} R \tag{425}$$

**(34b)** The definition of the robustness is:

$$\widehat{h} = \max \left\{ h : \left( \min_{R \in \mathcal{U}(h)} \mathsf{E}(\mathsf{PW}) \right) \geq \mathsf{PW}_{\text{c}} \right\} \tag{426}$$

Let $m(h)$ denote the inner minimum, which occurs for $R = \widetilde{R} - sh$ so:

$$m(h) = -C + p(1+i)^{-N}(\widetilde{R} - sh) \geq \mathsf{PW}_{\text{c}} \implies \boxed{\widehat{h} = \frac{-C + p(1+i)^{-N}\widetilde{R} - \mathsf{PW}_{\text{c}}}{sp(1+i)^{-N}} = \frac{\mathsf{PW}(\widetilde{R}) - \mathsf{PW}_{\text{c}}}{sp(1+i)^{-N}}} \tag{427}$$

or zero if this is negative.

**Solution to Problem 35, Artistic restoration** (p.31).

**(35a)** The discounted PW of the benefit, $B$, and the cost, $C$, and the BCR, are:

$$B = (1+i_b)^{-N} R \tag{428}$$

$$C = \sum_{k=1}^{N} (1+i_c)^{-k} A = \delta_f(i_c) A \tag{429}$$

$$\text{BCR}_1 = \frac{B}{C} = \frac{(1+i_b)^{-N} R}{\delta_f(i_c) A} \tag{430}$$

**(35b)** The discounted PW of the benefit, $B$, and the cost, $C$, and the BCR, are:

$$B = \sum_{k=1}^{N} (1+i_b)^{-k} \frac{R}{N} = \delta_f(i_b) \frac{R}{N} \tag{431}$$

$$C = \sum_{k=1}^{N} (1+i_c)^{-k} A = \delta_f(i_c) A \tag{432}$$

$$\text{BCR}_2 = \frac{B}{C} = \frac{\delta_f(i_b) R}{\delta_f(i_c) A N} \tag{433}$$

**(35c)** The ratio of the BCR of the 2nd to the 1st option is:

$$\frac{\text{BCR}_2}{\text{BCR}_1} = \frac{\delta_f(i_b)}{N(1+i_b)^{-N}} = \frac{\sum_{k=1}^{N}(1+i_b)^{-k}}{N(1+i_b)^{-N}} > \frac{N(1+i_b)^{-N}}{N(1+i_b)^{-N}} = 1 \tag{434}$$

The economic or time-value reason that the 2nd option has greater BCR is that its benefit is discounted less due to benefits accruing throughout the restoration period, while its cost has the same time profile.

**Solution to Problem 36, Earnings and investments** (p.32).

**(36a)** Present and future worth are related as PW $= (1+i)^{-k}F$, so $F$ must equal:

$$F = (1+i)^k \text{PW} = 1.07^5 \times 100,000 = \boxed{140,255.17} \tag{435}$$

**(36b)**

Year $k = 1$: $F$ is earned at the end of year 1. The present worth of this is:

$$\text{PW}_1 = (1+i)^{-1}F = 1.03^{-1}F = 0.97087F \tag{436}$$

Year $k = 2$: $1.1F$ is earned at the end of year 2. The present worth of this is:

$$\text{PW}_2 = (1+i)^{-2}1.1F = 1.03^{-2} \times 1.1F = 1.036855F \tag{437}$$

Year $k = 3$: $1.2F$ is earned at the end of year 3. The present worth of this is:

$$\text{PW}_3 = (1+i)^{-3}1.2F = 1.03^{-3} \times 1.2F = 1.0981699F \tag{438}$$

Thus:

$$\text{PW} = \text{PW}_1 + \text{PW}_2 + \text{PW}_3 = 3.10589F = 250,000 \quad \Longrightarrow \quad \boxed{F = 80,492.10} \tag{439}$$

**(36c)** Consider year $k$. The real earnings at the end of the year are $F = 50,000$ pesos. The annual peso inflation rate is $f_p = 0.07$. Thus the nominal peso earnings at the end of year $k$ are $(1+f_p)^k F$. This is invested for $N-k$ years, $N = 3$, at nominal annual interest of $i = 0.05$. Thus the nominal peso value of the year-$k$ earnings, at the end of year $N$, is:

$$A_{k,p} = (1+i)^{N-k}(1+f_p)^k F \tag{440}$$

This is exchanged to dollars at the rate of $r$ pesos/dollar, so the nominal dollar value, at the end of year $N$, of the earnings from year $k$, are:

$$A_{k,d} = A_{k,p}/r = (1+i)^{N-k}(1+f_p)^k F/r \tag{441}$$

These yearly nominal dollar values are:

$$\begin{aligned} A_{1,d} &= 1.05^2 \times 1.07^1 \times 50,000/30 = 1,966.13 &(442)\\ A_{2,d} &= 1.05^1 \times 1.07^2 \times 50,000/30 = 2,003.58 &(443)\\ A_{3,d} &= 1.05^0 \times 1.07^3 \times 50,000/30 = 2,041.74 &(444)\end{aligned}$$

The total nominal dollar value obtained at the end of year $N$ is:

$$A_d = A_{1,d} + A_{2,d} + A_{3,d} = \boxed{6,011.45} \tag{445}$$

**(36d)** The present worth of the earnings of year $k$ is:

$$\text{PW}_k = (1+i)^{-k}[1+(k-1)x]F \tag{446}$$

The present worth of the sum of the earnings is:

$$\begin{aligned} \text{PW} &= \sum_{k=1}^{N} \text{PW}_k = \sum_{k=1}^{N}(1+i)^{-k}[1+(k-1)x]F = F\underbrace{\sum_{k=1}^{N}(1+i)^{-k}}_{P_0} + xF\underbrace{\sum_{k=1}^{N}(k-1)(1+i)^{-k}}_{P_1} &(447)\\ &= P_0F + P_1Fx &(448)\end{aligned}$$

The definition of the robustness is:

$$\widehat{h} = \max\left\{h : \left(\min_{x\in\mathcal{U}(h)} \text{PW}\right) \geq \text{PW}_{\text{c}}\right\} \tag{449}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function, and occurs for $x = \widetilde{x} - sh$:

$$m(h) = P_0F + P_1F(\widetilde{x} - sh) \geq \text{PW}_{\text{c}} \quad\Longrightarrow\quad \boxed{\widehat{h} = \frac{P_0F + P_1F\widetilde{x} - \text{PW}_{\text{c}}}{P_1Fs} = \frac{\text{PW}(\widetilde{x}) - \text{PW}_{\text{c}}}{P_1Fs}} \tag{450}$$

or zero if this is negative.

**(36e)** With $F = F_{36b}$ we know that PW$(\widetilde{x}) = 250,000$. Hence $\widehat{h}(\text{PW}_{\text{c}} = 250,000) = 0$ by the zeroing theorem.

**Solution to Problem 37, Investment and expected returns** (p.33).

**(37a)** The return of project $j$, if it succeeds, is $r_j q_j / Q$. The present worth of this return is $\text{PW}_j = (1+i)^{-n_j} r_j q_j / Q$. The return is zero if the project fails. The probability of success is $p_j = q_j / Q$. Thus the expected return of the investment is:

$$\text{E(PW)} = p_1 \text{PW}_1 + p_2 \text{PW}_2 = \boxed{(1+i)^{-n_1} r_1 \frac{q_1^2}{Q} + (1+i)^{-n_2} r_2 \frac{(Q-q_1)^2}{Q}} \tag{451}$$

**(37b)** Define $\rho_j = (1+i)^{-n_j} r_j$, so:

$$\text{E(PW)} = \rho_1 \frac{q_1^2}{Q} + \rho_2 \frac{(Q-q_1)^2}{Q} \tag{452}$$

Thus:

$$\frac{\partial \text{E(PW)}}{\partial q_1} = \frac{2\rho_1 q_1}{Q} - \frac{2\rho_2 (Q-q_1)}{Q} \tag{453}$$

Assuming that $\rho_j > 0$, the slope is negative for small $q_1$, and positive for large $q_1$. Thus E(PW) vs. $q_1$ is U-shaped. Note that:

$$\text{E[PW}(q_1 = 0)] = 2\rho_2 Q \quad \text{and} \quad \text{E[PW}(q_1 = Q)] = 2\rho_1 Q \tag{454}$$

Consequently, the allocation that maximizes the expected worth is:

$$\boxed{q_1 = Q \text{ if } \rho_1 > \rho_2 \quad \text{and} \quad q_1 = 0 \text{ if } \rho_1 < \rho_2} \tag{455}$$

**(37c)** The definition of the robustness is:

$$\widehat{h}(\text{PW}_\text{c}) = \max \left\{ h : \left( \min_{r \in \mathcal{U}(h)} \text{E(PW)} \right) \geq \text{PW}_\text{c} \right\} \tag{456}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness, and occurs for $r_j = \widetilde{r}_j - s_j h$. From eq.(451) we obtain:

$$m(h) = (1+i)^{-n_1} (\widetilde{r}_1 - s_1 h) \frac{q_1^2}{Q} + (1+i)^{-n_2} (\widetilde{r}_2 - s_2 h) \frac{(Q-q_1)^2}{Q} \tag{457}$$

$$= \text{E[PW}(\widetilde{r})] - h\text{E[PW}(s)] \geq \text{PW}_\text{c} \tag{458}$$

where:

$$\text{E[PW}(\widetilde{r})] = (1+i)^{-n_1} \widetilde{r}_1 \frac{q_1^2}{Q} + (1+i)^{-n_2} \widetilde{r}_2 \frac{(Q-q_1)^2}{Q} \tag{459}$$

$$\text{E[PW}(s)] = (1+i)^{-n_1} s_1 \frac{q_1^2}{Q} + (1+i)^{-n_2} s_2 \frac{(Q-q_1)^2}{Q} \tag{460}$$

Hence:

$$\boxed{\widehat{h}(\text{PW}_\text{c}) = \frac{\text{E[PW}(\widetilde{r})] - \text{PW}_\text{c}}{\text{E[PW}(s)]}} \tag{461}$$

or zero if this is negative. It is evident that the allocation in eq.(455) does not necessarily maximize the robustness.

**(37d)** The present worth of the project is $\text{PW} = (1+i)^{-n} R$. The probability of failure is:

$$P_\text{f} = \text{Prob}(\text{PW} \leq \text{PW}_\text{c}) = \text{Prob}\left((1+i)^{-n} R \leq \text{PW}_\text{c}\right) = \text{Prob}(R \leq \underbrace{(1+i)^n \text{PW}_\text{c}}_{R_\text{c}}) \tag{462}$$

$$= \int_0^{R_\text{c}} \lambda \text{e}^{-\lambda R} \, \text{d}R = \boxed{1 - \text{e}^{-\lambda (1+i)^n PW_\text{c}}} \tag{463}$$

**(37e)** The definition of the robustness is:

$$\widehat{h}(\mathsf{PW}_{\rm c}) = \max\left\{h:\ \left(\max_{\lambda\in\mathcal{U}(h)} P_{\rm f}\right)\le P_{\rm fc}\right\} \tag{464}$$

Let $m(h)$ denote the inner maximum, which is the inverse of the robustness, and, in light of eq.(463), occurs for $\lambda=\widetilde{\lambda}+sh$, so:

$$m(h) = 1-\mathsf{e}^{-(\widetilde{\lambda}+sh)(1+i)^n PW_c} \le P_{\rm fc} \quad\Longrightarrow\quad -\ln(1-P_{\rm fc}) = (\widetilde{\lambda}+sh)(1+i)^n\mathsf{PW}_{\rm c} \tag{465}$$

Hence:

$$\boxed{\widehat{h}(P_{\rm fc}) = \frac{1}{s}\left(\frac{-\ln(1-P_{\rm c})}{(1+i)^n\mathsf{PW}_{\rm c}}-\widetilde{\lambda}\right)} \tag{466}$$

or zero if this is negative.

**(37f)** The discounted total cost and benefit are:

$$C = S+\sum_{j=1}^{N}(1+i_c)^{-j}c_j \tag{467}$$

$$B = \sum_{j=1}^{N}(1+i_b)^{-j}b_j \tag{468}$$

Thus the benefit-cost ratio is:

$$\mathsf{BCR} = \frac{\displaystyle\sum_{j=1}^{N}(1+i_b)^{-j}b_j}{\displaystyle S+\sum_{j=1}^{N}(1+i_c)^{-j}c_j} \tag{469}$$

In the special case of constant cost and benefit, eqs.(467) and (469) become:

$$C = S+\frac{1-(1+i_c)^{-N}}{i_c}c \tag{470}$$

$$B = \frac{1-(1+i_b)^{-N}}{i_b}b \tag{471}$$

$$\mathsf{BCR} = \frac{\dfrac{1-(1+i_b)^{-N}}{i_b}b}{S+\dfrac{1-(1+i_c)^{-N}}{i_c}c} \tag{472}$$

| Year | Debt at start of year | Interest accrued in year | Payment at end of year |
|---|---|---|---|
| 1 | 10,000 | 500 | 2,500 |
| 2 | 8,000 | 400 | 2,500 |
| 3 | 5,900 | 295 | 2,500 |
| 4 | 3,695 | 184.75 | 2,500 |
| 5 | 1,379.75 | **68.9875** | **1448.7375** |

Table 16: Solution of problem 38a. Currency is $.

**Solution to Problem 38, Income and uncertainty** (p.34).

**(38a)** We solve this problem one year at a time, as explained in the table 16. The payment in the last year is $1448.7375. The interest that accrues in the 5th year is $68.9875.

**(38b)** The present worth of the income stream is:

$$\text{PW} = \sum_{k=1}^{N}(1+i)^{-k}A = \frac{1-(1+i)^{-N}}{i}A \tag{473}$$

With $N = 10$, $i = 0.04$ and $A = 15,000$ we find:

$$\text{PW} = 8.11089 \times 15,000 = \boxed{121{,}663.44} \tag{474}$$

**(38c)** The present worth of the cash flow is:

$$\text{PW} = \sum_{k=1}^{N}(1+i)^{-k}(A-C) = \underbrace{\frac{1-(1+i)^{-N}}{i}}_{\delta}(A-C) \tag{475}$$

which defines the discount factor $\delta$. The robustness is defined as:

$$\widehat{h}(P_{\rm c}) = \max\left\{h:\ \left(\min_{A,C\in\mathcal{U}(h)}(A-C)\delta\right) \geq P_{\rm c}\right\} \tag{476}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function. This inner minimum occurs for $A = \widetilde{A} - s_A h$ and $C = \widetilde{C} + s_C h$. Thus:

$$m(h) = \left[\widetilde{A} - \widetilde{C} - h(s_A + s_C)\right]\delta\ \geq\ P_{\rm c} \quad\Longrightarrow\quad \boxed{\widehat{h}(P_{\rm c}) = \frac{(\widetilde{A}-\widetilde{C})\delta - P_{\rm c}}{(s_A+s_C)\delta}} \tag{477}$$

or zero if this is negative.

**(38d)** The future worth of the investment is:

$$F = (1+i)^N A \tag{478}$$

The probability of satisfying the future-worth criterion is:

$$P_{\rm s} = \text{Prob}(F \geq F_{\rm c}) = \text{Prob}\left((1+i)^N A \geq F_{\rm c}\right) = \text{Prob}\left(1+i \geq \left(\frac{F_{\rm c}}{A}\right)^{1/N}\right) \tag{479}$$

$$= \text{Prob}\left(i \geq \underbrace{\left(\frac{F_{\rm c}}{A}\right)^{1/N} - 1}_{i_c}\right) = \int_{i_c}^{\infty} \lambda \text{e}^{-\lambda i}\,\text{d}i = \boxed{\text{e}^{-\lambda i_c}} \tag{480}$$

where $i_c$ is defined in eq.(480).

**(38e)** The robustness is defined as:

$$\widehat{h}(P_c) = \max \left\{ h : \left( \min_{\lambda \in \mathcal{U}(h)} P_s(\lambda) \right) \geq P_c \right\} \tag{481}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function and occurs for $\lambda = (1+h)\widetilde{\lambda}$. Thus:

$$m(h) = \mathrm{e}^{-(1+h)\widetilde{\lambda} i_c} \geq P_c \implies (1+h)\widetilde{\lambda} i_c = -\ln P_c \implies \boxed{\widehat{h}(P_c) = -\frac{\ln P_c}{\widetilde{\lambda} i_c} - 1} \tag{482}$$

or zero if this is negative.

**(38f)** Option 1 guarantees a future worth of exactly $F_1$. This can be expressed as the following robustness function:

$$\widehat{h}_1(F_c) = \begin{cases} \infty & \text{if } F_c \leq F_1 \\ 0 & \text{else} \end{cases} \tag{483}$$

Comparing $\widehat{h}_1(F_c)$ with $\widehat{h}_2(F_c)$ in eq.(28), p.34, we see:

$$\widehat{h}_2(F_c) > \widehat{h}_1(F_c) \quad \text{if} \quad F_1 < F_c < 2F_1 \tag{484}$$

$$\widehat{h}_2(F_c) < \widehat{h}_1(F_c) \quad \text{if} \quad F_c \leq F_1 \tag{485}$$

$$\widehat{h}_2(F_c) = \widehat{h}_1(F_c) \quad \text{if} \quad F_c \geq 2F_1 \tag{486}$$

Thus, based on a robust preference ranking, we prefer option 2 if $F_1 < F_c < 2F_1$. We prefer option 1 if $F_c \leq F_1$. We are indifferent otherwise.

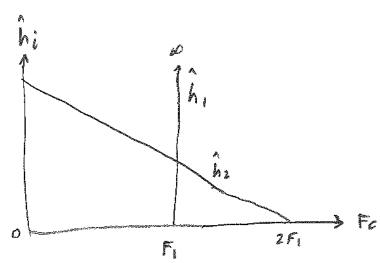


Figure 11: Robustness curves for problem 38f.

**Solution to Problem 39, Present worth, interest, inflation and uncertainty** (p.35).

**(39a)** The present worth is:

$$\text{PW} = -S + \sum_{k=1}^{N}(1+i)^{-k}(R-C) = \boxed{-S + (R-C)\underbrace{\frac{1-(1+i)^{-N}}{i}}_{\delta(i)}} \quad (487)$$

**(39b)** $R > C$, so eq.(487) implies that the present worth increases as the duration, $N$, increases. This makes sense economically, because longer duration implies more years in which positive net income balances the negative initial investment. Thus, to find the shortest time at which the present worth is non-negative, equate PW to zero and solve for $N$:

$$\text{PW} = 0 \implies S = (R-C)\delta(i) \implies \frac{1-(1+i)^{-N}}{i} = \frac{S}{R-C} \implies 1 - \frac{Si}{R-C} = (1+i)^{-N} \quad (488)$$

$$\implies N = -\frac{\ln\left(1 - \frac{Si}{R-C}\right)}{\ln(1+i)} = \boxed{17.79} \quad (489)$$

or 18 years if one wants an integer result for the shortest duration with non-negative PW.

**(39c)** $R > C$, so eq.(487) implies that the present worth increases as the duration, $N$, increases. Thus the present worth is maximal at infinite duration:

$$\lim_{N\to\infty} \text{PW} = -S + \frac{R-C}{i} \quad (490)$$

Equating this to zero and solving for $S$ yields the lowest initial investment at which the present worth is negative for all finite durations:

$$0 = -S + \frac{R-C}{i} \implies S = \frac{R-C}{i} = \frac{2000-1000}{.07} = \boxed{14,285.71} \quad (491)$$

**(39d)** The definition of the robustness is:

$$\widehat{h}(\text{PW}_{\text{c}}) = \max\left\{h : \left(\min_{i\in\mathcal{U}(h)} \text{PW}(i)\right) \ge \text{PW}_{\text{c}}\right\} \quad (492)$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function, $\widehat{h}(\text{PW}_{\text{c}})$. From eq.(487) we see that this minimum occurs when $i = \widetilde{i} + s_i h$. Thus:

$$\boxed{m(h) = -S + (R-C)\delta(\widetilde{i} + s_i h)} \quad (493)$$

**(39e)** The definition of the robustness is:

$$\widehat{h}(\text{PW}_{\text{c}}) = \max\left\{h : \left(\min_{R,C\in\mathcal{U}(h)} \text{PW}(R,C)\right) \ge \text{PW}_{\text{c}}\right\} \quad (494)$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function, $\widehat{h}(\text{PW}_{\text{c}})$. From eq.(487) we see that this minimum occurs when $R$ is minimal and $C$ is maximal:

$$R = \widetilde{R} - s_R h, \quad C = \widetilde{C} + s_C h \quad (495)$$

Thus the inner minimum becomes:

$$m(h) = -S + \left(\widetilde{R} - s_R h - \widetilde{C} - s_C h\right)\delta(i) = \text{PW}(\widetilde{R}, \widetilde{C}) - (s_R + s_C)\delta(i)h \ge \text{PW}_{\text{c}} \quad (496)$$

Equating $m(h)$ to $\text{PW}_\text{c}$ and solving for $h$ yields the robustness:

$$\boxed{\widehat{h}(\text{PW}_\text{c}) = \frac{\text{PW}(\widetilde{R}, \widetilde{C}) - \text{PW}_\text{c}}{(s_R + s_C)\delta(i)}} \tag{497}$$

or zero if this is negative.

**(39f)** The nominal net income at the end of year $k$ is $(1+f)^k(R-C)$. Thus the present worth is:

$$\text{PW} = -S + \sum_{k=1}^{N}(1+i)^{-k}(1+f)^k(R-C) \tag{498}$$

$$= -S + (R-C)\sum_{k=1}^{N}\left(\frac{1+f}{1+i}\right)^k \tag{499}$$

$$= \boxed{-S + (R-C)\frac{\left(\frac{1+f}{1+i}\right)^{N+1} - \frac{1+f}{1+i}}{\frac{1+f}{1+i} - 1}} \tag{500}$$

**(39g)** With $f = i$, we see from eq.(499) that:

$$\text{PW} = -S + (R-C)N \implies N = \frac{S}{R-C} = \frac{10^4}{1000} = \boxed{10} \tag{501}$$

**(39h)** The definition of the robustness is:

$$\widehat{h}(\text{PW}_\text{c}) = \max\left\{h : \left(\min_{i,f\in\mathcal{U}(h)} \text{PW}(i,f)\right) \geq \text{PW}_\text{c}\right\} \tag{502}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function, $\widehat{h}(\text{PW}_\text{c})$. From eq.(498) we see that this minimum occurs when $i = \widetilde{i} + s_i h$ and $f = (\widetilde{f} - s_f f h)^+$. Thus:

$$m(h) = -S + (R-C)\delta\left(\widetilde{i} + s_i h,\, (\widetilde{f} - s_f f h)^+\right) \tag{503}$$

where $\delta(i,f)$ is the fractional expression in eq.(500).

**(39i)** Let $\text{PW}_b$ and $\text{PW}_c$ denote the present worth of the benefit and the total cost (including initial investment), respectively. Thus the PW and the BCR of the project are defined as:

$$\text{PW} = \text{PW}_b - \text{PW}_c \tag{504}$$

$$\text{BCR} = \frac{\text{PW}_b}{\text{PW}_c} \tag{505}$$

We see that:

$$\text{PW} > 0 \quad \text{if and only if} \quad \text{BCR} > 1 \tag{506}$$

Thus Joe and Jane agree on accepting or rejecting the project.

**(39j)** Let $i_b$ denote Jane's discount rate for benefits. We know that $i_b > i$, so:

$$\text{PW}_{b,\text{Jane}} = R\sum_{k=1}^{N}(1+i_b)^{-k} \;<\; R\sum_{k=1}^{N}(1+i)^{-k} = \text{PW}_{b,\text{Joe}} \tag{507}$$

Combining this with eqs.(504) and (505) we see that:

$$\text{PW}_\text{Joe} < 0 \quad \text{implies} \quad \text{BCR}_\text{Jane} < 1 \tag{508}$$

Hence if Joe rejects the project, then so does Jane.

However, we also see that:

$$\text{PW}_{\text{Joe}} > 0 \quad \text{does not imply} \quad \text{BCR}_{\text{Jane}} > 1 \tag{509}$$

Indeed, the left hand condition (causing Joe to accept) can co-exist with the righthand condition in eq.(508) (causing Jane to reject).

**Solution to Problem 40, Present worth, interest, inflation and uncertainty** (p.37).

**(40(a)i)** The present worth is:

$$\begin{aligned}\text{PW} &= -S + \sum_{k=1}^{N}(1+i)^{-k}(R_k - C_k) && (510)\\ &= -S + R_0\sum_{k=1}^{N}(1+i)^{-k}(1+i)^k - C\sum_{k=1}^{N}(1+i)^{-k} && (511)\\ &= \boxed{-S + R_0 N - \underbrace{\frac{1-(1+i)^{-N}}{i}}_{\delta(i)} C} && (512)\end{aligned}$$

**(40(a)ii)** From eq.(512) we find:

$$\text{PW} = -gC + NC - \delta(i)C \geq 0 \quad\Longrightarrow\quad g \leq N - \delta(i) = 10 - \frac{1-1.04^{-10}}{0.04} = \boxed{1.88910} \tag{513}$$

**(40b)** From eq.(510), the present worth is:

$$\text{PW} = -S + \delta(i)(R - C) \tag{514}$$

where $\delta(i)$ is defined in eq.(512). The parameters for which the PW is zero are all positive, which implies that $R - C > 0$. From the geometric series that defines $\delta(i)$, e.g. the righthand sum in eq.(511), we conclude that:

$$\frac{\partial\delta(i)}{\partial i} < 0 \tag{515}$$

Hence, from eq.(514) we conclude that:

$$\frac{\partial \text{PW}}{\partial i} < 0 \tag{516}$$

Finally, we conclude that, if $i$ increases from a constellation of parameters at which PW $= 0$, we see that $R - C$ must increase in order for the PW to remain positive.

Thus statement 40(b)i is the only true statement.

**(40c)** The present worth is:

$$\text{PW} = -S + \delta(i)(R - C) \tag{517}$$

We note that:

$$\lim_{N\to\infty} \delta(i) = \frac{1}{i} \tag{518}$$

Hence the PW is:

$$\text{PW} = -S + \frac{R - C}{i} \tag{519}$$

This is finite for any positive $i$. Thus statement 40(c)iii is the only true statement.

**(40d)** The present worth is:

$$\text{PW} = -S + \delta(i)(gC - C) = -S + \delta(i)(g-1)C \tag{520}$$

The robustness is defined as:

$$\widehat{h}(\text{PW}_{\text{c}}) = \max\left\{h:\ \left(\min_{g\in\mathcal{U}(h)} \text{PW}\right) \geq \text{PW}_{\text{c}}\right\} \tag{521}$$

Let $m(h)$ denote the inner minimum, which occurs for $g = \widetilde{g} - wh$, so:

$$m(h) = -S + \delta(i)\left[(\widetilde{g} - wh)C - C\right] \geq \text{PW}_{\text{c}} \tag{522}$$

Hence:

$$\boxed{\widehat{h}(\mathsf{PW}_{\rm c}) = \frac{-S+\delta(i)(\widetilde{g}C-C)-\mathsf{PW}_{\rm c}}{\delta(i)wC} = \frac{\mathsf{PW}(\widetilde{g})-\mathsf{PW}_{\rm c}}{\delta(i)wC}} \quad (523)$$

or zero if this is negative.

**(40e)** The present worth is specified in eq.(520). The robustness is defined as:

$$\widehat{h}(\mathsf{PW}_{\rm c}) = \max\left\{h:\ \left(\min_{g,C\in\mathcal{U}(h)} \mathsf{PW}\right) \ge \mathsf{PW}_{\rm c}\right\} \quad (524)$$

Let $m(h)$ denote the inner minimum. The info-gap model requires that $g \ge 1$, so define $x^\star$ to equal $x$ if $x \ge 1$ and to equal 1 otherwise. Thus $m(h)$ occurs for:

$$g = (\widetilde{g} - w_g h)^\star, \quad C = \widetilde{C} - w_c h \quad (525)$$

Thus the inverse of the robustness function is:

$$\boxed{m(h) = -S + \delta(i)\left[(\widetilde{g}-w_g h)^\star - 1\right](\widetilde{C} - w_c h)} \quad (526)$$

**(40f)** Your nominal salary at the end of the $k$th month is:

$$\boxed{S_k = (1+f)^k R_0} \quad (527)$$

**(40g)** The robustness is defined as:

$$\widehat{h}(\mathsf{PW}_{\rm c}) = \max\left\{h:\ \left(\min_{f\in\mathcal{U}(h)} S_k\right) \ge S_{\rm c}\right\} \quad (528)$$

Let $m(h)$ denote the inner minimum which occurs for $f = \widetilde{f} - wf$:

$$m(h) = (1+\widetilde{f}-wh)^k R_0 \ge S_{\rm c} \implies \boxed{\widehat{h}(S_{\rm c}) = \frac{1}{w}\left[1+\widetilde{f}-\left(\frac{S_{\rm c}}{R_0}\right)^{1/k}\right]} \quad (529)$$

or zero if this is negative.

**(40h)** The nominal value at the end of $k$ years is:

$$A_k = (1+i_{\rm nom})^k S \implies \boxed{A_{12} = 1.08^{12}\times 1000 = \$2,518.17} \quad (530)$$

The real value at the end of $k$ years is:

$$R_k = (1+f)^{-k}A_k = (1+f)^{-k}(1+i_{\rm nom})^k S \implies \boxed{R_{12} = \left(\frac{1.08}{1.06}\right)^{12}\times 1000 = \$1,251.45} \quad (531)$$

The real interest rate is defined by:

$$R_k = (1+i_{\rm r})^k S \quad (532)$$

Comparing this with the left hand part of eq.(531) we see that the real interest rate, $i_{\rm r}$, is related to the inflation, $f$, and the nominal interest rate, $i_{\rm nom}$, by:

$$(1+i_{\rm r})^k = (1+f)^{-k}(1+i_{\rm nom})^k \implies (1+i_{\rm r})^{-k}(1+f)^{-k} = (1+i_{\rm nom})^{-k} \implies (1+i_{\rm r})(1+f) = 1+i_{\rm nom} \quad (533)$$

Hence:

$$i_{\rm r} = \frac{1+i_{\rm nom}}{1+f} - 1 = \frac{i_{\rm nom}-f}{1+f} \quad\Longrightarrow\quad \boxed{i_{\rm r} = \frac{0.02}{1.06} = 0.018867} \tag{534}$$

**(40(i)i)** The real value of the investment at the end of $k$ years is, from eq.(531):

$$R_k(i_{\rm nom}, f) = \left(\frac{1+i_{\rm nom}}{1+f}\right)^k S \tag{535}$$

The robustness is defined as:

$$\widehat{h}(R_{\rm c}) = \max\left\{h:\ \left(\min_{i_{\rm nom}, f\in\mathcal{U}(h)} R_k\right) \ge R_{\rm c}\right\} \tag{536}$$

Let $m(h)$ denote the inner minimum, which occurs for $i_{\rm nom} = (\widetilde{i}_{\rm nom} - w_i h)^+$ and $f = \widetilde{f} + w_f h$. Thus the inverse of the robustness function is:

$$\boxed{m(h) = \left(\frac{1+(\widetilde{i}_{\rm nom} - w_i h)^+}{1+\widetilde{f}+w_f h}\right)^k S} \tag{537}$$

**(40(i)ii)** From eq.(537) we see that $m(h)$ is positive for all finite positive $h$. Also:

$$\lim_{h\to\infty} m(h) = 0 \tag{538}$$

Thus $m(h) = R_{\rm c} = 0$ implies that $\widehat{h}(0) = \infty$.

**(40(i)iii)** From eq.(537) we see that $m(h) = R_{\rm c} = \left(\frac{1+\widetilde{i}_{\rm nom}}{1+\widetilde{f}}\right)^k S$ if $h=0$. Thus $\widehat{h}(R_{\rm c}) = 0$ for this value of $R_{\rm c}$.

**(40(j)i)** The BCR is:

$$\text{BCR} = \frac{\text{PW}_b}{\text{PW}_c} \quad\text{where}\quad \text{PW}_b = \sum_{k=1}^N (1+i)^{-k} R = \delta(i) R, \quad \text{PW}_c = S + \sum_{k=1}^N (1+i)^{-k} C = S + \delta(i) C \tag{539}$$

$S = 0$ implies that BCR $= R/C$. Thus:

$$\boxed{\text{BCR} \ge 1 \quad\text{if and only if}\quad \frac{R}{C} \ge 1} \tag{540}$$

**(40(j)ii)** $C = 0$ and eq.(539) imply:

$$\text{BCR} = \frac{\delta(i) R}{S} \tag{541}$$

$\delta(i)$ is:

$$\delta(i) = \sum_{k=1}^N (1+i)^{-k} = \frac{1-(1+i)^{-N}}{i} \tag{542}$$

From these relations we see that:

$$\frac{\partial \delta(i)}{\partial N} > 0 \quad\text{and}\quad \lim_{N\to\infty} \delta(i) = \frac{1}{i} \tag{543}$$

Hence:

$$\max_N \text{BCR} = \frac{R}{iS} \tag{544}$$

We require that $\frac{R}{iS} < 1$, so we can assert:

$$\boxed{\text{BCR} < 1 \quad\text{for all } N, \quad\text{if and only if}\quad S > \frac{R}{i}} \tag{545}$$

Solution to problem 41, **Present worth of yearly profit** (p.39).

**(41a)** The present worth is:

$$\text{PW} = \sum_{n=1}^{N} (1+i)^{-n} R_n = 1.09^{-1} 2,500 + 1.09^{-2} 3,500 + 1.09^{-3} 5,000 = \$9,100.4 \tag{546}$$

**(41b)** The total accumulated value at the end of year 3 is:

$$V = \sum_{n=1}^{N} (1+i_{\text{a}})^{N-n} R_n = 1.15^2 2,500 + 1.15^1 3,500 + 1.15^0 5,000 = \$12,331 \tag{547}$$

The present worth of this value is:

$$\text{PW}(V) = (1+i)^{-3} V = 1.09^{-3} 12,331 = \$9,521.8 \tag{548}$$

**(41c)** The present worth of the profit stream is:

$$\text{PW}(R) = \sum_{n=1}^{N} (1+i)^{-n} R_n \tag{549}$$

The definition of the robustness is:

$$\widehat{h} = \max \left\{ h : \left( \min_{R \in \mathcal{U}(h)} \text{PW}(R) \right) \geq \text{PW}_{\text{c}} \right\} \tag{550}$$

Let $m(h)$ denote the inner minimum, which occurs when each profit is as low as possible at horizon of uncertainty $h$:

$$m(h) = \sum_{n=1}^{N} (1+i)^{-n} (1-h) \widetilde{R}_n = (1-h) \text{PW}(\widetilde{R}) \tag{551}$$

Equating this to the critical value, $\text{PW}_{\text{c}}$, and solving for $h$ yields the robustness:

$$\widehat{h} = 1 - \frac{\text{PW}_{\text{c}}}{\text{PW}(\widetilde{R})} \tag{552}$$

or zero if this is negative.

**(41d)** The definition of the robustness is:

$$\widehat{h} = \max \left\{ h : \left( \min_{R_{\text{c}} \in \mathcal{U}(h)} P_{\text{s}}(R_{\text{c}}) \right) \geq P_{\text{c}} \right\} \tag{553}$$

Let $m(h)$ denote the inner minimum, which occurs when the critical value, $R_{\text{c}}$, is as large as possible at horizon of uncertainty $h$:

$$m(h) = \begin{cases} \dfrac{R_2 - (1+h)\widetilde{R}_{\text{c}}}{R_2 - R_1} & \text{if } (1+h)\widetilde{R}_{\text{c}} \leq R_2 \ \ (\text{equiv: } h \leq \dfrac{R_2}{\widetilde{R}_{\text{c}}} - 1) \\ 0 & \text{else} \end{cases} \tag{554}$$

Equating this to the critical value, $P_{\text{c}}$, and solving for $h$ yields the robustness:

$$\widehat{h} = \frac{R_2 - (R_2 - R_1) P_{\text{c}}}{\widetilde{R}_{\text{c}}} - 1 \tag{555}$$

or zero if this is negative. Note that $\widehat{h} \leq \dfrac{R_2}{\widetilde{R}_{\text{c}}} - 1$.

**(41e)** The present worth is:

$$\text{PW} = \sum_{n=1}^{N}(1+i)^{-n}R_n = 1.05^{-1}5,000 + 1.05^{-2}3,500 + 1.05^{-3}2,500 = \$10,096 \tag{556}$$

**(41f)** The total accumulated value at the end of year 3 is:

$$V = \sum_{n=1}^{N}(1+i_{\rm a})^{N-n}R_n = 1.1^{2}5,000 + 1.1^{1}3,500 + 1.1^{0}2,500 = \$12,400 \tag{557}$$

The present worth of this value is:

$$\text{PW}(V) = (1+i)^{-3}V = 1.04^{-3}12,400 = \$11,024 \tag{558}$$

**(41g)** The present worth of the profit stream is:

$$\text{PW}(R) = \sum_{n=1}^{N}(1+i)^{-n}R_n \tag{559}$$

The definition of the robustness is:

$$\widehat{h} = \max\left\{h:\ \left(\min_{R\in\mathcal{U}(h)} \text{PW}(R)\right) \ge \text{PW}_{\rm c}\right\} \tag{560}$$

Let $m(h)$ denote the inner minimum, which occurs when each profit is as low as possible at horizon of uncertainty $h$:

$$m(h) = \sum_{n=1}^{N}(1+i)^{-n}\left(\widetilde{R}_n - wh\right) = \text{PW}(\widetilde{R}) - h\text{PW}(w) \tag{561}$$

Equating this to the critical value, $\text{PW}_{\rm c}$, and solving for $h$ yields the robustness:

$$\widehat{h} = \frac{\text{PW}(\widetilde{R}) - \text{PW}_{\rm c}}{\text{PW}(w)} \tag{562}$$

or zero if this is negative.

**(41h)** The definition of the robustness is:

$$\widehat{h} = \max\left\{h:\ \left(\min_{R_{\rm c}\in\mathcal{U}(h)} P_{\rm s}(R_{\rm c})\right) \ge P_{\rm c}\right\} \tag{563}$$

Let $m(h)$ denote the inner minimum, which occurs when the critical value, $R_{\rm c}$, is as large as possible at horizon of uncertainty $h$:

$$m(h) = \begin{cases} \dfrac{R_2 - (\widetilde{R}_{\rm c} + wh)}{R_2 - R_1} & \text{if } \widetilde{R}_{\rm c} + wh \le R_2 \ \ (\text{equiv: } h \le \dfrac{R_2 - \widetilde{R}_{\rm c}}{w}) \\ 0 & \text{else} \end{cases} \tag{564}$$

Equating this to the critical value, $P_{\rm c}$, and solving for $h$ yields the robustness:

$$\widehat{h} = \frac{R_2 - \widetilde{R}_{\rm c} - (R_2 - R_1)P_{\rm c}}{w} \tag{565}$$

or zero if this is negative. Note that $\widehat{h} \le \dfrac{R_2 - \widetilde{R}_{\rm c}}{w}$.

Solution to problem 42, **Time, money and benefit.** (p.41).

**(42a).** The present worth of the cash flow is:

$$\begin{aligned} \text{PW} &= -S + \sum_{k=1}^{N}(1+i)^{-k} I_k && (566) \\ &= -10,000 + \frac{4,000}{1.09^1} + \frac{3,000}{1.09^2} + \frac{6,000}{1.09^3} && (567) \\ &= \boxed{\$827.87} && (568) \end{aligned}$$

**(42b).** The nominal income at the end of year $k$ is $I_k$. Correcting for inflation, the real value is $(1+f)^{-k} I_k$. Discounting this to time $t = 0$ is $(1+i)^{-k}(1+f)^{-k} I_k$. Thus the total present worth of the cash flow is:

$$\begin{aligned} \text{PW} &= -S + \sum_{k=1}^{N}(1+i)^{-k}(1+f)^{-k} I_k && (569) \\ &= -10,000 + \frac{4,000}{1.09^1 \times 1.07^1} + \frac{3,000}{1.09^2 \times 1.07^2} + \frac{6,000}{1.09^3 \times 1.07^3} && (570) \\ &= \boxed{-\$582.89} && (571) \end{aligned}$$

**(42c).** The benefit-cost ratio is:

$$\begin{aligned} \text{BCR} &= \frac{\sum_{k=1}^{N}(1+i)^{-k} I_k}{\sum_{k=1}^{N}(1+i_{\text{d}})^{-k} D_k} && (572) \\ &= \frac{\frac{20{,}000}{1.04^1} + \frac{15{,}000}{1.04^2} + \frac{45{,}000}{1.04^3}}{\frac{40}{1.15^1} + \frac{70}{1.15^2} + \frac{50}{1.15^3}} && (573) \\ &= \frac{\$73,104}{120.59} && (574) \\ &= 606.23 \ \boxed{\$/\text{customer}} && (575) \end{aligned}$$

**(42d).**

**(42(d)i).** The nominal euro income at the end of year 3 is:

$$A_{3,\text{for}} = (1+f_{\text{for}})^3 R_{r,\text{for}} \qquad (576)$$

**(42(d)ii).** This is exchanged to shekels to produce the nominal shekel income at the end of year 3:

$$\begin{aligned} A_{3,\text{dom}} &= r A_{3,\text{for}} && (577) \\ &= r(1+f_{\text{for}})^3 R_{r,\text{for}} && (578) \end{aligned}$$

**(42(d)iii).** The real shekel value of this sum, at the end of year 3, is:

$$\begin{aligned} R_{3,\text{dom}} &= (1+f_{\text{dom}})^{-3} A_{3,\text{dom}} && (579) \\ &= r(1+f_{\text{dom}})^{-3}(1+f_{\text{for}})^3 R_{r,\text{for}} && (580) \\ &= r\left(\frac{1+f_{\text{for}}}{1+f_{\text{dom}}}\right)^3 R_{r,\text{for}} && (581) \end{aligned}$$

**(42(d)iv).** The present worth of the entire project, in shekels, is:

$$\begin{aligned} \text{PW}_{\text{dom}} &= -rS + (1+i_{\text{r,dom}})^{-3} R_{3,\text{dom}} && (582) \\ &= -rS + r(1+i_{\text{r,dom}})^{-3} \left( \frac{1+f_{\text{for}}}{1+f_{\text{dom}}} \right)^3 R_{r,\text{for}} && (583) \\ &= -rS + r \left( \frac{1+f_{\text{for}}}{(1+i_{\text{r,dom}})(1+f_{\text{dom}})} \right)^3 R_{r,\text{for}} && (584) \end{aligned}$$

**(42(d)v).** The numerical value of the present worth in shekels is:

$$\text{PW}_{\text{dom}} = -4.2 \times 100,000 + 4.2 \left( \frac{1.07}{1.12 \times 1.03} \right)^3 250,000 = \boxed{\text{NIS}417,867} \qquad (585)$$

**(42e).** The definition of the robustness function is:

$$\widehat{h}(\text{FW}_{\text{c}}) = \max \left\{ h : \left( \min_{R \in \mathcal{U}(h)} \text{FW}(R) \right) \geq \text{FW}_{\text{c}} \right\} \qquad (586)$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function. The minimum occurs when $R$ is as small a possible at horizon of uncertainty $h$. Thus:

$$m(h) = (1+i)^N (\widetilde{R} - sh) \geq \text{FW}_{\text{c}} \quad \Longrightarrow \quad \boxed{\widehat{h}(\text{FW}_{\text{c}}) = \frac{1}{s} \left( \widetilde{R} - \frac{\text{FW}_{\text{c}}}{(1+i)^N} \right)} \qquad (587)$$

or zero if this is negative. We can re-write this as:

$$\widehat{h}(\text{FW}_{\text{c}}) = \frac{1}{s(1+i)^N} \left( (1+i)^N \widetilde{R} - \text{FW}_{\text{c}} \right) = \frac{1}{s(1+i)^N} \left( \text{FW}(\widetilde{R}) - \text{FW}_{\text{c}} \right) \qquad (588)$$

or zero if this is negative.

**(42f).** The definition of the robustness function is:

$$\widehat{h}(\text{FW}_{\text{c}}) = \max \left\{ h : \left( \min_{i \in \mathcal{U}(h)} \text{FW}(i) \right) \geq \text{FW}_{\text{c}} \right\} \qquad (589)$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function. This minimum occurs when $i$ is as small as possible, so:

$$m(h) = (1 + \widetilde{i} - wh)^N R \geq \text{FW}_{\text{c}} \quad \Longrightarrow \quad \boxed{\widehat{h}(\text{FW}_{\text{c}}) = \frac{1}{w} \left( 1 + \widetilde{i} - \left( \frac{\text{FW}_{\text{c}}}{R} \right)^{1/N} \right)} \qquad (590)$$

or zero if this is negative. We can re-write this as:

$$\widehat{h}(\text{FW}_{\text{c}}) = \frac{1}{wR^{1/N}} \left( (1+\widetilde{i}) R^{1/N} - \text{FW}_{\text{c}}{}^{1/N} \right) = \frac{1}{wR^{1/N}} \left( \text{FW}(\widetilde{i})^{1/N} - \text{FW}_{\text{c}}{}^{1/N} \right) \qquad (591)$$

or zero if this is negative.

**(42g).** The definition of the robustness function for option $k$ is:

$$\widehat{h}_k(\mathsf{FW}_\mathrm{c}) = \max\left\{h:\ \left(\min_{R\in\mathcal{U}_k(h)} \mathsf{FW}(i)\right) \ge \mathsf{FW}_\mathrm{c}\right\} \tag{592}$$

Let $m_k(h)$ denote the inner minimum, which is the inverse of the robustness function.

*Consider option 1.*

Note that $\mathcal{U}_1(h)$ can be written as the following interval of $R$ values:

$$\mathcal{U}_1(h) = \left[\widetilde{R}_1 + s_1 h,\ \infty\right) \tag{593}$$

This is **not** an info-gap model of uncertainty because it violates the nesting axiom which asserts:

$$h < h' \quad\Longrightarrow\quad \mathcal{U}(h) \subseteq \mathcal{U}(h') \tag{594}$$

From eq.(593) we see that as the horizon of uncertainty, $h$, *increases*, the uncertainty set *contracts*. This is the opposite of the behavior of info-gap models. Note that:

$$m_1(h) = (1+i)^N(\widetilde{R}_1 + s_1 h) \tag{595}$$

How should we understand the concept of robustness when the "uncertainty model" violates the axiom of nesting? The following seems reasonable.

The robustness of option 1, $\widehat{h}_1$, is the greatest horizon of uncertainty, $h$, up to which all realizations of $m_1(h)$ are no less than $\mathsf{FW}_\mathrm{c}$ for all $h \le \widehat{h}_1$.

If $h = 0$, then $m_1(0) = \mathsf{FW}(\widetilde{R})$. If $\mathsf{FW}(\widetilde{R}) \ge \mathsf{FW}_\mathrm{c}$, then $m_1(h) \ge \mathsf{FW}_\mathrm{c}$ for all $h \ge 0$ and the robustness is infinite.

If $\mathsf{FW}(\widetilde{R}) < \mathsf{FW}_\mathrm{c}$, then $m_1(0) < \mathsf{FW}_\mathrm{c}$ so the robustness is zero. In other words, the requirement is violated even without uncertainty so the robustness is zero.

We conclude:

$$\boxed{\widehat{h}_1(\mathsf{FW}_\mathrm{c}) = \begin{cases} \infty & \text{if } \mathsf{FW}_\mathrm{c} \le (1+i)^N \widetilde{R}_1 \\ 0 & \text{else} \end{cases}} \tag{596}$$

See fig. 12.

*Consider option 2.*

For option 2 the inner minimum occurs for $R = \widetilde{R}_2 - s_2 h$, so:

$$m_2(h) = (1+i)^N(\widetilde{R}_2 - s_2 h) \ge \mathsf{FW}_\mathrm{c} \quad\Longrightarrow\quad \boxed{\widehat{h}_2(\mathsf{FW}_\mathrm{c}) = \frac{1}{s_2}\left(\widetilde{R}_2 - \frac{\mathsf{FW}_\mathrm{c}}{(1+i)^N}\right)} \tag{597}$$

or zero if this is negative. See fig. 12.

The robustness curves cross one another because $\widetilde{R}_2 > \widetilde{R}_1$ as stated in eq.(48). From fig. 12 and eqs.(596) and (597) we see that the robust preference is for option 1 if $\mathsf{FW}_\mathrm{c} \le \mathsf{FW}(\widetilde{R}_1)$.

**(42h).** The definition of the robustness function for option $k$ is:

$$\widehat{h}_k(\mathsf{FW}_\mathrm{c}) = \max\left\{h:\ \left(\min_{R\in\mathcal{U}_k(h)} \mathsf{FW}(i)\right) \ge \mathsf{FW}_\mathrm{c}\right\} \tag{598}$$

Let $m_k(h)$ denote the inner minimum, which is the inverse of the robustness function.

*Consider option 1.*

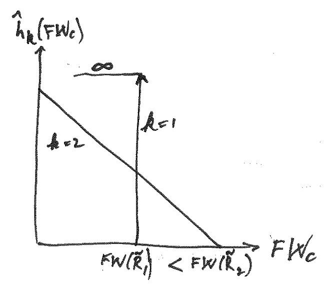


Figure 12: Robustness curves for problem 42g, based on eqs.(596) and (597).

From the info-gap model in eq.(49), p.42, we see that $m_1(h)$ occurs when $R = \widetilde{R}_1$. Thus:

$$m_1(h) = (1+i)^N \widetilde{R}_1 \geq \text{FW}_\text{c} \quad \Longrightarrow \quad \boxed{\widehat{h}_1(\text{FW}_\text{c}) = \begin{cases} \infty & \text{if FW}_\text{c} \leq (1+i)^N \widetilde{R}_1 \\ 0 & \text{else} \end{cases}} \tag{599}$$

which is the same as eq.(596).

The robustness for option 2 is the same as in question 42g, eq.(597).

Thus the robustness curves are the same as in fig. 12.

The robustness curves cross one another because $\widetilde{R}_2 > \widetilde{R}_1$ as stated in eq.(48). From fig. 12 and eqs.(599) and (597) we see that the robust preference is for option 1 if $\text{FW}_\text{c} \leq \text{FW}(\widetilde{R}_1)$. This is the same as in question 42g.

Solution to problem 43 **Investments over time.** (Based on exam 4.10.2018.) (p.43).

**(43a).** The present worth is:

$$\text{PW} = -S + \sum_{k=1}^{N} (1+i)^{-k}(R-C) = \boxed{-S + \varepsilon C \underbrace{\frac{1-(1+i)^{-N}}{i}}_{\delta(i)}} \tag{600}$$

**(43b).** PW $= 0$ in eq.(600) implies:

$$\varepsilon = \frac{S}{C\delta(i)} = \frac{10,000}{1,000 \times 8.11095} = \boxed{1.233} \tag{601}$$

**(43c).** PW $\geq 0$ in eq.(600) implies:

$$\frac{1-(1+i)^{-N}}{i} \geq \frac{S}{\varepsilon C} \iff (1+i)^{-N} \leq 1 - \frac{iS}{\varepsilon C} \iff -N\ln(1+i) \leq \ln\left(1 - \frac{iS}{\varepsilon C}\right) \tag{602}$$

$$\iff N \geq -\frac{\ln\left(1 - \frac{iS}{\varepsilon C}\right)}{\ln(1+i)} = 5.6894 \tag{603}$$

Thus the shortest integer duration for which the PW is non-negative is $\boxed{\text{6 years.}}$

**(43d).** The definition of the robustness function is:

$$\widehat{h}(\text{PW}_\text{c}) = \max\left\{h : \left(\min_{\varepsilon \in \mathcal{U}(h)} \text{PW}(\varepsilon)\right) \geq \text{PW}_\text{c}\right\} \tag{604}$$

Let $m(h)$ denote the inner minimum, which occurs when $\varepsilon$ is minimal at horizon of uncertainty $h$:

$$m(h) = -S + (1-h)\widetilde{\varepsilon}C\delta(i) \geq \text{PW}_\text{c} \implies \boxed{\widehat{h}(\text{PW}_\text{c}) = 1 - \frac{\text{PW}_\text{c} + S}{\widetilde{\varepsilon}C\delta(i)}} \tag{605}$$

or zero if this is negative.

**(43e).** The present worth is:

$$\text{PW} = -S + \sum_{k=1}^{N} (1+i)^{-k}(1+f)^{-k}(R-C) = \boxed{-S + \varepsilon C \sum_{k=1}^{N} [(1+i)(1+f)]^{-k}} \tag{606}$$

**(43f).** Define $\phi = (1+i)(1+f)$. We require that $\phi$ remains constant, which means that:

$$(1+i)(1+f) = (1+i')(1+f') \implies i' = \frac{(1+i)(1+f)}{1+f'} - 1 = \frac{1.04 \times 1.07}{1.10} - 1 = \boxed{0.0116} \tag{607}$$

**(43g).** Because there is no inflation in the U.S., real and nominal values are the same. Hence the future worth in \$ is:

$$F_1 = (1 + i_\$)^N R_1 \tag{608}$$

**(43h).** To calculate the future worth, in dollars, we proceed as follows, which answers the 4 questions asked.

1. **(43(h)i)**: Convert \$$R_1$ to euros to obtain $R_\text{e} = \boxed{rR_1}$ euros.
2. **(43(h)ii)**: The real rate of return of the project in the euro zone is $i_\text{e}$. Thus, after $N$ years, the real euro value of the project is $R_2 = (1+i_\text{e})^N R_\text{e} = \boxed{(1+i_\text{e})^N rR_1}$.
3. **(43(h)iii)**: The nominal value of these euros, an actual pile of euro bills, is $A_2 = (1+f_\text{e})^N R_2 = \boxed{(1+f_\text{e})^N (1+i_\text{e})^N rR_1}$.

4. **(43(h)iv)**: You convert these euros to dollars to obtain the nominal future worth of the project in dollars. This is $F_2 = A_2/r = \boxed{(1+f_\mathrm{e})^N(1+i_\mathrm{e})^N R_1}$.

**(43i).** We prefer project 2 if and only if $F_2 > F_1$, or if and only if:

$$\boxed{(1+f_\mathrm{e})^N(1+i_\mathrm{e})^N > (1+i_\$)^N} \tag{609}$$

Thus euro inflation increases the competitiveness of the euro project against the U.S. project. This of course assumes that we actually know the *real* rates of return of the two projects, $i_\mathrm{e}$ and $i_\$$, and that the exchange rate does not change. Neither assumption — especially the latter — is a good assumption in practice.

Solution to problem 44 **Two investment options.** (Based on exam, 27.5.2019.) (p.44).

**44a.** The present worth of the plan is:

$$P = \sum_{n=1}^{N} \frac{1}{(1+i)^n} C = C \frac{1-(1+i)^{-N}}{i} \tag{610}$$

For the stated values, $C = 1000$, $N = 10$ and $i = 0.07$, we find:

$$P = \$1000 \frac{1-1.07^{-10}}{0.07} = \boxed{\$7,023.6} \tag{611}$$

The value of the present worth is less than $NC$ because the present worth of each dollar earned later is less than one dollar due to the discount effect (unless $i = 0$).

**44b.** The first relation in eq.(51), $C_1 < C_2$, means that the yearly income in plan 1 is less than in plan 2. The present worths of these income streams are:

$$\boxed{P_1 = C_1 \underbrace{\sum_{n=1}^{N} \frac{1}{(1+i_1)^n}}_{I_1} \quad \text{and} \quad P_2 = C_2 \underbrace{\sum_{n=1}^{N} \frac{1}{(1+i_2)^n}}_{I_2}} \tag{612}$$

The second relation in eq.(51), $i_1 < i_2$, means that $I_1 > I_2$. Thus, in plan 1, the value of later earnings is not eroded as rapidly as the value of later earnings in plan 2. Stated differently, $i_1 < i_2$ means that more money would be needed at time $t = 0$ in plan 1 to reproduce, by interest alone, each future earned dollar. Thus the present worth of each dollar earned in plan 1 is greater than the present worth of each dollar earned in plan 2. However, the yearly income is greater in plan 2. Hence, which present worth is greater, $P_1$ or $P_2$, depends on the specific values of the 4 constants, $C_1$, $C_2$, $i_1$ and $i_2$. In short, if all we know about these plans is the two inequalities in eq.(51), then we face a dilemma: plan 1 looks better in terms of the time value of money, while plan 2 looks better in terms of quantity earned each year.

For $i_1$ and $i_2$ fixed, we prefer plan 1, based on the present worth, if and only if:

$$P_1 > P_2 \iff C_1 I_1 > C_2 I_2 \iff \boxed{\frac{C_1}{C_2} > \frac{I_2}{I_1}} \tag{613}$$

Note that:

$$I_j = \frac{1-(1+i_j)^{-N}}{i_j} \tag{614}$$

So, if $N = 10$, $i_1 = 0.05$ and $i_2 = 0.07$, we find:

$$I_1 = 7.7217 \quad \text{and} \quad I_2 = 7.0236 \tag{615}$$

Thus, we prefer plan 1 if and only if:

$$P_1 > P_2 \iff \frac{C_1}{C_2} > \frac{I_2}{I_1} = \frac{7.0236}{7.7217} = \boxed{0.9096} \tag{616}$$

**44c.** The future worth of plan $j$, $F_j$, is related to its present worth, $P_j$, as:

$$F_j = (1+i_j)^N P_j = (1+i_j)^N C_j I_j \tag{617}$$

Recall that $i_1 < i_2$.

**If:**

$$P_2 = C_2 I_2 > C_1 I_1 = P_1 \tag{618}$$

then:

$$F_2 = (1+i_2)^N C_2 I_2 > (1+i_1)^N C_1 I_1 = F_1 \tag{619}$$

because $i_2 > i_1$. In this case, present worth and future worth lead to the same choice.

**However, if:**

$$P_1 = C_1 I_1 > C_2 I_2 = P_2 \tag{620}$$

then:

$$F_1 = (1+i_1)^N C_1 I_1 > (1+i_2)^N C_2 I_2 = F_2 \tag{621}$$

if and only if:

$$\frac{1+i_1}{1+i_2} > \left(\frac{C_2 I_2}{C_1 I_1}\right)^{1/N} \tag{622}$$

which might hold, but need not hold because $i_2 > i_1$.

In short, present worth and future worth assessments may not lead to the same choice.

**44d.** As in eqs.(612) and (614), the present worth of plan $j$ is:

$$P_j = C_j \sum_{n=1}^{N_j} \frac{1}{(1+i)^{-n}} = C_j \underbrace{\frac{1-(1+i)^{-N_j}}{i}}_{I_j} \tag{623}$$

Based on present worth, we prefer plan 1 if and only if:

$$P_1 > P_2 \iff C_1 I_1 > C_2 I_2 \iff C_1 \frac{1-(1+i)^{-N_1}}{i} > C_2 \frac{1-(1+i)^{-N_2}}{i} \tag{624}$$

$$\iff 1-(1+i)^{-N_1} > \frac{C_2}{C_1}\left(1-(1+i)^{-N_2}\right) \tag{625}$$

$$\iff 1-\frac{C_2}{C_1}\left(1-(1+i)^{-N_2}\right) > (1+i)^{-N_1} \tag{626}$$

$$\iff \ln\left[1-\frac{C_2}{C_1}\left(1-(1+i)^{-N_2}\right)\right] > -N_1 \ln(1+i) \tag{627}$$

$$\iff \boxed{N_1 > -\frac{\ln\left[1-\frac{C_2}{C_1}\left(1-(1+i)^{-N_2}\right)\right]}{\ln(1+i)}} \tag{628}$$

For example, if $i = 0.05$, $N_2 = 10$ and $C_2/C_1 = 1.2$, we prefer plan 1 if and only if:

$$N_1 > -\frac{\ln\left[1-1.2\left(1-1.05^{-10}\right)\right]}{\ln 1.05} = \boxed{12.7551} \tag{629}$$

**44e.** The definition of the robustness function for option $j$ is:

$$\widehat{h}_j(v_c) = \max\left\{h : \left(\min_{v_j \in \mathcal{U}(h)} v_j\right) \geq v_c\right\} \tag{630}$$

Let $m(h)$ denote the inner minimum, which is the inverse of $\widehat{h}_j(v_c)$. This minimum occurs for $v_j$ minimal at horizon of uncertainty $h$. Thus:

$$m(h) = \widetilde{v}_j - s_j h \geq v_c \implies \boxed{\widehat{h}_j(v_c) = \frac{\widetilde{v}_j - v_c}{s_j}} \tag{631}$$

or zero if this is negative.

From the two relations in eq.(53), p.44, we conclude that the two robustness curves cross one another. Let $v_{\times}$ denote the value of $v_{\rm c}$ at which the robustness curves cross one another. We see that $\widehat{h}_1(v_{\rm c}) > \widehat{h}_2(v_{\rm c})$ for $v_{\rm c} < v_{\times}$. Thus we robust-prefer option 1 for $v_{\rm c} < v_{\times}$. The value of $v_{\times}$ is obtained as follows.

$$\widehat{h}_1(v_{\times}) = \widehat{h}_2(v_{\times}) \tag{632}$$

$$\iff \quad \frac{\widetilde{v}_1 - v_{\times}}{s_1} = \frac{\widetilde{v}_2 - v_{\times}}{s_2} \tag{633}$$

$$\iff \quad \frac{\widetilde{v}_1}{s_1} - \frac{\widetilde{v}_2}{s_2} = \left( \frac{1}{s_1} - \frac{1}{s_2} \right) v_{\times} \tag{634}$$

$$\iff \quad s_2\widetilde{v}_1 - s_1\widetilde{v}_2 = (s_2 - s_1) v_{\times} \tag{635}$$

$$\iff \quad \boxed{v_{\times} = \frac{s_2\widetilde{v}_1 - s_1\widetilde{v}_2}{s_2 - s_1}} \tag{636}$$

**44f.** Use Lagrange optimization. Define:

$$H = v^T r + \lambda \left[ h^2 - (v - \widetilde{v})^T W (v - \widetilde{v}) \right] \tag{637}$$

Condition for an extremum is:

$$0 = \frac{\partial H}{\partial v} = r - 2\lambda W (v - \widetilde{v}) \quad \Longrightarrow \quad v - \widetilde{v} = \frac{1}{2\lambda} W^{-1} r \tag{638}$$

Use the constraint to find the Lagrange multiplier:

$$h^2 = \frac{1}{4\lambda^2} r^T W^{-1} W W^{-1} r \quad \Longrightarrow \quad \frac{1}{2\lambda} = \pm \frac{h}{\sqrt{r^T W^{-1} r}} \tag{639}$$

Thus the extremal vectors are:

$$v = \widetilde{v} \pm \frac{h}{\sqrt{r^T W^{-1} r}} W^{-1} r \quad \Longrightarrow \quad \boxed{\min_{v \in \mathcal{U}(h)} v^T r = \widetilde{v}^T r - h\sqrt{r^T W^{-1} r}} \tag{640}$$

Solution to problem 45 **Investment and earnings.** (Based on exam, 16.7.2019.) (p.45).

**(45a)** The real value at time 0, in \$'s, of the \$ payment $A_N$ at the end of year $N$ is, from eq.(272) (see the derivation there, p.86):

$$R_{0,N} = A_{0,N} = A_N \prod_{j=1}^{N} (1+f_j)^{-1} \tag{641}$$

$$= \frac{\$10,000}{1.12 \times 1.06 \times 1.09} \tag{642}$$

$$= \boxed{\$7,727.69} \tag{643}$$

**(45b)** The foreign nominal value in year $N$ is:

$$A_{N,\text{for}} = r_N A_N \tag{644}$$

Eq.(641) applies to pesos when using the peso inflation rates, so the real value at time 0, in peso's, of the \$ payment $A_N$ after transferring to pesos at the end of year $N$ is:

$$R_{0,N,\text{for}} = A_{0,N,\text{for}} = A_{N,\text{for}} \prod_{j=1}^{N} (1+\phi_j)^{-1} \tag{645}$$

$$= r_N A_N \prod_{j=1}^{N} (1+\phi_j)^{-1} \tag{646}$$

$$= \frac{40 \times 10,000}{1.24 \times 1.14 \times 1.32} \tag{647}$$

$$= \boxed{\text{peso}214,367.79} \tag{648}$$

**(45c)** The robustness is defined as:

$$\widehat{h} = \max \left\{ h : \left( \min_{r_N \in \mathcal{U}(h)} R_{0,N,\text{for}}(r_N) \right) \geq R_\text{c} \right\} \tag{649}$$

Let $m(h)$ denote the inner minimum, which occurs for the lowest exchange rate at horizon of uncertainty $h$: $r_N = (1-h)^+ \widetilde{r}_N$. From eq.(646):

$$m(h) = (1-h)^+ R_{0,N,\text{for}}(\widetilde{r}_N) \tag{650}$$

Consider first the case $h \leq 1$. Equating $m(h)$ to $R_\text{c}$ and solving for $h$ yields the robustness:

$$\widehat{h} = 1 - \frac{R_\text{c}}{R_{0,N,\text{for}}(\widetilde{r}_N)} \tag{651}$$

which is less than 1. Hence this is the robustness, or zero if this is negative.

Solution to problem 46 **Interest, inflation and uncertainty.** (Based on exam, 16.7.2019.) (p.46).

**(46a)** The present worth is:

$$\text{PW} = -S + \sum_{k=1}^{N}(1+i)^{-k}(R-C) \;=\; -S + (R-C)\underbrace{\frac{1-(1+i)^{-N}}{i}}_{\delta(i)} \tag{652}$$

$$= -7,500 + (5,000-3,000)\frac{1-1.12^{-9}}{0.12} \tag{653}$$

$$= \boxed{\$3,156.50} \tag{654}$$

**(46b)** From part 46a the PW is \$3,156.50. We can increase the initial investment by this amount and the PW is zero. Thus:

$$S_{\max} = \$7,500 + \$3,156.50 = \boxed{\$10,656.50} \tag{655}$$

**(46c)** $R > C$, so eq.(652) implies that the present worth increases as the duration, $N$, increases. This makes sense economically, because longer duration implies more years in which positive net income balances the negative initial investment. Thus, to find the shortest time at which the present worth is non-negative, equate PW to zero and solve for $N$:

$$\text{PW} = 0 \implies S = (R-C)\delta(i) \implies \frac{1-(1+i)^{-N}}{i} = \frac{S}{R-C} \implies 1 - \frac{Si}{R-C} = (1+i)^{-N} \tag{656}$$

$$\implies N = -\frac{\ln\left(1-\frac{Si}{R-C}\right)}{\ln(1+i)} = -\frac{\ln\left(1-\frac{7,500\times 0.12}{5,000-3,000}\right)}{\ln(1.12)} = -\frac{\ln(1-0.45)}{\ln(1.12)} = \boxed{5.2752} \tag{657}$$

or 6 years if one wants an integer result for the shortest duration with non-negative PW.

**(46d)** The definition of the robustness is:

$$\widehat{h}(\text{PW}_{\text{c}}) = \max\left\{h:\ \left(\min_{i\in\mathcal{U}(h)} \text{PW}(i)\right) \ge \text{PW}_{\text{c}}\right\} \tag{658}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function, $\widehat{h}(\text{PW}_{\text{c}})$. From eq.(652) we see that this minimum occurs when $i = \widetilde{i} + s_i h$ because $R - C > 0$. Thus:

$$\boxed{m(h) = -S + (R-C)\delta(\widetilde{i} + s_i h)} \tag{659}$$

**(46e)** The definition of the robustness is:

$$\widehat{h}(\text{PW}_{\text{c}}) = \max\left\{h:\ \left(\min_{R,C\in\mathcal{U}(h)} \text{PW}(R,C)\right) \ge \text{PW}_{\text{c}}\right\} \tag{660}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function, $\widehat{h}(\text{PW}_{\text{c}})$. From eq.(652) we see that this minimum occurs when $R$ is minimal and $C$ is maximal:

$$R = \widetilde{R} - s_R h, \quad C = \widetilde{C} + s_C h \tag{661}$$

Thus the inner minimum becomes:

$$m(h) = -S + \left(\widetilde{R} - s_R h - \widetilde{C} - s_C h\right)\delta(i) = \text{PW}(\widetilde{R},\widetilde{C}) - (s_R + s_C)\delta(i)h \ge \text{PW}_{\text{c}} \tag{662}$$

Equating $m(h)$ to $\text{PW}_\text{c}$ and solving for $h$ yields the robustness:

$$\boxed{\widehat{h}(\text{PW}_\text{c}) = \frac{\text{PW}(\widetilde{R}, \widetilde{C}) - \text{PW}_\text{c}}{(s_R + s_C)\delta(i)}} \tag{663}$$

or zero if this is negative.

**(46f)** The nominal net income at the end of year $k$ is $(1+f)^k(R-C)$.
The present worth of this net income is $(1+i)^{-k}(1+f)^k(R-C)$.
Thus the present worth of the project is:

$$\text{PW} = -S + \sum_{k=1}^{N}(1+i)^{-k}(1+f)^k(R-C) \tag{664}$$

$$= -S + (R-C)\sum_{k=1}^{N}\left(\frac{1+f}{1+i}\right)^k \tag{665}$$

$$= \boxed{-S + (R-C)\frac{\left(\frac{1+f}{1+i}\right)^{N+1} - \frac{1+f}{1+i}}{\frac{1+f}{1+i} - 1}} \tag{666}$$

Solution to problem 47 **Uncertain but correlated investments.** (Based on exam, 16.7.2019.) (p.46).

The robustness of allocation $q$, with required return $R_{\rm c}$, is defined as:

$$\widehat{h}(q, R_{\rm c}) = \max \left\{ h : \min_{u \in \mathcal{U}(h, \widetilde{u})} R(q, u) \ge R_{\rm c} \right\} \tag{667}$$

The return is $R(q, u) = q^T u$.

To evaluate the inner minimum in the robustness function use Lagrange optimization and define:

$$H = q^T u + \lambda \left[ h^2 - (u - \widetilde{u})^T V^{-1} (u - \widetilde{u}) \right] \tag{668}$$

Differentiating:

$$0 = \frac{\partial H}{\partial u} = q - 2\lambda V^{-1}(u - \widetilde{u}) \quad \Longrightarrow \quad u - \widetilde{u} = \frac{1}{2\lambda} V q \tag{669}$$

Use the constraint to determine $\lambda$

$$h^2 = \frac{1}{4\lambda^2} q^T V V^{-1} V q \quad \Longrightarrow \quad \frac{1}{2\lambda} = \frac{\pm h}{\sqrt{q^T V q}} \tag{670}$$

Hence the optimizing return vector is:

$$u = \widetilde{u} \pm \frac{h}{\sqrt{q^T V q}} V q \tag{671}$$

The inner minimum in eq.(667) is:

$$\min_{u \in \mathcal{U}(h, \widetilde{u})} q^T u = q^T \widetilde{u} - h \sqrt{q^T V q} \tag{672}$$

Equating this to $R_{\rm c}$ and solving for $h$ yields the robustness:

$$\boxed{\widehat{h}(q, R_{\rm c}) = \begin{cases} \dfrac{q^T \widetilde{u} - R_{\rm c}}{\sqrt{q^T V q}} & \text{if } R_{\rm c} \le q^T \widetilde{u} \\ 0 & \text{else} \end{cases}} \tag{673}$$

Solution to problem 48 **Price indices.** (Based on exam, 3.10.2019.) (p.47).

**48a.** The real value, at the start of year 1, of the price of commodity $i$ in year $j$, is:

$$\left( \prod_{k=1}^{j} \frac{1}{1+f_k} \right) p_i^{(j)} \tag{674}$$

The inflation is the same for all prices (we are not considering distinct sectorial inflations). Thus the real value, at the start of year 1, of the arithmetic average price, is:

$$\boxed{R_0 = \left( \prod_{k=1}^{j} \frac{1}{1+f_k} \right) \left( \frac{1}{N} \sum_{i=1}^{N} p_i^{(j)} \right)} \tag{675}$$

**48b.** Like in eq.(674), the real value, at the start of year 1, of the price of commodity $i$ in year 2, is $p_i^{(2)}/[(1+f_1)(1+f_2)]$. Similarly, the real value, at the start of year 1, of the price of commodity $i$ in year 1, is $p_i^{(1)}/(1+f_1)$. Thus the real value, at the start of year 1, of the arithmetic average of the price ratios is:

$$R_0 = \frac{1}{N} \sum_{i=1}^{N} \frac{p_i^{(2)}/[(1+f_1)(1+f_2)]}{p_i^{(1)}/(1+f_1)} = \boxed{\frac{1}{N(1+f_2)} \sum_{i=1}^{N} \frac{p_i^{(2)}}{p_i^{(1)}}} \tag{676}$$

**48c.** The definition of the robustness function is:

$$\widehat{h} = \max \left\{ h : \left( \min_{p^{(1)}, p^{(2)} \in \mathcal{U}(h)} \rho \right) \geq \rho_{\rm c} \right\} \tag{677}$$

where $\rho$ is defined in eq.(61) on p.47. Let $m(h)$ denote the inner minimum in eq.(677), which is the inverse of the robustness function. This minimum occurs for $p_i^{(2)} = (1-wh)^+ \widetilde{p}_i^{(2)}$, and $p_i^{(1)} = (1+wh)\widetilde{p}_i^{(1)}$. Thus:

$$m(h) = \frac{(1-wh)^+}{1+wh} \underbrace{\sum_{i=1}^{N} \frac{\widetilde{p}_i^{(2)}}{\widetilde{p}_i^{(1)}}}_{\widetilde{\rho}} \geq \rho_{\rm c} \tag{678}$$

We obtain the robustness by solving eq.(678) for $h$ at equality. For $h \leq 1/w$:

$$(1-wh)\widetilde{\rho} = (1+wh)\rho_{\rm c} \iff \widetilde{\rho} - \rho_{\rm c} = (\widetilde{\rho} + \rho_{\rm c})wh \iff \boxed{\widehat{h} = \frac{\widetilde{\rho} - \rho_{\rm c}}{(\widetilde{\rho} + \rho_{\rm c})w}} \tag{679}$$

or zero if this is negative. Note that this robustness is no greater than $1/w$.

**48d.** The definition of the robustness function is:

$$\widehat{h} = \max \left\{ h : \left( \min_{p \in \mathcal{U}(h)} w^T p \right) \geq CPI_c \right\} \tag{680}$$

Let $m(h)$ denote the inner minimum, which we derive by Lagrange optimization. Define:

$$H = w^T p + \lambda \left[ h^2 - (p - \widetilde{p})^T V^{-1} (p - \widetilde{p}) \right] \tag{681}$$

Differentiating with respect to $p$ and equating to zero:

$$\frac{\mathrm{d}H}{\mathrm{d}p} = 0 = w - 2\lambda V^{-1}(p - \widetilde{p}) \implies p - \widetilde{p} = \frac{1}{2\lambda} V w \implies h^2 = \frac{1}{4\lambda^2} w^T V V^{-1} V w \tag{682}$$

Thus:

$$\frac{1}{2\lambda} = \pm \frac{h}{\sqrt{w^T V w}} \implies p = \widetilde{p} \pm \frac{h}{\sqrt{w^T V w}} V w \implies m(h) = w^T \widetilde{p} - h\sqrt{w^T V w} \geq CPI_c \quad (683)$$

So the robustness function is:

$$\boxed{\widehat{h} = \frac{w^T \widetilde{p} - CPI_c}{\sqrt{w^T V w}}} \quad (684)$$

or zero if this is negative.

**48e.** The definition of the robustness function is:

$$\widehat{h} = \max \left\{ h : \left( \min_{p^{(2)} \in \mathcal{U}(h)} f_2 \right) \geq f_c \right\} \quad (685)$$

Let $m(h)$ denote the inner minimum, which occurs when each $p_i^{(2)}$ is as small as possible at horizon of uncertainty $h$: $p_i^{(2)} = \left(\widetilde{p}_i^{(2)} - v_i h\right)^+$. Thus the inverse of the robustness function is:

$$\boxed{m(h) = \frac{\sum_{i=1}^N w_i \left(\widetilde{p}_i^{(2)} - v_i h\right)^+}{w^T \widetilde{p}^{(1)}} - 1} \quad (686)$$

**48f.** The definition of the robustness function is:

$$\widehat{h} = \max \left\{ h : \left( \max_{p^{(2)} \in \mathcal{U}(h)} \left[ f_2\left(p^{(2)c}, \widetilde{p}^{(1)}\right) - f_2\left(p^{(2)}, \widetilde{p}^{(1)}\right) \right] \right) \leq \delta \right\} \quad (687)$$

Let $m(h)$ denote the inner maximum.

Note that:

$$f_2\left(p^{(2)c}, \widetilde{p}^{(1)}\right) - f_2\left(p^{(2)}, \widetilde{p}^{(1)}\right) = \frac{w^T p^{(2)c}}{w^T \widetilde{p}^{(1)}} - \frac{w^T p^{(2)}}{w^T \widetilde{p}^{(1)}} \quad (688)$$

$$= \frac{w^T \left(p^{(2)c} - p^{(2)}\right)}{w^T \widetilde{p}^{(1)}} \quad (689)$$

The elements of $w$ are known and positive, so the inner maximum occurs when each $p_i^{(2)}$ is as small as possible at horizon of uncertainty $h$. According to the info-gap model in eq.(68) this is: $p_i^{(2)} = \widetilde{p}_i^{(2)}$. Thus, from eq.(689), we find:

$$m(h) = \frac{w^T \left(p^{(2)c} - \widetilde{p}^{(2)}\right)}{w^T \widetilde{p}^{(1)}} \quad (690)$$

This is constant, independent of $h$. Call this value $M$ for convenience. We find that the robustness function is:

$$\widehat{h}(\delta) = \begin{cases} \infty & \text{if } \delta \geq M \\ 0 & \text{else} \end{cases} \quad (691)$$

**48g.** The definition of the robustness function is:

$$\widehat{h} = \max \left\{ h : \left( \max_{p^{(2)} \in \mathcal{U}(h)} \left[ f_2\left(p^{(2)}, \widetilde{p}^{(1)}\right) - f_2\left(p^{(2)c}, \widetilde{p}^{(1)}\right) \right] \right) \leq \delta \right\} \quad (692)$$

Let $m(h)$ denote the inner maximum. In analogy to eqs.(688) and (689), we write:

$$f_2\left(p^{(2)}, \widetilde{p}^{(1)}\right) - f_2\left(p^{(2)c}, \widetilde{p}^{(1)}\right) = \frac{w^T \left(p^{(2)} - p^{(2)c}\right)}{w^T \widetilde{p}^{(1)}} \quad (693)$$

Thus $m(h)$ occurs when each $p_i^{(2)}$ is as large as possible at horizon of uncertainty $h$: $p_i^{(2)} = \widetilde{p}_i^{(2)} + v_i h$. Thus $m(h)$ becomes:

$$m(h) = \frac{w^T \left( \widetilde{p}^{(2)} + hv - p^{(2)c} \right)}{w^T \widetilde{p}^{(1)}} = \underbrace{\frac{w^T \left( \widetilde{p}^{(2)} - p^{(2)c} \right)}{w^T \widetilde{p}^{(1)}}}_{\delta^c} + h \frac{w^T v}{w^T \widetilde{p}^{(1)}} \leq \delta \tag{694}$$

which defines $\delta^c$. Solving for $h$ yields the robustness function:

$$\widehat{h}(\delta) = \frac{\delta - \delta^c}{w^T v / w^T \widetilde{p}^{(1)}} \tag{695}$$

or zero if this is negative.

Solution to problem 49 **Foreign investment.** (Based on exam, 3.10.2019.) (p.49).

**49a.** The nominal pesos needed to cover costs at the end of year $n$ are:

$$(1+f)^n C \tag{696}$$

The nominal dollars needed to buy these pesos at the end of year $n$ are:

$$A_n = \frac{(1+f)^n C}{r} \tag{697}$$

The present worth with dollar discount rate $i$ is:

$$\text{PW}_c = \sum_{n=1}^{N} (1+i)^{-n} A_n = \frac{C}{r} \sum_{n=1}^{N} \left(\frac{1+f}{1+i}\right)^n = \frac{C}{r} \frac{\left(\frac{1+f}{1+i}\right)^{N+1} - \frac{1+f}{1+i}}{\frac{1+f}{1+i} - 1} = \boxed{\frac{C}{r} \frac{\left(\frac{1+f}{1+i}\right)^{N} - 1}{1 - \frac{1+i}{1+f}}} \tag{698}$$

**49b.** The nominal pesos earned at the end of year $n$ are:

$$(1+\varepsilon)^n R_0 \tag{699}$$

The nominal dollars earned at the end of year $n$ are:

$$A_n = \frac{(1+\varepsilon)^n R_0}{r} \tag{700}$$

The present worth with dollar discount rate $i$ is:

$$\text{PW}_r = \sum_{n=1}^{N} (1+i)^{-n} A_n = \frac{R_0}{r} \sum_{n=1}^{N} \left(\frac{1+\varepsilon}{1+i}\right)^n = \frac{R_0}{r} \frac{\left(\frac{1+\varepsilon}{1+i}\right)^{N+1} - \frac{1+\varepsilon}{1+i}}{\frac{1+\varepsilon}{1+i} - 1} = \boxed{\frac{R_0}{r} \frac{\left(\frac{1+\varepsilon}{1+i}\right)^{N} - 1}{1 - \frac{1+i}{1+\varepsilon}}} \tag{701}$$

**49c.** Eqs.(698) and (701) and $f < i < \varepsilon$ imply:

$$\lim_{N\to\infty} \text{PW}_c = \text{finite} \tag{702}$$

$$\lim_{N\to\infty} \text{PW}_r = \infty \tag{703}$$

The total present worth is $\text{PW}_r - \text{PW}_c$. Thus:

$$\lim_{N\to\infty} \text{PW} = \infty \tag{704}$$

**49d.** Eqs.(698) and (701) and $\varepsilon < i < f$ imply:

$$\lim_{N\to\infty} \text{PW}_c = \infty \tag{705}$$

$$\lim_{N\to\infty} \text{PW}_r = \text{finite} \tag{706}$$

The total present worth is $\text{PW}_r - \text{PW}_c$. Thus:

$$\lim_{N\to\infty} \text{PW} = -\infty \tag{707}$$

**49e.** The definition of the robustness function for country $j$ is:

$$\widehat{h}_j(\text{PW}_\text{c}) = \max\left\{h : \left(\min_{\varepsilon_j \in \mathcal{U}(h)} \text{PW}_j\right) \geq \text{PW}_\text{c}\right\} \tag{708}$$

Let $m(h)$ denote the inner minimum, which occurs when $\varepsilon_j$ is minimal at horizon of uncertainty $h$: $\varepsilon_j = \widetilde{\varepsilon} - w_j h$. Thus:

$$m(h) = (\widetilde{\varepsilon} - w_j h)\mathsf{PW}_{0j} \ge \mathsf{PW}_{\rm c} \quad \Longrightarrow \quad \widehat{h}_j(\mathsf{PW}_{\rm c}) = \left(\widetilde{\varepsilon} - \frac{\mathsf{PW}_{\rm c}}{\mathsf{PW}_{0j}}\right)\frac{1}{w_j} \tag{709}$$

or zero if this is negative. These robustness functions cross at the value of $\mathsf{PW}_\times$ satisfying:

$$\widehat{h}_1(\mathsf{PW}_\times) = \widehat{h}_2(\mathsf{PW}_\times) \iff \left(\widetilde{\varepsilon} - \frac{\mathsf{PW}_\times}{\mathsf{PW}_{01}}\right)\frac{1}{w_1} = \left(\widetilde{\varepsilon} - \frac{\mathsf{PW}_\times}{\mathsf{PW}_{02}}\right)\frac{1}{w_2} \tag{710}$$

$$\iff \widetilde{\varepsilon}\left(\frac{1}{w_1} - \frac{1}{w_2}\right) = \mathsf{PW}_\times\left(\frac{1}{w_1\mathsf{PW}_{01}} - \frac{1}{w_2\mathsf{PW}_{02}}\right) \iff \boxed{\mathsf{PW}_\times = \frac{(w_2 - w_1)\mathsf{PW}_{01}\mathsf{PW}_{02}}{w_2\mathsf{PW}_{02} - w_1\mathsf{PW}_{01}}\widetilde{\varepsilon}} \tag{711}$$

We prefer country 2 for $\mathsf{PW}_\times < \mathsf{PW}_{\rm c} \le \widetilde{\varepsilon}\mathsf{PW}_{02}$. We prefer country 1 for $\mathsf{PW}_{\rm c} < \mathsf{PW}_\times$ and we are indifferent for $\widetilde{\varepsilon}\mathsf{PW}_{02} \le \mathsf{PW}_{\rm c}$ because then both robustnesses are zero.

Solution to problem 50 **Future worth.** (Based on midterm exam, 26.6.2023.) (p.50).

**50a**. Future and present worth are related as:

$$F = (1+i)^N P = \$1.15^{25} \times 100,000 = \$3.2929 \times 10^6 \tag{712}$$

**50b**. The future worths of the two alternatives are:

$$F_1 = (1+i_1)^{N_1} P, \quad F_2 = (1+i_2)^{N_2} P \tag{713}$$

Preference for the 1st investment is expressed as follows:

$$\frac{F_1}{F_2} = \frac{(1+i_1)^{N_1}}{(1+i_2)^{N_2}} > 1 \quad \Longleftrightarrow \quad (1+i_1)^{N_1} > (1+i_2)^{N_2} \tag{714}$$

$$\Longleftrightarrow \quad N_1 \ln(1+i_1) > N_2 \ln(1+i_2) \tag{715}$$

$$\Longleftrightarrow \quad \frac{N_1}{N_2} > \frac{\ln(1+i_2)}{\ln(1+i_1)} \tag{716}$$

We prefer the 1st investment for all values of $\frac{N_1}{N_2}$ satisfying eq.(716).

**50c**. The definition of the robustness function is:

$$\widehat{h}(F_c) = \max \left\{ h : \left( \min_{P \in \mathcal{U}(h)} F \right) \ge F_c \right\} \tag{717}$$

where $F = (1+i)^N P$.

Let $m(h)$ denote the inner minimum in eq.(717). $m(h)$ occurs for $P = \widetilde{P} - wh$ so:

$$m(h) = (1+i)^N (\widetilde{P} - wh) \ge F_c \quad \Longrightarrow \quad \widehat{h}(F_c) = \frac{(1+i)^N \widetilde{P} - F_c}{(1+i)^N w} = \frac{F(\widetilde{P}) - F_c}{(1+i)^N w} \tag{718}$$

or zero if this is negative.

Solution to problem 51 **Forecasting.** (p.51).

**51a**. The info-gap model in eq.(77) asserts that the true transition coefficient, $\lambda_1$ is no less than $\widetilde{\lambda}$. This expresses our confidence that $\widetilde{\lambda}$ is an underestimate. The unbounded upper horizon of uncertainty, $h$, expresses our lack of a reliable upper bound for $\lambda_1$.

**51b**. The original info-gap model, in eq.(77) is:

$$\mathcal{U}(h) = \left\{ \lambda_1 : \ \widetilde{\lambda} \le \lambda_1 \le \widetilde{\lambda} + \widetilde{\lambda} h \right\}, \quad h \ge 0 \tag{719}$$

This can be expressed as the following asymmetric fractional-error info-gap model:

$$\mathcal{U}(h) = \left\{ \lambda_1 : \ 0 \le \frac{\lambda_1 - \widetilde{\lambda}}{\widetilde{\lambda}} \le h \right\}, \quad h \ge 0 \tag{720}$$

Given the new information, one could modify eq.(720) this as:

$$\mathcal{U}(h) = \left\{ \lambda_1 : \ 0 \le \frac{\lambda_1 - \widetilde{\lambda}}{w} \le h \right\}, \quad h \ge 0 \tag{721}$$

Or, equivalently:

$$\mathcal{U}(h) = \left\{ \lambda_1 : \ \widetilde{\lambda} \le \lambda_1 \le \widetilde{\lambda} + wh \right\}, \quad h \ge 0 \tag{722}$$

In short, the fractional-error with respect to $\widetilde{\lambda}$, eq.(720), is replaced by a fractional-error with respect to $w$, eq.(721).

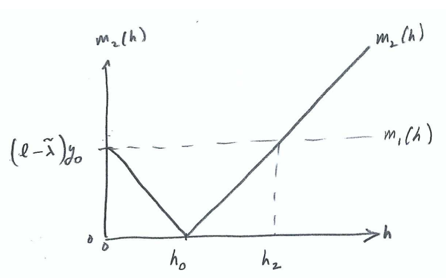


Figure 13: Functions $m_1(h)$ and $m_2(h)$ for problem 51c.

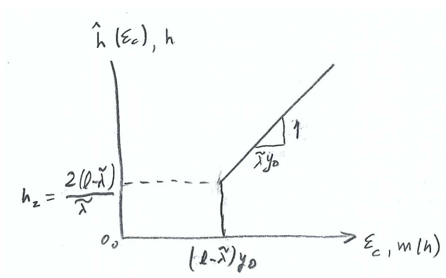


Figure 14: Robustness function for problem 51c.

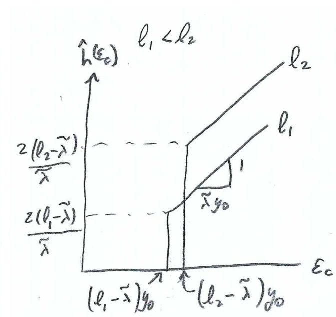


Figure 15: Robustness functions for two values of $\ell$, for problem 51c.

**51c**. The definition of the robustness is:

$$\widehat{h}(\varepsilon_c) = \max \left\{ h : \ \left( \max_{\lambda_1 \in \mathcal{U}(h)} \varepsilon(\lambda_1) \right) \le \varepsilon_c \right\} \tag{723}$$

Let $m(h)$ denote the inner maximum in this definition of the robustness. $m(h)$ is the inverse function of $\widehat{h}(\varepsilon_c)$.

Recall that:

$$\varepsilon(\lambda_1) = |y_1^s - y_1| = |(\ell - \lambda_1) y_0| \tag{724}$$

$m(h)$ occurs when $\lambda_1$ is either minimal ($\lambda_1 = \widetilde{\lambda}$) or maximal ($\lambda_1 = (1+h)\widetilde{\lambda}$). Denote the two possible values of $m(h)$ as follows:

$$m_1(h) = |(\ell - \widetilde{\lambda}) y_0| \tag{725}$$

$$m_2(h) = |(\ell - (1+h)\widetilde{\lambda}) y_0| \tag{726}$$

The inner maximum is the greater of these two terms:

$$m(h) = \max[m_1(h),\ m_2(h)] \tag{727}$$

$m_1(h)$ is constant, while $m_2(h)$ decreases linearly as $h$ increases until it reaches zero, and then it increases linearly as seen in fig. 13.

One determines the values of $h_0$ and $h_2$, appearing in fig. 13, as follows.

$h_0$: $m_2(h)$ reaches the horizontal axis when:

$$m_2(h) = 0 \quad \Longrightarrow \quad \ell - (1+h)\widetilde{\lambda} = 0 \quad \Longrightarrow \quad h_0 = \frac{\ell}{\widetilde{\lambda}} - 1 \tag{728}$$

$h_2$: $m_2(h)$ crosses $m_1(h)$ at a positive value of $h$ when:

$$m_2(h) = m_1(h) \quad \Longrightarrow \quad \ell - \widetilde{\lambda} = (1+h)\widetilde{\lambda} - \ell \quad \Longrightarrow \quad h_2 = \frac{2(\ell - \widetilde{\lambda})}{\widetilde{\lambda}} \tag{729}$$

Thus $m(h) = m_2(h)$ for all $h$ values for which:

$$\underbrace{[(1+h)\widetilde{\lambda} - \ell]y_0}_{m_2(h)} > \underbrace{(\ell - \widetilde{\lambda})y_0}_{m_1(h)} \quad \Longrightarrow \quad h > \frac{2(\ell - \widetilde{\lambda})}{\widetilde{\lambda}} = h_2 \tag{730}$$

Summarizing, we can express $m(h)$ as follows:

$$m(h) = \begin{cases} (\ell - \widetilde{\lambda})y_0 & \text{if} \quad h \le h_2 \\ [(1+h)\widetilde{\lambda} - \ell]y_0 & \text{else} \end{cases} \tag{731}$$

This inverse robustness function is plotted in fig. 14.

We now illustrate how the robustness function assists in choosing the value of $\ell$, illustrated in fig. 15. The slope $\ell_2$ is more robust than the slope $\ell_1$ for values of $\varepsilon_{\rm c}$ exceeding $(\ell_2 - \widetilde{\lambda})y_0$, so $\ell_2$ is preferred in this range of $\varepsilon_{\rm c}$ values. However, $\ell_2$ has zero robustness for lower values of $\varepsilon_{\rm c}$ while $\ell_1$ has positive robustness for part of this lower range of $\varepsilon_{\rm c}$ values. In other words, the robustness curves cross one another, indicating the potential for a reversal of preference between these values of the forecasting slope, depending on the required accuracy, $\varepsilon_{\rm c}$.

**51d**. Probability distributions can be formulated based on different types of information. The two most prominent types are (1) frequency of occurrence of events and (2) subjective or contextual assessments of likelihood. These two approaches can also be combined in various ways.

Let us suppose that we estimate the pdf based on frequency of occurrence. Data are limited and the tails of the distribution are highly uncertain. We make the judgement that the estimated distribution is fairly reliable up to 2 standard deviations, $2\sigma$, around the mean, $\mu$. Beyond that interval the pdf is info-gap uncertain. An info-gap model that reflects this information is:

$$\mathcal{U}(h) = \{p(x) : \ |p(x) - \widetilde{p}(x)| \le w(x)h\}, \quad h \ge 0 \tag{732}$$

where we define $w(x)$ as:

$$w(x) = \begin{cases} 0 & \text{if} \quad x \in [\mu - 2\sigma,\ \mu + 2\sigma] \\ \widetilde{p}(x) & \text{else} \end{cases} \tag{733}$$

This info-gap model reflects fairly moderate uncertainty because the envelope decays on the tails.

Deeper uncertainty would be represented by the following envelope function:

$$w(x) = \begin{cases} 0 & \text{if} \quad x \in [\mu - 2\sigma,\ \mu + 2\sigma] \\ 1 & \text{else} \end{cases} \tag{734}$$

There are also many other info-gap models for representing info-gap uncertainty in probability distributions.

Solution to problem 52 **Future Worth and More: 1.** (p.52).

**52a**. The future worth at the end of $N$ years is:

$$\text{FW} = F\sum_{k=1}^{N}(1+i)^k = F\frac{(1+i)^{N+1}-(1+i)}{i} = \frac{1.05^{26}-1.05}{0.05}F = 50.1135F = 10^5 \tag{735}$$

Thus:

$$\boxed{F = \$1995.47} \tag{736}$$

**52b**. The definition of the robustness is:

$$\widehat{h}(\text{FW}_{\text{c}}) = \max\left\{h:\ \left(\min_{c\in\mathcal{U}(h)} F\right) \geq \text{FW}_{\text{c}}\right\} \tag{737}$$

Let $m(h)$ denote the inner minimum, which occurs at $c = \widetilde{c} - wh$. Thus:

$$(\widetilde{c}-wh)A \geq \text{FW}_{\text{c}} \quad\Longrightarrow\quad \boxed{\widehat{h}(\text{FW}_{\text{c}}) = \frac{1}{w}\left(\widetilde{c} - \frac{\text{FW}_{\text{c}}}{A}\right)} \tag{738}$$

or zero if this is negative.

**52c**. The sketch of the robustness curves is in fig. 16. Project 1 is robust-preferred for $\text{FW}_{\text{c}} \leq \text{FW}_{\times}$ where:

$$\widehat{h}_1(\text{FW}_{\text{c}}) = \widehat{h}_2(\text{FW}_{\text{c}}) \implies \frac{\widetilde{FW}_1 - \text{FW}_{\times}}{s_1} = \frac{\widetilde{FW}_2 - \text{FW}_{\times}}{s_2} \tag{739}$$

which implies:

$$\frac{\widetilde{FW}_1}{s_1} - \frac{\widetilde{FW}_2}{s_2} = \left(\frac{1}{s_1} - \frac{1}{s_2}\right)\text{FW}_{\times} \implies \text{FW}_{\times} = \frac{\widetilde{FW}_1 s_2 - \widetilde{FW}_2 s_1}{s_2 - s_1} \tag{740}$$

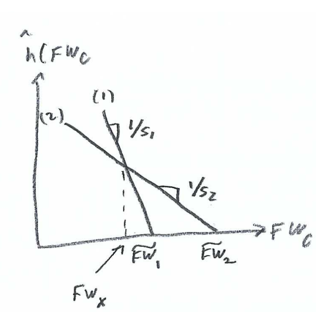


Figure 16: Fig. for solution of problem 52c.

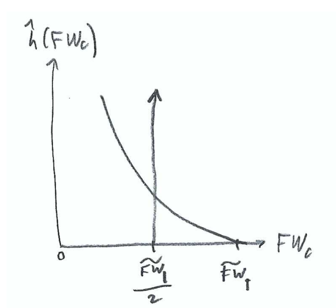


Figure 17: Fig. for solution of problem 53c.

**52d**. The definition of the robustness is:

$$\widehat{h}(f_{\text{c}}) = \max\left\{h:\ \left(\max_{p_{jk}\in\mathcal{U}(h)} f\right) \leq f_{\text{c}}\right\} \tag{741}$$

Let $m(h)$ denote the inner maximum, which occurs when the $p_{jk}$ are maximal:

$$m(h) = \frac{\sum_{j=1}^{N} w_j p_{jk}}{\sum_{j=1}^{N} w_j p_{jk-1}} - 1 = \frac{(1+h)\sum_{j=1}^{N} w_j \widetilde{p}_{jk}}{\sum_{j=1}^{N} w_j p_{jk-1}} - 1 = \frac{(1+h)\widetilde{I}_k}{I_{k-1}} - 1 \leq f_{\text{c}} \tag{742}$$

Solving for $h$ at equality yields the robustness function:

$$(1+h)\widetilde{I}_k = (1+f_{\text{c}})I_{k-1} \quad\Longrightarrow\quad \boxed{\widehat{h}(f_{\text{c}}) = (1+f_{\text{c}})\frac{I_{k-1}}{\widetilde{I}_k} - 1} \tag{743}$$

or zero if this is negative.

**52e**. The initial investment, at time $t = 0$, is $S$ and the real annual income is $R = \frac{S}{N}$ at the end of each of $N$ years. It is true that $S = NR$. However, the present worth of $N$ increments of future income, with positive inflation less than the nominal interest, will be less than the initial investment. Clearly this project is not economically justified.

We can see this more explicitly with the expression for present worth, though this is not necessary to answer the question of whether or this project is economically justified. There is no need to actually calculate the numerical value of the present worth: it will be negative.

The real interest rate is:

$$i_{\rm r} = \frac{i_{\rm nom} - f}{1 + f} \tag{744}$$

which is positive because $f < i_{\rm nom}$.

The present worth of this project is:

$$\text{PW} \;=\; -S + \sum_{n=1}^{N} (1 + i_{\rm r})^{-k} R \tag{745}$$

The $N$ terms $(1 + i_{\rm r})^{-k}$ are all less than 1, so the righthand side of eq.(745) is negative: the project is not economically justified.

Solution to problem 53 **Future Worth and More: 2.** (p.53).

**53a**. The future worth at the end of $N$ years is:

$$\text{FW} = f\sum_{k=1}^{N}(1+i)^k = f\frac{(1+i)^{N+1}-(1+i)}{i} = \frac{1.07^{N+1}-1.07}{0.07}1000 = 10^6 \tag{746}$$

Thus:

$$1.07^{N+1} = 0.07\times 1000+1.07 = 71.07 \implies (N+1)\ln 1.07 = \ln 71.07 \implies N = \frac{\ln 71.07}{\ln 1.07} - 1 = \boxed{62.017} \tag{747}$$

**53b**. The definition of the robustness is:

$$\widehat{h}(\text{FW}_\text{c}) = \max\left\{h:\ \left(\min_{c\in\mathcal{U}(h)}\text{PW}\right)\geq \text{PW}_\text{c}\right\} \tag{748}$$

where $\text{PW} = \frac{A}{c}$. Let $m(h)$ denote the inner minimum, which occurs at $c = \widetilde{c} + wh$. Thus:

$$\frac{A}{\widetilde{c}+wh} \geq \text{PW}_\text{c} \implies \frac{A}{\text{PW}_\text{c}} \geq \widetilde{c}+wh \implies \boxed{\widehat{h}(\text{PW}_\text{c}) = \frac{1}{w}\left(\frac{A}{\text{PW}_\text{c}} - \widetilde{c}\right)} \tag{749}$$

or zero if this is negative.

**53c**. See fig. 17 on p.146 where the vertical line to infinity represents the robustness curve of the known outcome of project 2, and the sloped curve is $\widehat{h}_1(\text{FW}_c)$.

The robustness is zero for project 2 in the interval $(\widetilde{FW}_1/2, \infty)$.

The robustness is positive for project 1 in the interval $(\widetilde{FW}_1/2, \widetilde{FW}_1)$.

Thus project 1 is robust-preferred in the interval $(\widetilde{FW}_1/2, \widetilde{FW}_1)$.

**53d**. The definition of the robustness is:

$$\widehat{h}(f_\text{c}) = \max\left\{h:\ \left(\max_{I_{k-1},I_k\in\mathcal{U}(h)} f\right)\leq f_\text{c}\right\} \tag{750}$$

Let $m(h)$ denote the inner maximum in the definition of the robustness. This maximum occurs, at horizon of uncertainty $h$, when $I_k$ is maximal and $I_{k-1}$ is minimal:

$$m(h) = \frac{(1+h)\widetilde{I}_k - (1-h)\widetilde{I}_{k-1}}{(1-h)\widetilde{I}_{k-1}} \leq f_\text{c} \tag{751}$$

Thus:

$$\left(\widetilde{I}_k + \widetilde{I}_{k-1}\right)h + \widetilde{I}_k - \widetilde{I}_{k-1} \leq f_\text{c}\widetilde{I}_{k-1} - hf_\text{c}\widetilde{I}_{k-1} \tag{752}$$

Thus:

$$\left(\widetilde{I}_k + (1+f_\text{c})\widetilde{I}_{k-1}\right)h \leq (1+f_\text{c})\widetilde{I}_{k-1} - \widetilde{I}_k \tag{753}$$

Thus:

$$\boxed{\widehat{h}(f_\text{c}) = \frac{(1+f_\text{c})\widetilde{I}_{k-1} - \widetilde{I}_k}{\widetilde{I}_k + (1+f_\text{c})\widetilde{I}_{k-1}}} \tag{754}$$

or zero if this is negative.

**53e**. We answer the question by comparing the present worths of the two projects.

The real discount (interest) rate of project $k$ is:

$$i_{\text{r}k} = \frac{i_\text{nom} - f_k}{1+f_k} \tag{755}$$

which is positive because $f_k < i_{\rm nom}$. Also, because $f_1 > f_2$, we see that:

$$0 < i_{\rm r1} < i_{\rm r2} \tag{756}$$

The present worth of project $k$ is:

$$\text{PW}_k = -S + \sum_{n=1}^{N} (1 + i_{\rm r}k)^{-n} R \tag{757}$$

Thus:

$$\text{PW}_1 > \text{PW}_2 \tag{758}$$

Thus project 1 is preferred.

Equivalently, we can calculate the PW of the project with the nominal earnings in each period, which are $(1+f_k)^n R$ for $n = 1, \ldots, N$. Thus the PW is:

$$\text{PW}_k = -S + \sum_{n=1}^{N} (1 + i_{\rm nom})^{-n} (1 + f_k)^n R = -S + \sum_{n=1}^{N} \left( \frac{1 + i_{\rm nom}}{1 + f_k} \right)^{-n} R \tag{759}$$

Now note that:

$$1 + i_{rk} = \frac{i_{\rm nom} - f_k}{1 + f_k} + 1 = \frac{i_{\rm nom} - f_k}{1 + f_k} + \frac{1 + f_k}{1 + f_k} = \frac{1 + i_{\rm nom}}{1 + f_k} \tag{760}$$

Thus the expressions for the PW in eqs.(757) and (759) are identical.

Solution to problem 54 **Loans and More.** (p.54).

**54a.** At time $t = 0$ you take a loan of $L$. After 1 year the accumulated interest, at rate $i$, is:

$$I_1 = iL \tag{761}$$

You now return the amount $R_1$ and the interest $I_1$, so you now hold a loan of $L - R_1$. At the end of the 2nd year you have held this loan for 1 year, so the accumulated interest, at rate $i$, is:

$$I_2 = i(L - R_1) \tag{762}$$

You now return the amount $R_2$ and the interest $I_2$, so you now hold a loan of $L - R_1 - R_2$. At the end of the 3rd year you have held this loan for 1 year, so the accumulated interest, at rate $i$, is:

$$I_3 = i(L - R_1 - R_2) \tag{763}$$

In summary, the 3 interest payments are:

$$I_1 = iL, \quad I_2 = i(L - R_1), \quad I_1 = i(L - R_1 - R_2) \tag{764}$$

**54b.** The future worth of these investments (total value at the end of year 3) is:

$$T = \text{FW} = (1+i)^2 R_1 + (1+i)R_2 + R_3 \tag{765}$$

The present worth (at the same rate of return) is:

$$\begin{aligned} \text{PW} &= (1+i)^{-3}\left[(1+i)^2 R_1 + (1+i)R_2 + R_3\right] && (766) \\ &= (1+i)^{-1}R_1 + (1+i)^{-2}R_2 + (1+i)^{-3}R_3 && (767) \end{aligned}$$

**54c.** The definition of the robustness function is:

$$\widehat{h}(\text{PW}_{\text{c}}) = \max\left\{h : \left(\min_{R\in\mathcal{U}(h)} \text{PW}\right) \ge \text{PW}_{\text{c}}\right\} \tag{768}$$

Let $m(h)$ denote the inner minimum, which occurs when $R = \widetilde{R} - wh$. Thus:

$$m(h) = b(\widetilde{R} - wh) \ge \text{PW}_{\text{c}} \quad \Longrightarrow \quad \widehat{h}(\text{PW}_{\text{c}}) = \frac{b\widetilde{R} - \text{PW}_{\text{c}}}{bw} \tag{769}$$

or zero if this is negative.

**54d.** The robustness curves are shown schematically in fig. 18, p.151.

Equate the two robustness functions to find the value of $\text{FW}_{\text{c}}$ at which the robustness curves intersect:

$$\widehat{h}_1(\text{FW}_\times) = \widehat{h}_2(\text{FW}_\times) \iff \frac{a - \text{FW}_\times}{b} = \frac{c - \text{FW}_\times}{d} \iff ad - d\text{FW}_\times = cb - b\text{FW}_\times \tag{770}$$

Hence:

$$(d - b)\text{FW}_\times = ad - cb \iff \text{FW}_\times = \frac{ad - cb}{d - b} \tag{771}$$

Thus project 1 is robust-preferred for:

$$\text{FW}_\times < \text{FW} < a \tag{772}$$

Note the strict inequalities.

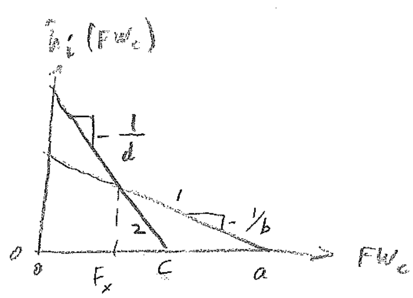


Figure 18: Schematic robustness curves for solution of problem 54d.

Solution to problem 55 **Time and money.** (p.55).

**55a**. The present worth is:

$$\text{PW} = \sum_{k=1}^{N} (1+i)^{-k} \left[ (1+i)^k R - C \right] \tag{773}$$

$$= NR - C \sum_{k=1}^{N} (1+i)^{-k} \tag{774}$$

$$= \boxed{NR - C \frac{1-(1+i)^{-N}}{i}} \tag{775}$$

The numerical value of the present worth, with the given parameters, is:

$$\text{PW} = 3000 - 2000 \frac{1 - 1.1^{-30}}{0.1} \tag{776}$$

$$= 3000 - 2000 \times 9.4269 \tag{777}$$

$$= 3000 - 18,853.83 \tag{778}$$

$$= \boxed{-15,853.83} \tag{779}$$

**55b**. The present worths of the two projects are:

$$\text{PW}_1 = \sum_{k=1}^{N} (1+i)^{-k} R = R \frac{1-(1+i)^{-N}}{i} \tag{780}$$

$$\text{PW}_2 = \sum_{k=1}^{M} (1+i)^{-k} 2R = 2R \frac{1-(1+i)^{-M}}{i} \tag{781}$$

Equating these two expressions for present worth:

$$\text{PW}_1 = \text{PW}_2 \iff \frac{1-(1+i)^{-N}}{i} = 2\frac{1-(1+i)^{-M}}{i} \iff -(1+i)^{-N} = 1 - 2(1+i)^{-M} \tag{782}$$

Hence:

$$(1+i)^{-N} = -1 + 2(1+i)^{-M} \iff -N \ln(1+i) = \ln\left[-1 + 2(1+i)^{-M}\right] \tag{783}$$

Hence:

$$\boxed{N = -\frac{\ln\left[-1 + 2(1+i)^{-M}\right]}{\ln(1+i)}} \tag{784}$$

For the values specified we find:

$$N = -\frac{\ln\left[-1 + 2 \times 1.05^{-10}\right]}{\ln 1.05} = \boxed{30.3169} \tag{785}$$

The first project must run much longer than the 2nd project because the income from the first project is much lower.

**55c**. The definition of the robustness function is:

$$\widehat{h}(\text{PW}_\text{c}) = \max\left\{ h : \left( \min_{i \in \mathcal{U}(h)} \text{PW} \right) \ge \text{PW}_\text{c} \right\} \tag{786}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function. The present worth is:

$$\text{PW} = \sum_{k=1}^{N} (1+i)^{-k} R = R \frac{1-(1+i)^{-N}}{i} \tag{787}$$

The inner minimum occurs when $i$ is maximal at horizon of uncertainty $h$, namely, for $i = \widetilde{i} + sh$. Thus:

$$\boxed{m(h) = R\frac{1-(1+\widetilde{i}+sh)^{-N}}{\widetilde{i}+sh}} \tag{788}$$

**55d.** The definition of the robustness function is:

$$\widehat{h}(R_{\rm c}) = \max\left\{h:\ \left(\min_{x\in\mathcal{U}(h)} R\right) \ge R_{\rm c}\right\} \tag{789}$$

Let $m(h)$ denote the inner minimum, which is the inverse of the robustness function. $R$ is a monotonically decreasing function of $x$, so $m(h)$ occurs for maximal $x$ at horizon of uncertainty $h$, namely, at $x = \widetilde{x} + wh$. Thus:

$$m(h) = \frac{\alpha(\widetilde{x}+wh)+\beta}{\widetilde{x}+wh} \ge R_{\rm c} \tag{790}$$

Solving this relation for $h$ at equality yields the robustness:

$$\frac{\alpha(\widetilde{x}+wh)+\beta}{\widetilde{x}+wh} = R_{\rm c} \iff \alpha(\widetilde{x}+wh)+\beta = R_{\rm c}(\widetilde{x}+wh) \iff \alpha\widetilde{x}-R_{\rm c}\widetilde{x}+\beta = (-\alpha w+R_{\rm c}w)h \tag{791}$$

Hence:

$$\boxed{\widehat{h}(R_{\rm c}) = \frac{\alpha\widetilde{x}+\beta-R_{\rm c}\widetilde{x}}{-\alpha w+R_{\rm c}w}} \tag{792}$$

or zero if this is negative. Note that the robustness vanishes for $R_{\rm c} \ge \alpha + \frac{\beta}{\widetilde{x}}$ which is the nominal value of the return. Also, the robustness approaches infinity at $R_{\rm c}$ approaches $\alpha$ from above.

**55e.** The real value of the nominal quantity $A_k$ is:

$$R_k = (1+f)^{-k}A_k \tag{793}$$

The present worth of this real value of $A_k$ is:

$$\text{PW}_k = (1+i_{\rm r})^{-k}R_k = (1+i_{\rm r})^{-k}(1+f)^{-k}A_k \tag{794}$$

Thus the present worth of the total income is:

$$\boxed{\text{PW} = \sum_{k=1}^{N}\text{PW}_k = \sum_{k=1}^{N}(1+i_{\rm r})^{-k}(1+f)^{-k}A_k} \tag{795}$$

**55f.** The nominal payment at the end of the $k$th year is:

$$P_k = (1+r)^{k-1}P \tag{796}$$

The real value, in $t = 0$ dollars, is:

$$R_k = (1+f)^{-k}P_k = (1+f)^{-k}(1+r)^{k-1}P \tag{797}$$

Thus, for the values given:

$$\boxed{R_7 = 1.09^{-7}1.15^6 \times \$50,000 = \$63,266.17} \tag{798}$$